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Conic Homographies and Bitangent Pencils

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Abstract. Conic homographies are homographies of the projective plane preserving a given conic. They are naturally associated with bitangent pencils of conics, which are pencils containing a double line. Here we study this connection and relate these pencils to various groups of homographies associated with a conic. A detailed analysis of the automorphisms of a given pencil specializes to the description of affinities preserving a conic. While the algebraic structure of the groups involved is simple, it seems that a geometric study of the various questions is lacking or has not been given much attention. In this respect the article reviews several well known results but also adds some new points of view and results, leading to a detailed description of the group of homographies preserving a bitangent pencil and, as a consequence, also the group of affinities preserving an affine conic.

1. Introduction

Deviating somewhat from the standard definition I call *bitangent* the pencils $\mathcal{P}$ of conics which are defined in the projective plane through equations of the form

$$\alpha c + \beta e^2 = 0.$$

Here $c(x, y, z) = 0$ and $e(x, y, z) = 0$ are the equations in *homogeneous coordinates* of a non-degenerate conic and a line and α, β are arbitrary, no simultaneously zero, real numbers. To be short I use the same symbol for the set and an equation representing it. Thus c denotes the set of points of a conic and $c = 0$ denotes an equation representing this set in some system of homogeneous coordinates. To denote bitangent pencils I use the letter $\mathcal{P}$ but also the more specific symbol (c, e). For any other member-conic c' of the pencil (c', e) represents the same pencil. I call line e and the pole E of e with respect to c respectively *invariant line* and *center* of the pencil. The intersection points $c \cap e$, if any, are called *fixed* or *base* points of the pencil. As is seen from the above equation, if such points exist, they lie on every member-conic of the pencil.

Traditionally the term *bitangent* is used only for pencils (c, e) for which line e either intersects c or is disjoint from it. This amounts to a second order (real or complex) contact between the members of the pencil, wherefore also the stem of the term. Pencils for which c and e^2 are tangent have a fourth order contact between their members and are classified under the name *superosculating* pencils

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([4, vol.II, p.188], [12, p.136]) or *penosculating* pencils ([9, p.268]). Here I take the liberty to incorporate this class of pencils into the bitangent ones, thus considering as a distinguished category the class of pencils which contain among their members a double line. This is done under the perspective of the tight relationship of conic homographies with bitangent pencils under this wider sense. An inspiring discussion in synthetic style on pencils of conics, which however, despite its wide extend, does not contain the relationship studied here, can be found in Steiner's lectures ([11, pp.224–430]).

Every homography[1] f of the plane preserving the conic c defines a bitangent pencil (c, e) to which conic c belongs as a member and to which f acts by preserving each and every member of the pencil. The pencil contains a double line e, which coincides with the *axis* of the homography. In this article I am mainly interested in the investigation of the geometric properties of four groups: $\mathcal{G}(c)$, $\mathcal{G}(c, e)$, $\mathcal{K}(c, e)$ and $\mathcal{A}(c)$, consisting respectively of homographies (i) preserving a conic c, (ii) preserving a pencil (c, e), (iii) permuting the members of a pencil, and (iv) preserving an affine conic. Last group is identical with a group of type $\mathcal{G}(c, e)$ in which line e is identified with the line at infinity. In Section 2 (*Conic homographies*) I review the well-known basic facts on homographies of conics stating them as propositions for easy reference. Their proofs can be found in the references given (especially [4, vol.II, Chapter 16], [12, Chapter VIII]). Section 3 (*Bitangent pencils*) is a short review on the classification of bitangent pencils. In Section 4 (*The isotropy at a point*) I examine the isotropy of actions of the groups referred above. In this, as well as in the subsequent sections, I supply the proofs of propositions for which I could not find a reference. Section 5 (*Automorphisms of pencils*) is dedicated to an analysis of the group $\mathcal{G}(c, e)$. Section 6 (*Bitangent flow*) comments on the vector-field point of view of a pencil and the characterization of its flow through a simple configuration on the invariant line. Section 7 (*The perspectivity group of a pencil*) contains a discussion on the group $\mathcal{K}(c, e)$ permuting the members of a pencil. Finally, Section 8 (*Conic affinities*) applies the results of the previous sections to the description of the group of affinities preserving an affine conic.

2. Conic homographies

Conic homographies are by definition restrictions on c of homographies of the plane that preserve a given conic c. One can define also such maps intrinsically, without considering their extension to the ambient plane. For this fix a point A on c and a line m and define the image $Y = f(X)$ of a point X by using its representation $f' = p \circ f \circ p^{-1}$ through the (stereographic) projection p of the conic onto line m centered at A.

[1] I use this term coming from my native language (greek) as an alternative equivalent to *projectivity*.

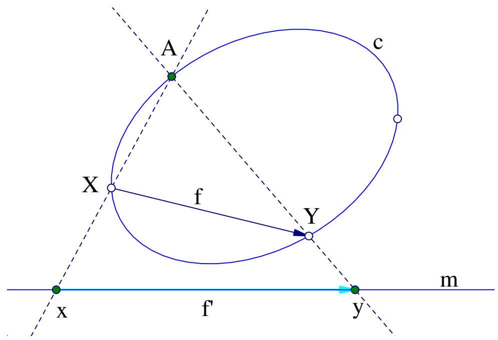

Figure 1. Conic homography

Homography f is the defined using a Moebius transformation ([10, p.40]) (see Figure 1)

$$y = f'(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

It can be shown ([4, vol.II, p.179]) that the two definitions are equivalent. Depending on the kind of the question one can prefer the first definition, through the restriction of a global homography, or the second through the projection. Later point of view implies the following ([10, p.47]).

Proposition 1. *A conic homography on the conic c is completely determined by giving three points A, B, C on the conic and their images A', B', C'. In particular, if a conic homography fixes three points on c it is the identity.*

The two ways to define conic homographies on a conic c reflect to the representation of their group $\mathcal{G}(c)$. In the first case, since every conic can be brought in appropriate homogeneous coordinates to the form ([2, p.209])

$$x^2 + y^2 - z^2 = 0$$

their group is represented through the group preserving this quadratic form which is $O(2,1)$. By describing homographies through Moebius transformations $\mathcal{G}(c)$ is represented with the group $PGL(2,\mathbb{R})$. The two representations are isomorphic but not *naturally* isomorphic ([4, vol.II, p.180]). An isomorphism between them can be established by fixing $A \in c$ and associating to each $f \in O(2,1)$ the corresponding induced in m transformation $f' \in PGL(2,\mathbb{R})$ ([15, p.235]), in the way this was defined above through the stereographic projection from A onto some line m (see Figure 1).

Next basic property of conic homographies is the existence of their *homography axis* ([4, vol.II, p.178]).

Proposition 2. *Given a conic homography f of the conic c, for every pair of points A, B on c, lines AB' and BA', with $A' = f(A), B' = f(B)$ intersect on a fixed line e, the homography axis of f. The fixed points of f, if any, are the intersection points of c and e.*

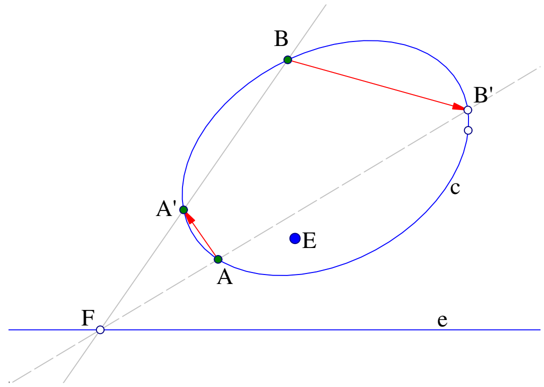

Figure 2. Homography axis e

Remark. This property implies (see Figure 2) an obvious geometric construction of the image B' of an arbitrary point B under the homography once we know the axis and a single point A and its image A' on the conic: Draw $A'B$ to find its intersection $F = A'B \cap e$ and from there draw line FA to find its intersection $B' = FA \cap c$.

Note that the existence of the axis is a consequence of the existence of at least a fixed point P for every homography f of the plane ([15, p.243]). If f preserves in addition a conic c, then it is easily shown that the polar e of P with respect to the conic must be invariant and coincides either with a tangent of the conic at a fixed point of f or coincides with the axis of f.

Next important property of conic homographies is the preservation of the whole bitangent family (c, e) generated by the conic c and the axis e of the homography. Here the viewpoint must be that of the restriction on c (and e) of a global homography of the plane.

Proposition 3. *Given a conic homography f of the conic c with homograpy axis e, the transformation f preserves every member $c' = \alpha c + \beta e^2$ of the pencil generated by c and the (double) line e. The pole E of the axis e with respect to c is a fixed point of the homography. It is also the pole of e with respect to every conic of the pencil. Line e is the axis of the conic homography induced by f on every member c' of the pencil.*

To prove the claims show first that line e is preserved by f (see Figure 3). For this take on c points A, B collinear with the pole E and consider their images $A' = f(A), B' = f(B)$. Since AB contains the pole of e, the pole Q of AB will be on line e. By Proposition 2 lines AB', BA' intersect at a point G of line e. It follows that the intersection point F of AA', BB' is also on e and that $A'B'$ passes through E. Hence the pole Q' of $A'B'$ will be on line e. Since homographies preserve polarity it must be $Q' = f(Q)$ and f preserves line e. From this follow easily all other claims of the proposition.

I call pencil $\mathcal{P} = (c, e)$ the *associated to f bitangent pencil*. I use also for E the name *center of the pencil* or/and *center of the conic homography f*.

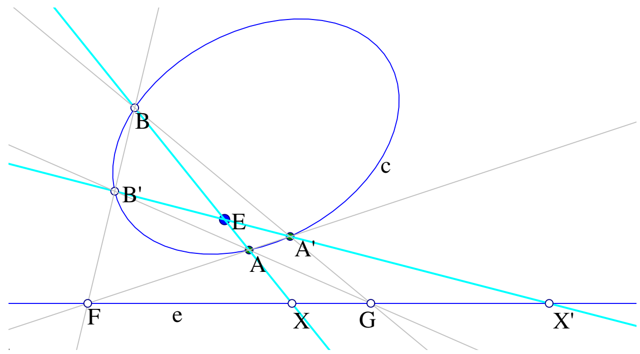

Figure 3. Invariance of axis e

Next deal with conic homographies is their distinction in *involutive* and non-involutive, i.e. homographies of period two and all others ([4, vol.II, p.179], [12, p.223]). Following proposition identifies involutive homographies with *harmonic homologies* (see Section 7) preserving a conic.

