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Some Triangle Centers Associated with the Tritangent Circles

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Abstract. We investigate two interesting special cases of the classical Apollonius problem, and then apply these to the tritangent circles of a triangle to find pair of perspective (or homothetic) triangles. Some new triangle centers are constructed.

1. An interesting construction

We begin with a simple construction of a special case of the classical Apollonius problem. Given two circles $O(r)$, $O'(r')$ and an external tangent $\mathcal{L}$, to construct a circle $O_1(r_1)$ tangent to the circles and the line, with point of tangency X between A and A', those of (O), (O') and $\mathcal{L}$ (see Figure 1). A simple calculation shows that $AX = 2\sqrt{r_1 r}$ and $XA' = 2\sqrt{r_1 r'}$, so that $AX : XA' = \sqrt{r} : \sqrt{r'}$. The radius of the circle is

$$r_1 = \frac{1}{4}\left(\frac{AA'}{\sqrt{r}+\sqrt{r'}}\right)^2.$$

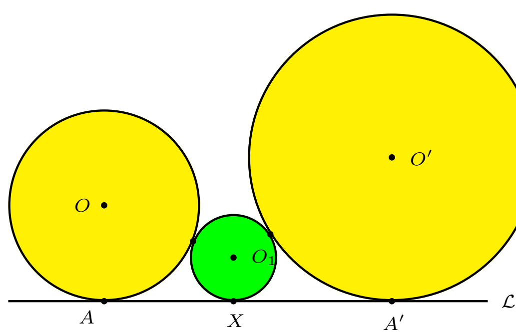

Figure 1

From this we design the following construction.

Construction 1. *On the line $\mathcal{L}$, choose two points P and Q be points on opposite sides of A such that $PA = r$ and $AQ = r'$. Construct the circle with diameter PQ to intersect the line OA at F such that O and F are on opposite sides of $\mathcal{L}$. The intersection of $O'F$ with $\mathcal{L}$