Proposition 4. *Every involutive conic homography f of the conic c fixes every point of its axis e. Inversely if it fixes its axis and $E \notin e$ it is involutive. Equivalently for each point $P \in c$ with $P' = f(P)$, line PP' passes through E the pole of the axis e of f. Point E is called in this case the center or Fregier point of the involution.*

Involutions are important because they can represent through their compositions every conic homography. The bitangent pencils (c, e) of interest, though, are those created by non-involutive conic homographies $f : c \to c$, and it will be seen that the automorphisms of such pencils consist of all homographies of the conic which commute with f. The following proposition clarifies the decomposition of every conic homography in two involutions ([4, vol.II, p.178], [12, p.224]).

Proposition 5. *Every conic homography f of a conic c can be represented as the product $f = I_2 \circ I_1$ of two involutions I_1, I_2. The centers of the involutions are necesserilly on the axis e of f. In addition the center of one of them may be any arbitrary point $P_1 \in e$ (not a fixed point of f), the center of the other $P_2 \in e$ is then uniquely determined.*

Following well known proposition signals also an important relation between a non-involutive conic homography and the associated to it bitangent pencil. I call the method suggested by this proposition the *tangential generation* of a non-involutive conic homography. It expresses for non-involutive homographies the counterpart of the property of involutive homographies to have all lines PP', with $(P \in c,\ P' = f(P))$, passing through a fixed point.

Proposition 6. *For every non-involutive conic homography f of a conic c and every point $P \in c$ and $P' = f(P)$ lines PP' envelope another conic c'. Conic c' is a member of the associated to f bitangent pencil. Inversely, given two member* *conics c, c' of a bitangent pencil the previous procedure defines a conic homography on c having its axis identical with the invariant line e of the bitangent pencil. Further the contact point Q' of line PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection point Q of PP' with the axis e of f.*

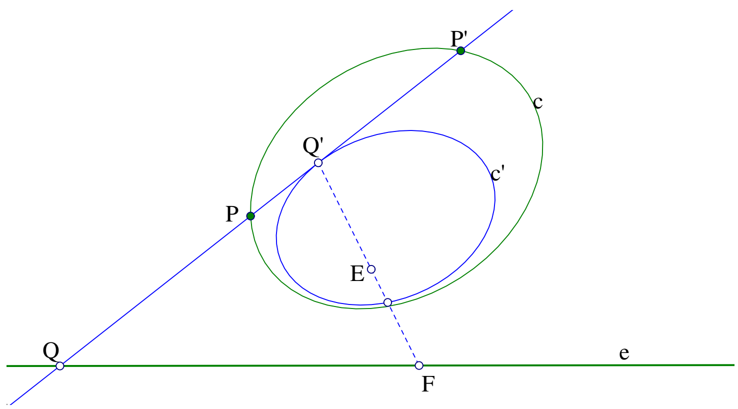

Figure 4. Tangential Generation

An elegant proof of these statements up to the last is implied by a proposition proved in [8, p.253], see also [4, vol.II, p.214] and [5, p.245]. Last statement follows from the fact that Q is the pole of line $Q'E$ (see Figure 4).

Propositions 5 and 6 allow a first description of the *automorphism group* $\mathcal{G}(c, e)$ of a given pencil (c, e) i.e. the group of homographies mapping every member-conic of the pencil onto itself. The group consists of homographies of two kinds. The first kind are the involutive homographies which are completely defined by giving their center on line e or their axis through E. The other homographies preserving the pencil are the non-involutive, which are compositions of pairs of involutions of the previous kind. Since we can put the center of one of the two involutions anywhere on e (except the intersection points of e and c), the homographies of this kind are parameterized by the location of their other center.

Before to look closer at these groups I digress for a short review of the classification of bitangent pencils and an associated naming convention for homographies.

3. Bitangent pencils

There are three cases of bitangent pencils in the real projective plane which are displayed in Figure 5. They are distinguished by the relative location of the invariant line and the conic generating the pencil.

Proposition 7. *Every bitangent pencil of conics is projectively equivalent to one generated by a fixed conic c and a fixed line e in one of the following three possible configurations.*
(I) *The line e non-intersecting the conic c (elliptic).*
(II) *The line e intersecting the conic c at two points (hyperbolic).*
(III) *The line e being tangent to the conic c (parabolic).*

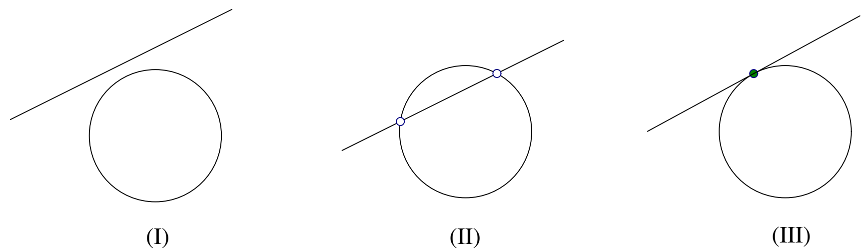

Figure 5. Bitangent pencils classification

The proof follows by reducing each case to a kind of *normal* form. For case (I) select a projective basis A, B, C making a *self-polar* triangle with respect to c. For this take A to be the pole of e with respect to c, take then B arbitrary on line e and define C to be the intersection of e and the polar p_B of B with respect to c. The triangle ABC thus defined is self-polar with respect to c and the equations of c and e take the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0, \quad x = 0.$$

In this we can assume that $\alpha > 0, \beta > 0$ and $\gamma < 0$. Applying then a simple projective transformation we reduce the equations in the form

$$x^2 + y^2 - z^2 = 0, \quad x = 0.$$

For case (II) one can define a projective basis A, B, C for which the equation of c and e take respectively the form

$$x^2 - yz = 0, \quad x = 0.$$

For this it suffices to take for A the intersection of the two tangents t_B, t_C to the conic at the intersections B, C of the line e with the conic c and the *unit* point of the basis on the conic. The projective equivalence of two such systems is obvious. Finally a system of type (III) can be reduced to one of type (II) by selecting again an appropriate projective base A, B, C. For this take B to be the contact point of the line and the conic. Take then A to be an arbitrary point on the conic and define C to be the intersection point of the tangents t_A, t_B. This reduces again the equations to the form ([4, vol.II, p.188])

$$xy - z^2 = 0, \quad x = 0.$$

The projective equivalence of two such *normal forms* is again obvious.

Remark. The disctinction of the three cases of bitangent pencils leads to a natural distinction of the non-involutive homographies in four general classes. The first class consists of homographies preserving a conic, such that the associated bitangent pencil is elliptic. It is natural to call these homographies *elliptic*. Analogously homographies preserving a conic and such that the associated bitangent pencil is hyperbolic or parabolic can be called respectively *hyperbolic* or *parabolic*. All other non-involutive homographies, not falling in one of these categories (i.e. not preserving a conic), could be called *loxodromic*. Simple arguments related to the set of fixed points of an homography show easily that the four classes are disjoint. In addition since, by Proposition 2, the fixed points of a homography f preserving a conic are its intersection points with the respective homography axis e, we see that the three classes of non-involutive homographies preserving a conic are characterized by the number of their fixed points on the conic ([12, p.101], [15, p 243]). This naming convention of the first three classes conforms also with the traditional naming of the corresponding kinds of real Moebius transformations induced on the invariant line of the associated pencil ([10, p.68]).

4. The isotropy at a point

Next proposition describes the structure of the isotropy group $\mathcal{G}_{AB}(c,e)$ for a hyperbolic pencil (c,e) at each one of the two intersection points $\{A,B\} = c \cap e$.

Proposition 8. *Every homography preserving both, a conic c, an intersecting the conic line e, and fixing one (A say) of the two intersection points A, B of c and e belongs to a group $\mathcal{G}_{AB}(c,e)$ of homographies, which is isomorphic to the multiplicative group $\mathbb{R}^*$ and can be parameterized by the points of the two disjoint arcs into which c is divided by A, B.*

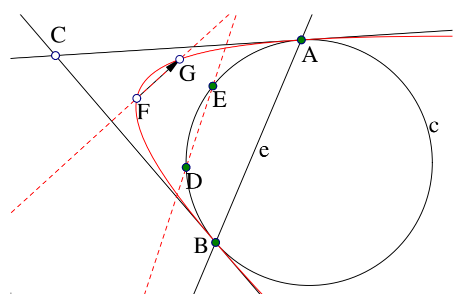

Figure 6. Isotropy of type IIb

Figure 6 illustrates the proof. Assume that homography f preserves both, the conic c, the line e, and also fixes A. Then it fixes also the other point B and also the pole C of line AB. Consequently f is uniquely determined by prescribing its value $f(D) = E \in c$ at a point $D \in c$. I denote this homography by f_{DE}. This map has a simple matrix representation in the projective basis $\{C, A, B, D\}$ in which conic c is represented by the equation $yz - x^2 = 0$ and line AB by $x = 0$, the unit point $D(1,1,1)$ being on the conic. In this basis and for $E \in c$ with coordinates (x,y,z) map f_{DE} is represented by non-zero multiples of the matrix

$$F_{DE} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

This representation shows that $\mathcal{G}_{AB}(c,e)$ is isomorphic to the multiplicative group $\mathbb{R}^*$ which has two connected components. The group $\mathcal{G}_{AB}(c,e)$ is the union of two cosets $\mathcal{G}_1, \mathcal{G}_2$ corresponding to the two arcs on c, defined by the two points

A, B. The arc containing point D corresponds to subgroup $\mathcal{G}_1$, coinciding with the connected component containing the identity. The other arc defined by AB corresponds to the other connected component $\mathcal{G}_2$ of the group. For points E on the same arc with D the corresponding homography f_{DE} preserves the two arcs defined by A, B, whereas for points E on the other arc than the one containing D the corresponding homography f interchanges the two arcs.

Obviously point D can be any point of c different from A and B. Selecting another place for D and varying E generates the same group of homographies. Clearly also there is a symmetry in the roles of A, B and the group can be identified with the group of homographies preserving conic c and fixing both points A and B.

Remark. Note that there is a unique involution I_0 contained in $\mathcal{G}_{AB}(c, e)$. It is the one having axis AB and center C, obtained for the position of E for which line DE passes through C, the corresponding matrix being then the $diagonal(-1, 1, 1)$.

Following proposition deals with the isotropy of pencils (c, e) at *normal* points of the conic c, i.e. points different from its intersection point(s) with the invariant line e.

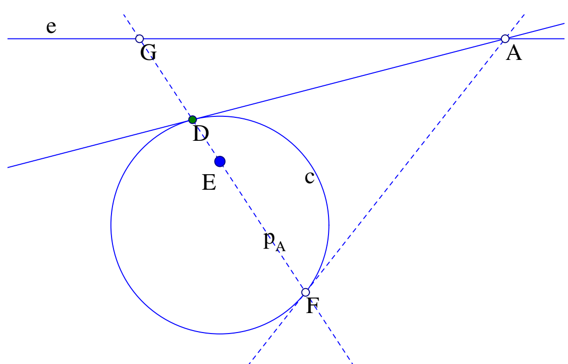

Figure 7. Isotropy at normal points

Proposition 9. *For every normal point D of the conic c the isotropy group $\mathcal{G}_D(c, e)$ is isomorphic to $\mathbb{Z}_2$. The different from the identity element of this group is the involution I_D with axis DE.*

For types (I) and (II) of pencils a proof is the following. Let the homography f preserve the conic c, the line e and fix point D. Then it preserves also the tangent t_D at D and consequently fixes also the intersection point A of this line with the axis e (see Figure 7). It is easily seen that the polar DF of A passes through the center E of the pencil and that f preserves DF. Thus the polar DF carries three points, which remain fixed under f. Since f has three fixed points on line DF it leaves the whole line fixed, hence it coincides with the involution with axis DF and center A.

For type (III) pencils the proof follows from the previous proposition. In fact, assuming $B = c \cap e$ and $A \in c, A \neq B$ an element f of the isotropy group $\mathcal{G}_A(c, e)$ fixes points A, B hence $f \in \mathcal{G}_{AB}$. But from all elements f of the last group only the involution I_B with axis AB preserves the members of the pencil (c, e). This is immediately seen by considering the decomposition of f in two involutions. Would f preserve the member-conics of the bitangent family (c, e) then, by Proposition 5, the centers of these involutions would be points of e but this is impossible for $f \in \mathcal{G}_{AB}$, since the involutions must in this case be centered on line AB.

A byproduct of the short investigation on the isotropy group $\mathcal{G}_{AB}$ of a hyperbolic pencil (c, e) is a couple of results concerning the orbits of $\mathcal{G}_{AB}$ on points of the plane, other than the fixed points A, B, C. To formulate it properly I adopt for triangle ABC the name of *invariant triangle*.

Proposition 10. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC the orbit $\mathcal{G}_{AB}F$ is the member conic c_F of the hyperbolic bitangent pencil (c, e) which passes through F.*

In fact, $\mathcal{G}_{AB}F \subset c_F$ since all $f \in \mathcal{G}_{AB}$ preserve the member-conics of the pencil (see Figure 6). By the continuity of the action the two sets must then be identical. The second result that comes as byproduct is the one suggested by Figure 8. In its formulation as well the formulation of next proposition I use the maps introduced in the course of the proof of Proposition 8.

Proposition 11. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, the intersection point H of lines DE and FG, where $G = f_{DE}(F)$, as E varies on the conic c, describes a conic passing through points A, B, C, D and F.*

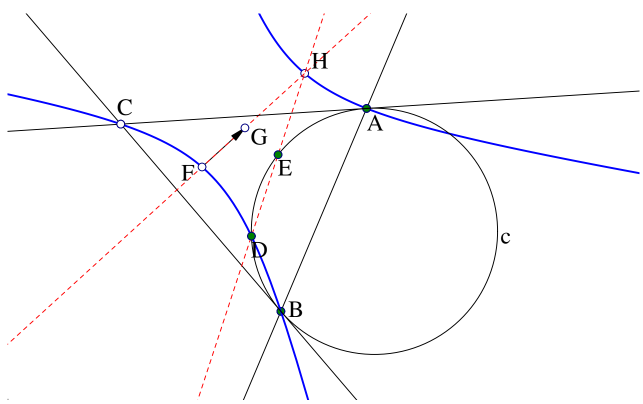

Figure 8. A triangle conic

To prove this consider the projective basis and the matrix representation of f_{DE} given above. It is easy to describe in this basis the map sending line DE to FG. Indeed let $E(x, y, z)$ be a point on the conic. Line DE has coefficients $(y - z, z - x, x - y)$. Thus, assuming F has coordinates (α, β, γ), its image will be described by the coordinates $(\alpha x, \beta y, \gamma z)$. The coefficients of the line FG will be then $(\beta\gamma(y-z), \gamma\alpha(z-x), \alpha\beta(x-y))$. Thus the correspondence of line FG to line DE will be described in terms of their coefficients by the projective transformation

$$(y-z, z-x, x-y) \mapsto (\,\beta\gamma(y-z),\ \gamma\alpha(z-x),\ \alpha\beta(x-y)\,).$$

The proposition is proved then by applying the *Chasles-Steiner* theorem, according to which the intersections of homologous lines of two pencils related by a homography describe a conic ([3, p.73], [4, vol.II, p.173]). According to this theorem the conic passes through the vertices of the pencils D, F. It is also easily seen that the conic passes through points A, B and C.

Proposition 12. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, lines EG with $G = f_{DE}(F)$ as E varies on c envelope a conic which belongs to the bitangent pencil $(c, e = AB)$.*

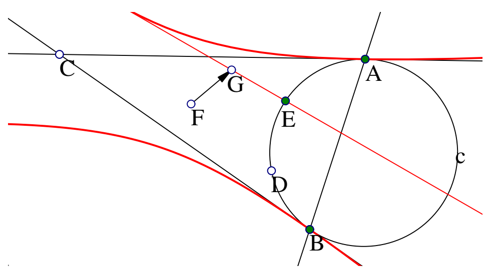

Figure 9. Bitangent member as envelope

The proof can be based on the dual of the argument of *Chasles-Steiner* ([3, p.89]), according to which the lines joining homologous points of a homographic relation between two ranges of points envelope a conic. Here lines EG (see Figure 9) join points (x, y, z) on the conic c with points $(\alpha x, \beta y, \gamma z)$ on the conic c_F, hence their coefficients are given by

$$(\,(\gamma-\beta)yz,\ (\alpha-\gamma)zx,\ (\beta-\alpha)xy\,).$$

Taking the traces of these lines on $x = 0$ and $y = 0$ we find that the corresponding coordinates $(0, y', z')$ and $(x'', 0, z'')$ satisfy an equation of the form $\tau'\tau'' = \kappa$, where $\tau' = y'/z',\ \tau'' = x''/z''$ and κ is a constant. Thus lines EG join points on $x = 0$ and $y = 0$ related by a homographic relation hence they envelope a conic. It is also easily seen that this conic passes through A, B and has there tangents CA, CB hence it belongs to the bitangent family.

Continuing the examination of possible isotropies, after the short digression on the three last propositions, I examine the isotropy group $\mathcal{G}_A(c, e)$ of a parabolic pencil (c, e), for which the axis e is tangent to the conic c at a point A. An element $f \in \mathcal{G}_A(c, e)$ may have A as its unique fixed point or may have an additional fixed point $B \neq A$.

An element $f \in \mathcal{G}_A(c, e)$ having A as a unique fixed point cannot leave invariant another line through A, since this would create a second fixed point on c. Also there is no other fixed point on the tangent e since this would also create another fixed point on c.

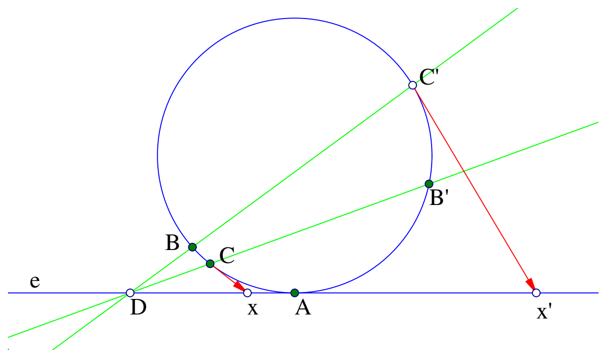

Figure 10. Parabolic isotropy

Proposition 13. *The group $\mathcal{G}_A^0$ including the identity and all homographies f, which preserve a conic c and have A as a unique fixed point, is isomorphic to the additive group $\mathbb{R}$. Every non-indetity homography in this group induces in the tangent e at A a parabolic transformation, which in line coordinates with origin at A is described by a function of the kind $x' = ax/(bx + a)$ or equivalently, by setting $d = b/a$, through the relation*

$$\frac{1}{x'} - \frac{1}{x} = d.$$

This function uniquely describes the conic homography from which it is induced in line e. All elements of this group are non-involutive.

In fact consider the induced Moebius transformation on line e with respect to coordinates with origin at A (see Figure 10). Since A is a fixed point this transformation will have the form $x' = ax/(bx + c)$. Since this is the only root of the equation $x(bx + c) = ax \Leftrightarrow bx^2 + (c - a)x = 0$, it must be $c = a$. Since for every point B other than A the tangents $t_B, t_{B'}$ where $B' = f(B)$ intersect line e at corresponding points $C, C' = f(C)$ the definition of f from its action on line e is complete and unique. The statement on the isomorphism results from the above representation of the transformation. The value $d = 0$ corresponds to the identity transformation. Every other value $d \in \mathbb{R}$ defines a unique parabolic transformation and the product of two such transformations corresponds to the sum $d + d'$ of these constants.

The group $\mathcal{G}_A$ of all homographies preserving a conic c and fixing a point A contains obviously the group $\mathcal{G}_A^0$. The other elements of this group will fix an additional point B on the conic. Consequently the group will be represented as a union $\mathcal{G}_A = \mathcal{G}_A^0 \cup_{B \neq A} \mathcal{G}_{AB}$. For another point C different from A and B the corresponding group $\mathcal{G}_{AC}$ is conjugate to $\mathcal{G}_{AB}$, by an element of the group $\mathcal{G}_A^0$. In fact, by the previous discussion there is a unique element $f \in \mathcal{G}_A^0$ mapping B to C. Then $Ad_f(\mathcal{G}_{AB}) = \mathcal{G}_{AC}$ i.e. every element $f_C \in \mathcal{G}_{AC}$ is represented as $f_C = f \circ f_B \circ f^{-1}$ with $f_B \in \mathcal{G}_{AB}$. These remarks lead to the following proposition.

Proposition 14. *The isotropy group $\mathcal{G}_A$ of conic homographies fixing a point A of the conic c is the semi-direct product of its subgroups $\mathcal{G}_A^0$ of all homographies f which preserve c and have the unique fixed point A on c and the subgroup $\mathcal{G}_{AB}$ of conic homographies which fix simultaneously A and another point $B \in c$ different from A.*

To prove this apply the criterion ([1, p.285]) by which such a decomposition of the group is a consequence of the following two properties: (i) Every element g of the group $\mathcal{G}_A$ is expressible in a unique way as a product $g = g_B \circ g_A$ with $g_A \in \mathcal{G}_A^0, \ g_B \in \mathcal{G}_{AB}$ and (ii) Group $\mathcal{G}_A^0$ is a normal subgroup of $\mathcal{G}_A$. Starting from property (ii) assume that $f \in \mathcal{G}_A$ has the form $f = g_B \circ g_A \circ g_B^{-1}$. Should f fix a point $C \in c$ different from A then it would be $g_A(g_B^{-1}(C)) = g_B^{-1}(C)$ i.e. $g_B^{-1}(C)$ would be a fixed point of g_A, hence $g_B^{-1}(C) = A$ which is impossible. To prove (i) show first that every element in $\mathcal{G}_A$ is expressible as a product $g = g_B \circ g_A$. This is clear if $g \in \mathcal{G}_A^0$ or $g \in \mathcal{G}_{AB}$. Assume then that g in addition to A fixes also the point $C \in c$ different from A. Then as remarked above g can be written in the form $g = g_A \circ g_B \circ g_A^{-1}$, hence $g = g_B \circ (g_B^{-1} \circ g_A \circ g_B \circ g_A^{-1})$ and the parenthesis is an element of $\mathcal{G}_A^0$. That such a representation is also unique follows trivially, since the equation $g_A \circ g_B = g'_A \circ g'_B$ would imply $g_A^{-1} \circ g'_A = g'_B \circ g_B^{-1}$ implying $g_A = g'_A$ and $g_B = g'_B$, since the two subgroups $\mathcal{G}_A^0$ and $\mathcal{G}_{AB}$ have in common only the identity element.

5. Automorphisms of pencils

In this section I examine the automorphism group $\mathcal{G}(c, e)$ of a pencil (c, e) and in particular the non-involutive automorphisms. Every such automorphism is a conic homography f of conic c preserving also the line e. Hence it induces on line e a homography which can be represented by a Moebius transformation

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Inversely, knowing the induced homography on line e from a non-involutive homography one can reconstruct the homography on every other member-conic c of the pencil. Figure 11 illustrates the construction of the image point $B' = f(B)$ by drawing the tangent t_B of c at B and finding its intersection C with e. The image $f(B)$ is found by taking the image point $C' = f(C)$ on e and drawing from there the tangents to c and selecting the appropriate contact point B' or B'' of the tangents from C'. The definition of the homography on c is unambiguous only for pencils of type (III). For the other two kinds of pencils one can construct two homographies f and f^*, which are related by the involution I_0 with center E and

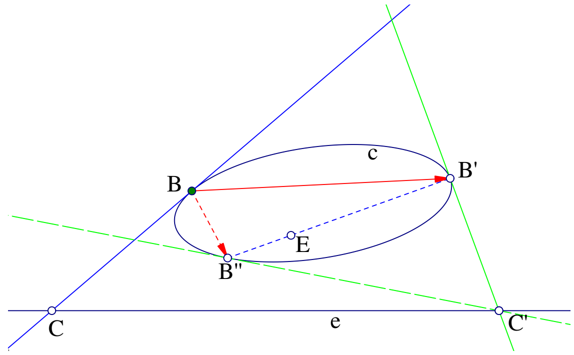

Figure 11. Using line e

axis e. The relation is $f^* = f \circ I_0 = I_0 \circ f$ (last equality is shown in Proposition 16).

Using this method one can easily answer the question of periodic conic homographies.

Proposition 15. *Only the elliptic bitangent pencils have homographies periodic of period $n > 2$. Inversely, if a conic homography is periodic, then it is elliptic.*

In fact, in the case of elliptic pencils, selecting the homography on e to be of the kind

$$x' = \frac{cos(\phi)x - sin(\phi)}{sin(\phi)x + cos(\phi)}, \quad \phi = \frac{2\pi}{n},$$

we define by the procedure described above an n-periodic homography preserving the pencil. For the cases of hyperbolic and parabolic pencils it is impossible to define a periodic homography with period $n > 2$. This because, for such pencils, every homography preserving them has to fix at least one point. If it fixes exactly one, then it is a parabolic homography, hence by Proposition 13 can not be periodic. If it fixes two points, then as we have seen in Proposition 8, the homography can be represented by a real *diagonal* matrix and this can not be periodic for $n > 2$. The inverse is shown by considering the associated bitangent pencil and applying the same arguments.

Since a general homography preserving a conic c can be written as the composition of two involutions, it is of interest to know the structure of the set of involutions preserving a given bitangent pencil. For non-parabolic pencils there is a particular involution I_0, namely the one having for axis the invariant line e of the family and for center E the pole of this line with respect to c.

If I is an arbitrary, other than I_0, involution preserving the bitangent family (c, e) then, since e is invariant by I, either its center Q is on line e or its axis coincides with e. Last case can be easily excluded by showing that the composition $f = I_0 \circ I$ is then an elation with axis e and drawing from this a contradiction. Consequently the axis $e_I = EF$ (see Figure 12) of the involution must pass through the pole E of e with respect to c. It follows that I commutes with I_0. A consequence of this is

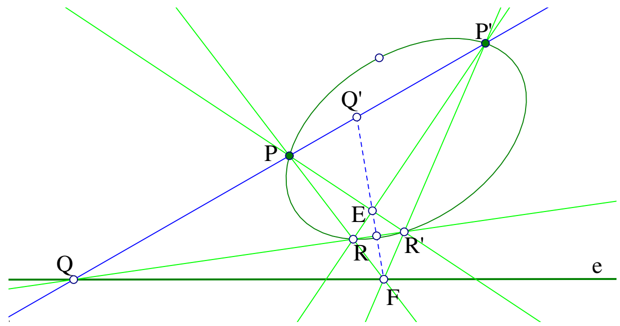

Figure 12. Involutive automorphisms

that $I' = I_0 \circ I$ is another involution the axis of which is line EQ and its center is F. Since by Proposition 5 every homography f preserving the bitangent pencil is a product of two involutions with centers on the axis e it follows that I_0 commutes with f. We arrive thus at the following.

Proposition 16. *The group $\mathcal{G}(c,e)$ of all homographies preserving a non-parabolic bitangent pencil is a subgroup of the group of homographies of the plane preserving line e, fixing the center E of the pencil and commuting with involution I_0.*

For the rest of the section I omit the reference to (c,e) and write simply $\mathcal{G}$ instead of $\mathcal{G}(c,e)$. Involution I_0 is a singularium and should be excluded from the set of all other involutions. It can be represented in infinite many ways as a product of involutions. In fact for any other involutive automorphism of the pencil I the involution $I' = I \circ I_0 = I_0 \circ I$ represents it as a product $I_0 = I \circ I'$. Counting it to the non-involutive automorphisms, it is easy to see that we can separate the group $\mathcal{G}$ into two disjoint sets. The set of non-involutive automorphisms $\mathcal{G}' \subset \mathcal{G}$ containing the identity and I_0 as particular elements, and the set $\mathcal{G}'' \subset \mathcal{G}$ of all other involutive automorphisms.

Proposition 17. *For non-parabolic pencils two involutions I, I' commute, if and only if their product is I_0. Further if the product of two involutions is an involution, then this involution is I_0. For parabolic pencils $I \circ I'$ is never commutative.*

For the first claim notice that $I' \circ I = I_0$ implies $I' = I_0 \circ I = I \circ I_0$. Last because every element of $\mathcal{G}$ commutes with I_0. Last equation implies $I \circ I' = I' \circ I$. Inversely, if last equation is valid it is readily seen that the two involutions have common fixed points on e and fix E hence their composition is $I' \circ I = I_0$. Next claim is a consequence of the previous, since $I' \circ I$ being involution implies $(I' \circ I) \circ (I' \circ I) = 1 \Rightarrow I' \circ I = I \circ I'$. Last claim is a consequence of the fact that $I \circ I'$ and $I' \circ I$ are inverse to each other and non-involutive, according to Proposition 13.

Proposition 18. *The automorphism group $\mathcal{G}$ of a pencil (c,e) is the union of two cosets $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$. $\mathcal{G}'$ consists of the non-involutive automorphisms (and I_0 for non-parabolic pencils) and builds a subgroup of $\mathcal{G}$. $\mathcal{G}''$ consists of all involutive* *automorphisms of the pencil (which are different from I_0 for non-parabolic pencils) and builds a coset of $\mathcal{G}'$ in $\mathcal{G}$. Further it is $\mathcal{G}''\mathcal{G}'' \subset \mathcal{G}'$ and $\mathcal{G}'\mathcal{G}'' \subset \mathcal{G}''$.*

In fact, given an involutive $I \in \mathcal{G}''$ and a non-involutive $f \in \mathcal{G}'$, we can, according to Proposition 5, represent f as a product $f = I \circ I'$ using involution I and another involution I' completely determined by f. Then $I \circ f = I' \in \mathcal{G}''$. This shows $\mathcal{G}''\mathcal{G}' \subset \mathcal{G}''$. The inclusion $\mathcal{G}'\mathcal{G}' \subset \mathcal{G}'$ proving $\mathcal{G}'$ a subgroup of $\mathcal{G}$ is seen similarly. The other statements are equally trivial.

Regarding commutativity, we can easily see that the (co)set of involutions contains non-commuting elements in general ($I' \circ I$ is the inverse of $I \circ I'$), whereas the subgroup $\mathcal{G}'$ is always commutative. More precisely the following is true.

Proposition 19. *The subgroup $\mathcal{G}' \subset \mathcal{G}$ of non-involutive automorphisms of the bitangent pencil (c, e) is commutative.*

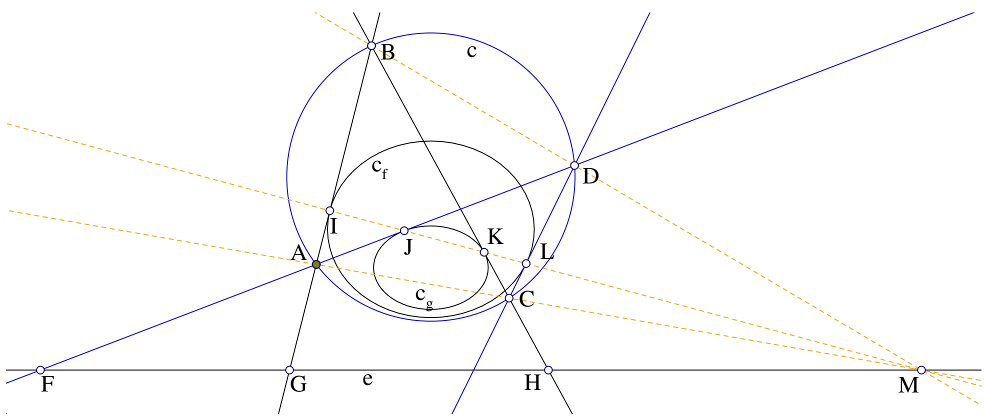

Figure 13. Commutativity for type I

The proof can be given on the basis of Figure 13, illustrating the case of elliptic pencils, the arguments though being valid also for the other types of pencils. In this figure the two products $f \circ g$ and $g \circ f$ of two non-involutive automorphisms of the pencil $f \in \mathcal{G}'$ and $g \in \mathcal{G}'$ are represented using the *tangential generation* of Proposition 6. For $A \in c$ point $B = f(A)$ has line AB tangent at I to a conic c_f of the pencil. Analogously $C = g(B)$ defines line BC tangent at K to a second conic c_g of the pencil. Let $D = g(A)$ and consequently AD be tangent at point J to c_g. It must be shown that $f(D) = C$ or equivalently that line DC is tangent at a point L to c_f. For this note first that lines $\{BD, IJ\}$ intersect at a point M on e. This happens because of the harmonic ratios $(A, B, G, I) = -1$ and $(A, D, F, J) = -1$. Similarly lines AC, IK intersect at a point M' of e. This follows again by the harmonic ratios $(B, A, I, G) = (B, C, K, H) = -1$. Hence $M' = M$ and consequently lines AC, BD intersect at M, hence according to Proposition 2, $C = f(D)$.

For hyperbolic pencils the result is also a consequence of the representation of these homographies through diagonal matrices, as in Proposition 8. For parabolic pencils the proof follows also directly from Proposition 13.

Note that for pencils (c, e) of type (II) for which c and e intersect at two points $\{A, B\}$, the involutions I_A, I_B with axes respectively BE, AE, do not belong to $\mathcal{G}$ but define through their composition $I_A \circ I_B = I_0$. This is noticed in Proposition 5 which represents every automorphism as the product of two involutions. It is though a case to be excluded in the following proposition, which results from Proposition 5 and the previous discussion.

Proposition 20. *If an automorphism $f \in \mathcal{G}$ of a pencil (c, e) is representable as a product of two involutions $f = I_2 \circ I_1$, then with the exception of $I_0 = I_A \circ I_B$ in the case of an hyperbolic pencil, in all other cases I_1 and I_2 are elements of $\mathcal{G}$.*

Regarding the transitivity of $\mathcal{G}(c, e)$ on the conics of the pencil, the following result can be easily proved.

Proposition 21. *(i) For elliptic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on the points of the conic c.*
(ii) For hyperbolic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A, B\}$, where $\{A, B\} = c \cap e$. All elements of $\mathcal{G}''$ interchange (A, B), whereas all elements of $\mathcal{G}'$ fix them.
(iii) For parabolic pencils each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A\}$ where $A = c \cap e$ and all of them fix point A.

6. Bitangent flow

Last proposition shows that every non-involutive conic homography f of a conic c is an element of a one-dimensional Lie group ([6, p.210], [13, p.82]) $\mathcal{G}$ acting on the projective plane . The invariant conic c is then a union of orbits of the action of this group. Group $\mathcal{G}$ is a subgroup of the Lie group $PGL(3, \mathbb{R})$ of all projectivities of the plane and contains a one-parameter group ([13, p.102]) of this group, which can be easily identified with the connected component of the subgroup $\mathcal{G}'$ containing the identity. Through the one-parameter group one can define a vector field on the plane, the integral curves of which are contained in the conics of the bitangent pencil associated to the non-involutive homography. Thus the bitangent pencil represents the flow of a vector field on the projective plane ([6, p.139], [14, p.292]). The fixed points correspond to the singularities of this vector field.

This point of view rises the problem of the determination of the simplest possible data needed in order to define such a flow on the plane. The answer (Proposition 26) to this problem lies in a certain involution on e related to the coset $\mathcal{G}''$ of the involutive automorphisms of the bitangent pencil. I start with non-parabolic pencils, characterized by the existence of the particular involution I_0.

Proposition 22. *For every non-parabolic pencil the correspondence $\mathcal{J} : Q \mapsto F$ between the centers of the involutions I and $I \circ I_0$ defines an involutive homography on line e. The fixed points of $\mathcal{J}$ coincide with the intersection points $\{A, B\} = c \cap e$.*

In fact considering the pencil E^* of lines through E it is easy to see that the correspondence $\mathcal{J} : F \mapsto Q$ (see Figure 14) is projective and has period two. The identification of the fixed points of $\mathcal{J}$ with $\{A, B\} = c \cap e$ is equally trivial.

Proposition 23. *The automorphism group $\mathcal{G}(c, e)$ of a non-parabolic pencil is uniquely determined by the triple*

$$(e, E, \mathcal{J})$$

consisting of a line e a point $E \notin e$ and an involutive homography on line e.

In fact $\mathcal{J}$ completely determines the involutive automorphisms I_Q of the pencil, since for each point Q on e point $F = \mathcal{J}(Q)$ defines the axis FE of the involution I_Q. The involutive automorphisms in turn, through their compositions, determine also the non involutive elements of the group.

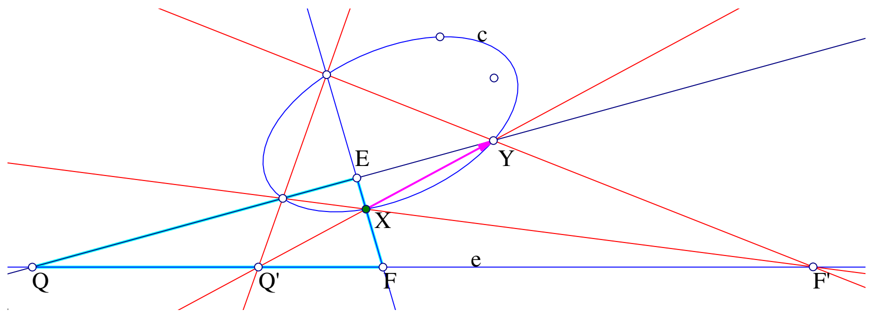

Figure 14. Quadrilateral in case I

Remark. For elliptic pencils involution $\mathcal{J}$ induces on every member-conic c of the pencil a correspondence of points $X \mapsto Y$ through its intersection with lines (EF, EQ) (see Figure 14). This defines an automorphism of the pencil of order 4 and through it infinite many convex quadrangles, each of which completely determines the pencil. Inversely, by the results of this section it will follow that for each convex quadrangle there is a well defined bitangent pencil having a member c circumscribed and a member c' inscribed in the quadrangle. Conic c is characterized by having its tangents at opposite vertices intersect on line e. Conic c' contacts the sides of the quadrangle at their intersections with lines $\{EI, EJ\}$ (see Figure 15). Note that for cyclic quadrilaterals in the euclidean plane the corresponding conic c does not coincide in general with their circumcircle. It is instead identical with the image of the circumcircle of the square under the unique projective map sending the vertices of the square to those of the given quadrilateral (e is the image under this map of the line at infinity).

Knowing the group $\mathcal{G}$ of its automorphisms, one would expect a complete reconstruction of the whole pencil, through the orbits $\mathcal{G}X$ of points X of the plane under the action of this group. Before to proceed to the proof of this property I modify slightly the point of view in order to encompass also parabolic pencils. For this consider the map $\mathcal{I} : e \mapsto E^*$ induced in the pencil E^* of lines emanating from E, the pole of the invariant line of the pencil. This map associates to every point $Q \in e$ the axis EF of the involution centered at Q. Obviously for non-parabolic pencils $\mathcal{I}$ determines $\mathcal{J}$ and vice versa. The first map though can be defined also

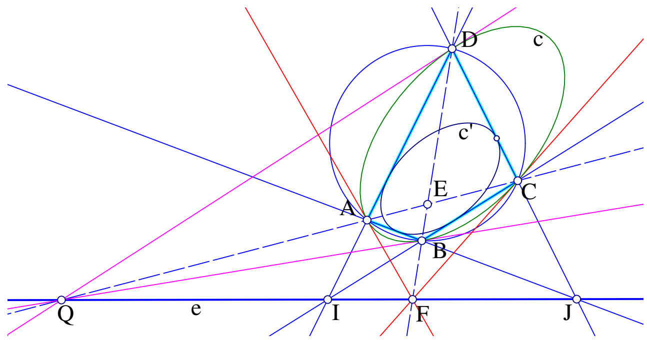

Figure 15. Circumcircle and circumconic

for parabolic pencils, since also in this case, for each point $Q \in e$ there is a unique line FQ representing the axis of the unique involutive automorphism of the pencil centered at Q. Following general fact is on the basis of the generation of the pencil through orbits.

Proposition 24. *Given a line e and a point E consider a projective map $\mathcal{I} : e \mapsto E^*$ of the line onto the pencil E^* of lines through E. Let e' denote the complement in e of the set $e'' = \{Q \in e : Q \in \mathcal{I}(Q)$. For every $Q \in e'$ denote the involution with center Q and axis $\mathcal{I}(Q)$. Then for every point $X \notin e$ of the plane the set $\{I_Q(X) : Q \in e'\} \cup e''$ is a conic.*

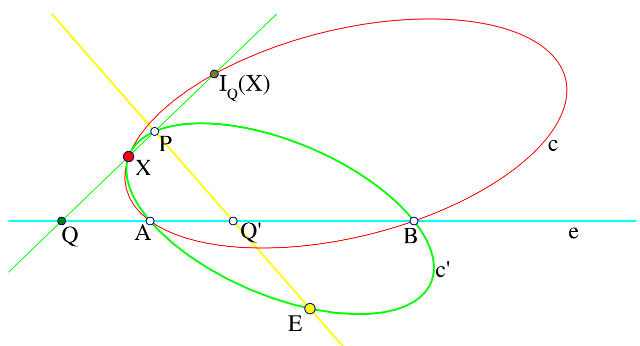

Figure 16. Orbits of involutions

In fact, by the Chasles-Steiner construction method of conics ([3, p.73]), lines XQ and $\mathcal{I}(Q)$ intersect at a point P describing a conic c', which passes through X and E. Every point $Q \in e''$ i.e. satisfying $Q \in \mathcal{I}(Q)$ coincides with a point of the intersection $c' \cap e$ and vice versa. Thus e'' has at most two points ($\{A, B\}$ in Figure 16).

The locus $\{I_Q(X) : Q \in e\}$ coincides then with the image c of the conic c', under the perspectivity p_X with center at X, axis the line e and homology coefficient $k = 1/2$.

Proposition 25. *The conics generated by the previous method belong to a bitangent pencil with axis* e *and center* E *if and only if they are invariant by all involutions* I_Q *for* $Q \in e'$. *The points in* e'' *are the fixed points of the pencil.*

The necessity of the condition is a consequence of Proposition 21. To prove the sufficiency assume that c is invariant under all $\{I_Q : Q \in e'\}$. Then for every $Q \in e'$ line $\mathcal{I}(Q)$ is the polar of Q with respect to c. Consequently line e is the polar of E with respect to c and, if $E \notin e$, the involution I_0 with axis e and center E leaves invariant c. Since the center of each involution from the pair (I_Q, I_0) is on the axis of the other the two involutions commute and $I_Q \circ I_0$ defines an involution with center at the intersection $Q' = e \cap \mathcal{I}(Q)$ and axis the polar of this point with respect to c, which, by the previous arguments, coincides with $\mathcal{I}(Q')$. This implies that the map induced in line e by $\mathcal{J}' : Q \mapsto Q' = \mathcal{I}(Q) \cap e$ is an involution. Consider now the pencil (c, e). It is trivial to show that its member-conics coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$ and $\mathcal{J}'$ is identical with the involution $\mathcal{J}$ of the pencil. This completes the proof of the proposition for the case $E \notin e$.

The proof for the case $E \in e$ is analogous with minor modifications. In this case the assumption of the invariance of c under I_Q implies that line $\mathcal{I}(Q)$ is the polar of Q with respect to c. From this follows that c is tangent to e at E and $\mathcal{I}(E) = e$. Thus e'' contains the single element E. Then it is again trivial to show that the conics of the pencil (c, e) coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$.

The arguments in the previous proof show that non-parabolic pencils are completely determined by the involution $\mathcal{J}$ on line e, whereas parabolic pencils are completely defined by a projective map $\mathcal{I} : e \to E^*$ with the property $\mathcal{I}(E) = e$. Following proposition formulates these facts.

Proposition 26. *(i) Non-parabolic pencils correspond bijectively to triples* $(e, E, \mathcal{J})$ *consisting of a line* e, *a point* $E \notin e$ *and an involution* $\mathcal{J} : e \to e$. *The fixed points of the pencil coincide with the fixed points of* $\mathcal{J}$.
(ii) Parabolic pencils correspond bijectively to triples $(e, E, \mathcal{I})$ *consisting of a line* e, *a point* $E \in e$ *and a projective map* $\mathcal{I} : e \to E^*$ *onto the pencil* E^* *of lines through* E, *such that* $\mathcal{I}(E) = e$.

7. The perspectivity group of a pencil

Perspectivities are homographies of the plane fixing a line e, called the *axis* and leaving invariant every line through a point E, called the *center* of the perspectivity. If $E \in e$ then the perspectivity is called an *elation*, otherwise it is called *homology*. Tightly related to the group $\mathcal{G}$ of automorphisms of the pencil (c, e) is the group $\mathcal{K}$ of perspectivities, with center E the center of the pencil and axis the axis e of the pencil. As will be seen, this group acts on the pencil (c, e) by permuting its members. For non-parabolic pencils the perspectivities of this group are *homologies*, and for parabolic pencils the perspectivities are *elations*. The basic facts about perspectivities are summarized by the following three propositions ([12, p.72], [15, p.228], [7, p.247]).

Proposition 27. *Given a line* e *and three collinear points* E, X, X'*, there is a unique perspectivity* f *with axis* e *and center* E *and* $f(X) = X'$*.*

Proposition 28. *For any perspectivity* f *with axis* e *and center* E *and two points* (X, Y) *with* $(X' = f(X),\ Y' = f(Y))$*, lines* XY *and* $X'Y'$ *intersect on* e*. For homologies the cross ratio* $(X, X', E, X_e) = \kappa$*, where* $X_e = XX' \cap e$*, is a constant* κ *called homology coefficient. Involutive homographies are homologies with* $\kappa = -1$ *and are called harmonic homologies.*

Proposition 29. *The set of homologies having in common the axis* e *and the center* E *builds a commutative group* $\mathcal{K}$ *which is isomorphic to the multiplicative group of real numbers.*

That the composition $h = g \circ f$ of two homologies with the previous characteristics is a homology follows directly from their definition. The homology coefficients multiply homomorphically $\kappa_h = \kappa_g \kappa_f$, this being a consequence of Proposition 27 and the well-known identity for cross ratios of five points (X, Y, Z, H) on a line d ([2, p.174])

$$(X, Y, E, H)(Y, Z, E, H)(Z, X, E, H) = 1,$$

where $H = d \cap e$. This implies also the commutativity.

Whereas the previous isomorphism is canonical, the following one, easily proved by using coordinates is not canonical. The usual way to realize it is to send e to infinity and have the elations conjugate to translations parallel to the direction determined by E ([4, vol.II, p.191]). The representation of the elation as a composition, given below follows directly from the definitions.

Proposition 30. *The set of all elations having in common the axis* e *and the center* E *builds a commutative group* $\mathcal{K}$ *which is isomorphic to the additive group of real numbers. Every elation* f *can be represented as a composition of two harmonic homologies* $f = I_B \circ I_A$*, which share with* f *the axis* e *and have their centers* $\{A, B\}$ *collinear with* E*. In this representation the center* $A \notin e$ *can be arbitrary, the other center* B *being then determined by* f *and lying on line* AE*.*

Returning to the pencil (c, e), the group $\mathcal{G}$ of its automorphisms and the corresponding group $\mathcal{K}$ of perspectivities, which are homologies in the non-parabolic case and elations in the parabolic, combine in the way shown by the following propositions.

Proposition 31. *For every bitangent pencil* (c, e) *the elements of* $\mathcal{K}(c, e)$ *commute with those of* $\mathcal{G}(c, e)$*.*

The proposition is easily proved first for involutive automorphisms of the pencil, characterized by having their centers Q on the perspectivity axis e and their axis q passing through the perspectivity center E. Figure 17 suggests the proof of the commutativity of such an involution f_Q with a homology f_E with center at E and axis the line e. Point $Y = f_E(X)$ satisfies the cross-ratio condition of the perspectivity $(X, Y, E, H) = \kappa$, where κ is the homology coefficient of the perspectivity.

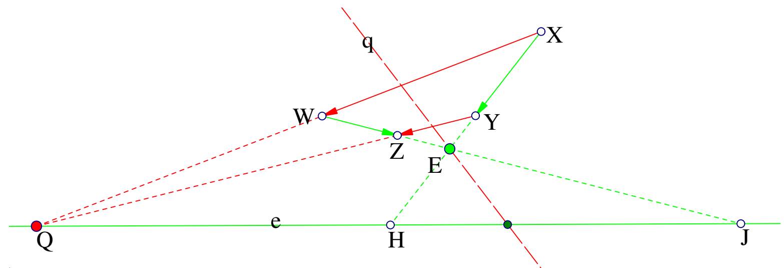

Figure 17. Homology commuting with involution

Then taking $Z = f_Q(Y)$ and the intersection W of line ZE with XQ it is readily seen that $f_Q(f_E(X)) = f_E(f_Q(X))$. Thus perspectivity f_E commutes with all involutive automorphisms of the pencil.

In the case of parabolic pencils, if EF is the axis of the involutive automorphism f_Q, $Q \in e$ of the pencil, according to Proposition 30, one can represent the elation f_E as a composition $I_B \circ I_A$ of two involutions with centers lying on EF and axis the invariant line e. Each of these involutions commutes then with f_Q, hence their composition will commute with f_Q too. Since the involutive automorphisms generate all automorphisms of the pencil it follows that f_E commutes with every automorphism of the pencil.

Proposition 32. *For non-hyperbolic pencils and every two member-conics (c, c') of the pencil there is a perspectivity with center at E and axis the line e, which maps c to c'. For hyperbolic pencils this is true if c and c' belong to the same connected component of the plane defined by lines (EA, EB), where $\{A, B\} = c \cap e$ are the base points of the pencil and E the center of the pencil.*

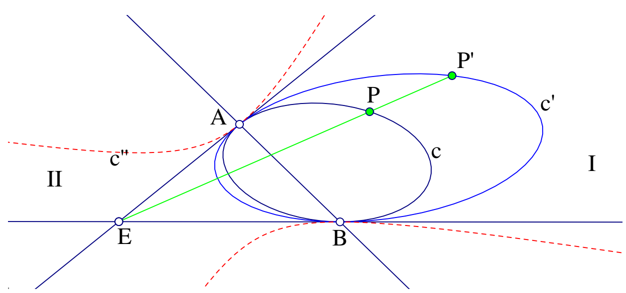

Figure 18. Perspectivity permuting member-conics

To prove the claim consider a line through E intersecting two conics of the pencil at points $P \in c$, $P' \in c'$ (see Figure 18). By Proposition 27 there is a perspectivity f mapping P to P'. By the previous proposition f commutes with all $g \in \mathcal{G}$ which can be used to map P to any other (than the base points of the pencil) point Q of c and point P' to $Q' \in c' \cap EQ$. This implies that $f(c) = c'$. The restriction for hyperbolic pencils is obviously necessary, since perspectivities leave invariant the lines through their center E.

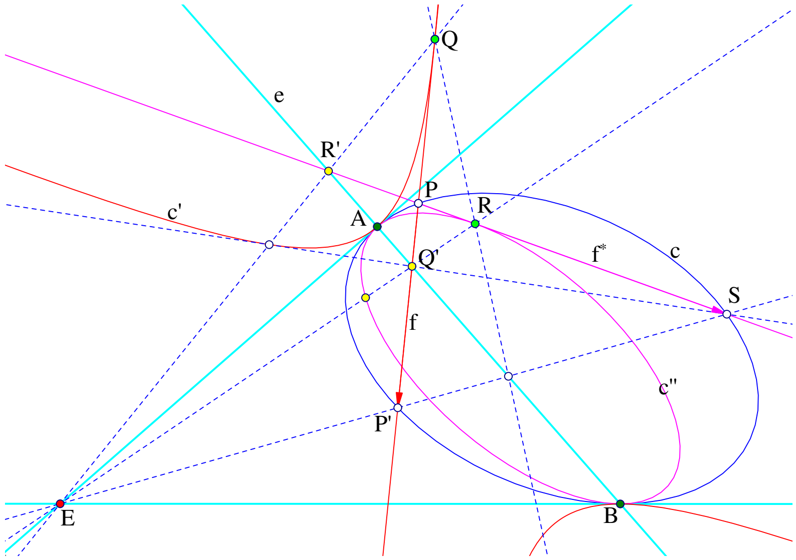

Figure 19. Conjugate member-conic

Proposition 33. *Let f be a non-involutive automorphism of the pencil (c, e) and c' be the member-conic determined by its tangential generation with respect to c(proposition-6). Then, for non-hyperbolic pencils there is a perspectivity $p_f \in \mathcal{K}$ mapping c to c'. This is true also for hyperbolic pencils provided f preserves the components of c cut out by e. Further p_f is independent of c.*

In fact, given $f \in \mathcal{G}$, according to *proposition-6*, there is a conic c' of the bitangent pencil such that lines PP', $P' = f(P)$ are tangent to c'. The exceptional case for pencils of type (II) occurs when f interchanges the two components cut out from c by the axis e. In this case conics c and c' are on different connected components of the plane defined by lines $\{EA, EB\}$. This is due to the fact (ibid) that the contact point Q of PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection $Q' = PP' \cap e$. Figure 19 illustrates this case and shows that for such automorphisms the resulting automorphism $f^* = f \circ I_0 = I_0 \circ f$, mapping P to $S = f^*(P) = I_0(f(P)) = I_0(P')$, defines through its corresponding tangential generation a kind of *conjugate* conic c'' to c' with respect to c.

To come back to the proof, first claim follows from the previous proposition. Last claim means that if the pencil is represented through another member-conic d by the pair (d, e), and the tangential generation of f is determined by a conic d', then the corresponding p'_f mapping d to d' is identical to p_f. The property is indeed a trivial consequence of the commutativity between the members of the groups $\mathcal{G}$ and $\mathcal{K}$. To see this consider a point $P \in c$ and its image $P' = f(P) \in c$. Consider also the perspectivity $g \in \mathcal{K}$ sending c to d and let $Q = g(P), Q' = g(P')$. By the commutativity of f, g it is $f(Q) = f(g(P)) = g(f(P)) = g(P') = Q'$. Thus the envelope c' of lines PP' maps via g to the envelope d' of lines QQ'. Hence if $p_f(c) = c'$ and $p'_f(d) = d'$ then $d' = g(c') = g(p_f(c))$ implying $p'_f(g(c)) = g(p_f(c))$ and from this $p'_f = g \circ p_f \circ g^{-1} = p_f$ since g and p_f commute.

Remark. Given a bitangent pencil (c, e) the correspondence of p_f to f considered above is univalent only for parabolic pencils. Otherwise it is bivalent, since both p_f and $p_f \circ I_0 = I_0 \circ p_f$ do the same job. Even in the univalent case the correspondence is not a homomorphism, since it is trivially seen that f and $g = f^{-1}$ have $p_f = p_g$. This situation is reflected also in simple configurations as, for example, in the case of the bitangent pencil (c, e) of concentric circles with common center E, the invariant line e being the line at infinity.

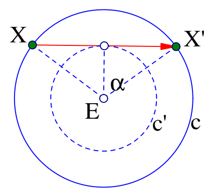

Figure 20. A case of $\mathcal{G}' \ni f \mapsto p_f \in \mathcal{K}$

In this case the rotation R_α by angle $\alpha \in (0, \pi)$ at E (see Figure 20), which is an element of the corresponding $\mathcal{G}'$, maps to the element $H_{cos(\frac{\alpha}{2})}$ of $\mathcal{K}$, which is the homothety with center E and ratio $cos(\frac{\alpha}{2})$.

8. Conic affinities

By identifying the invariant line e of a bitangent pencil (c, e) with the line at infinity all the results of the previous sections translate to properties of affine maps preserving affine conics ([2, p.184], [4, vol.II, p.146]). The automorphism group $\mathcal{G}$ of the pencil (c, e) becomes the group $\mathcal{A}$ of affinities preserving the conic c. Elliptic pencils correspond to *ellipses*, hyperbolic pencils correspond to *hyperbolas* and parabolic pencils to *parabolas*. The center E of the pencils becomes the *center* of the conic, for ellipses and hyperbolas, called collectively *central* conics. For these kinds of conics involution I_0 becomes the *symmetry* or *half turn* at the center of the conic. Every involution I other that I_0, becomes an *affine reflection* ([7, p.203]) with respect to the corresponding axis of the involution, which coincides with a *diameter* d of the conic. The center of the affine reflection I is a point at infinity defining the *conjugate* direction of lines XX' ($X' = f(X)$) of the reflection. This direction coincides with the one of the conjugate diameter to d. For an affine reflection I with diameter d the reflection $I \circ I_0$ is the reflection with respect to the *conjugate* diameter d' of the conic. Products of two affine reflexions are called *equiaffinities* ([7, p.208]) or *affine rotations*. For central conics the group $\mathcal{K}$ of perspectivities becomes the group of homotheties centered at E.

For parabolas the center of the pencil E is the contact point of the curve with the line at infinity. All affine reflections have in this case their axes passing through E

i.e. they are parallel to the direction defined by this point at infinity, which is also the contact point of the conic with e. The group $\mathcal{K}$ of elations in this case becomes the group of translations parallel to the direction defined by E.

In order to stress the differences between the three kinds of affine conics I translate the results of the previous paragraphs for each one separately.

By introducing an euclidean metric into the plane ([4, vol.I, p.200]) and taking for c the unit circle $c: x^2+y^2=1$, the group of affinities of an ellipse becomes equal to the group of isometries of the circle. The subgroup $\mathcal{G}'$ equals then the group of rotations about the center of the circle and the coset $\mathcal{G}''$ equals the coset of reflections on diameters of the circle. I_0 is the symmetry at the center of the circle and the map $I \mapsto I \circ I_0$ sends the reflection on a diameter d to the reflection on the orthogonal diameter d' of the circle. An affine rotation is identified with an euclidean rotation and in particular a periodic affinity is identified with a periodic rotation. This and similar simple arguments lead to the following well-known results.

Proposition 34. (1) *The group $\mathcal{G}$ of affinities preserving the ellipse c is isomorphic to the rotation group of the plane.*
(2) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(3) *For every $n > 2$ there is a unique cyclic group of n elements $\{f, f^2, ..., f^n = 1\} \subset \mathcal{G}'$ with f periodic of period n.*
(4) *For every affine rotation f of an ellipse c the corresponding axis e is the line at infinity and the center E is the center of the conic.*
(5) *The pencil (c, e) consists of the conics which are homothetic to c with respect to its center.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the ellipse. To each affine rotation f of the conic corresponds a real number $r_f \in [0, 1]$ which is the homothety ratio of the element $p_f \in \mathcal{K} : c' = p_f(c)$, where c' realizes through its tangents the tangential generation of f.*

In the case of hyperbolas the groups differ slightly from the corresponding ones for ellipses in the connectedness of the cosets $\mathcal{G}'$, $\mathcal{G}''$ which now have two components. The existence of two components has a clear geometric meaning. The components result from the two disjoint parts into which is divided the axis e through its intersection points A, B with the conic c. Involutions I_P which have their center P in one of these parts have their axis non-intersecting the conic. These involutions are characterized in the affine plane by diameters non-intersecting the hyperbola. They represent affine reflections which have no fixed points on the hyperbola. The other connected component of the coset of affine reflections is characterized by the property of the corresponding diameters to intersect the hyperbola, thus defining two fixed points of the corresponding reflection.

Group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$ corresponding to $\mathcal{G}_{AB}$ of Proposition 8. This group is the disjoint union of the subgroup $\mathcal{G}'_0$ of affine *hyperbolic* rotations that preserve the components of the hyperbola and its coset $\mathcal{G}'_1 = I_0\mathcal{G}'_0$ of affine *hyperbolic crossed* rotations that interchange the two components of the hyperbola ([7, p.206]). By identifying c with the hyperbola $xy = 1$ one can describe the elements of $\mathcal{G}'_0$ through the affine maps $\{(x, y) \mapsto (\mu x, \frac{1}{\mu}y),\ \mu > 0\}$. The other component $\mathcal{G}'_1$ is then identified with the set of maps $\{(x, y) \mapsto (-\mu x, -\frac{1}{\mu}y),\ \mu > 0\}$. Following proposition summarizes the results.

Proposition 35. (1) *The group of affinities $\mathcal{G}$ of a given affine hyperbola c consists of affine reflections and affine rotations which are compositions of two reflections.*
(2) *The affine rotations build a commutative subgroup $\mathcal{G}' \subset \mathcal{G}$ and the affine reflections build the unique coset $\mathcal{G}'' \subset \mathcal{G}$ of this group. $\mathcal{G}'$ and $\mathcal{G}''$ are each homeomorphic to the pointed real line $\mathbb{R}^*$ and group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$.*
(3) *Group $\mathcal{G}' = \mathcal{G}'_0 \cup \mathcal{G}'_1$ has two components corresponding to rotations that preserve the components of the hyperbola and the others $\mathcal{G}'_1 = I_0\mathcal{G}'_0$, called crossed rotations, that interchange the two components. There are no periodic affinities preserving the hyperbola for a period $n > 2$.*
(4) *The coset of affine reflections of the hyperbola is the union $\mathcal{G}'' = \mathcal{G}''_0 \cup \mathcal{G}''_1$ of two components. Reflections $I \in \mathcal{G}''_0$ preserve hyperbola's components and have fixed points on them, whereas reflections $I \in \mathcal{G}''_1 = I_0\mathcal{G}''_0$ interchange the two components and have no fixed points.*
(5) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the hyperbola.*
(7) *For each non-involutive affinity f of an hyperbola c preserving the components the tangential generation of f defines another hyperbola c' homothetic to c with respect to E. If f permutes the components of c then $f^* = f \circ I_0$ defines through tangential generation c' homothetic c. The homotety ratios for the two cases are correspondingly $r_f > 1$ and $r_{f^*} \in (0, 1)$.*

The case of affine parabolas demonstrates significant differences from ellipses and hyperbolas. Since the affinities preserving a parabola fix its point at infinity E, their group is isomorphic to the group $\mathcal{G}_E$ discussed in Proposition 14. This group contains the subgroup $\mathcal{G}^0_E$ of so-called *parabolic rotations*, which are products of two *parabolic reflections*. These are the only affine reflections preserving the parabola and their set is a coset of $\mathcal{G}^0_E$ in $\mathcal{G}_E$. The most important addition in the case of parabolas are the isotropy groups $\mathcal{G}_{EB}$ fixing the point at infinity E of the parabola and an additional point B of it. Figure 21 illustrates an example of such a group $\mathcal{G}_{EB}$ in which the parabola is described in an affine frame by the equation $y = x^2$ and point B is the origin of coordinates. In this example the group $\mathcal{G}_{EB}$ is represented by affine transformations of the form

$$(x, y) \mapsto (rx, r^2y),\ \ r \in \mathbb{R}^*.$$

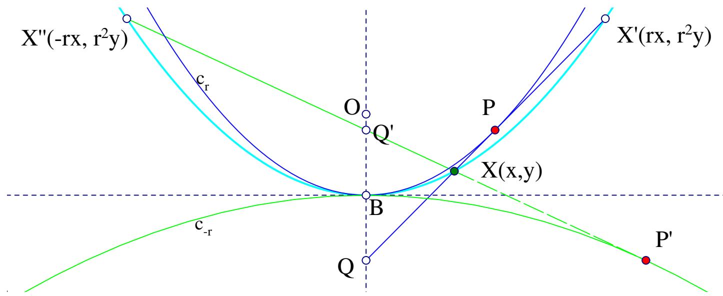

Figure 21. The group $\mathcal{G}_{EB}$

The figure displays also two other parabolas c_r, c_{-r}. These are the conics realizing the tangential generations (Proposition 6) for the corresponding affinities $(x, y) \mapsto (rx, r^2y)$ and $(x, y) \mapsto (-rx, r^2y)$ for $r > 0$. Note that, in addition to the unique affine reflection and the unique affine rotation sending a point X to another point X' and existing for affine conics of all kinds, there are for parabolas infinite more affinities doing the same job. In fact, in this case, by Proposition 14, for every two points (X, X') different from B there is precisely one element $f \in \mathcal{G}_{EB}$ mapping X to X'. This affinity preserves the parabola, fixes B and is neither an affine reflection nor an affine rotation.

Figure 22 shows the decomposition of the previous affinity into two involutions $f_r = I \circ I'$ with centers Q_1, Q_2 lying on the axis ($x = 0$) of the parabola. These are not affine since they do not preserve the line at infinity. They have though their axis parallel to the tangent d at B and their intersections with the axis R_i are symmetric to Q_i with respect to B. Thus, both of them map the line at infinity onto the tangent d at B, so that their composition leaves the line at infinity invariant. Since for another point $C \in c$ the corresponding group $\mathcal{G}_{EC} = Ad_f(\mathcal{G}_{EB})$ is conjugate to $\mathcal{G}_{EB}$ by an affine rotation $f \in \mathcal{G}_E^0$ the previous analysis transfers to the isotropy at C.

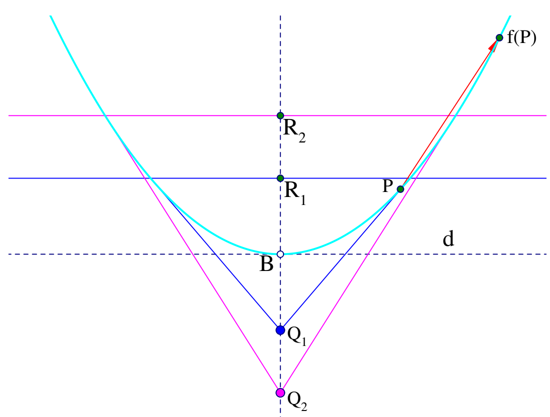

Figure 22. f_r as product of involutions

Another issue to be discussed when comparing the kinds of affine conics is that of area. Area in affine planes is defined up to a multiplicative constant ([4, vol.I, p.59]). To measure areas one fixes an affine frame, thus fixing simultaneously the *orientation* of the plane, and refers everything to this frame. Affinities preserving the area build a subgroup of the group of affinities of the plane, to which belong all affine rotations. Affine reflections reverse the sign of the area. Thus affine reflections and rotations, considered together build a group preserving the unsigned area. The analysis in the previous sections shows that affinities preserving an affine conic are automatically also unsigned-area-preserving in all cases with the exception of some types of affinities of parabolas. These affinities are the elements of the subgroups $\mathcal{G}_{EB}$ with the exception of the identity and the parabolic reflection I_B fixing B. Thus, while for all conics there are exactly two area preserving affinities mapping a point X to another point X' (an affine rotation and an affine reflection), for parabolas there is in addition a one-parameter infinity of area non-preserving affinities mapping X to X'.

Proposition 36. *In the following e denotes the line at infinity and E its unique common point with the parabola c.*
(1) *The group $\mathcal{G}$ of affinities preserving parabola c is the union $\mathcal{G} = \mathcal{G}_E^0 \cup_{B \in c} \mathcal{G}_{EB}$. This group is also the semidirect product of its subgroups $\mathcal{G}_E^0$ and $\mathcal{G}_{EB}$, the first containing all parabolic rotations and the second being the isotropy group at a point $B \in c$ of $\mathcal{G}$.*
(2) *$\mathcal{G}_E^0$ is the group of affine rotations, which are products of two affine reflections preserving the conic. This group is isomorphic to the additive group of real numbers. There are no periodic affinities preserving a parabola for a period $n > 2$.*
(3) *The set $\mathcal{G}''$ of all affine reflections preserving the parabola consists of affinities having their axis parallel to the axis of the parabola, which is the direction determined by its point at infinity E. This is a coset of the previous subgroup of $\mathcal{G}$ acting simply transitively on c.*
(4) *The group $\mathcal{G}_{EB}$ is isomorphic to the multiplicative group $\mathbb{R}^*$ and its elements, except the affine reflection $I_B \in \mathcal{G}_{EB}$, the axis of which passes through B, though they preserve c, are neither affine reflections nor affine rotations and do not preserve areas. This group acts simply transitively on $c - \{B\}$.*
(5) *For every pair of points $B \in c, C \in c$ there is a unique affine rotation $f \in \mathcal{G}_E^0$ such that $f(B) = C$. This element conjugates the corresponding isotropy groups: $Ad_f(\mathcal{G}_{EB}) = \mathcal{G}_{EC}$.*
(6) *Every coset of $\mathcal{G}_E^0$ intersects each subgroup $\mathcal{G}_{EB} \subset \mathcal{G}$ in exactly one element.*
(7) *For each affine rotation $f \in \mathcal{G}_E^0$ the tangential generation defines an element $p_f \in \mathcal{K}$. Last group coincides with the group of translations parallel to the axis of the parabola.*
(8) *For each element $f \in \mathcal{G}_{EB}$ the tangential generation defines a parabola which is a member of the bitangent pencil (c, EB). This pencil consists of all parabolas sharing with c the same axis and being tangent to c at B.*

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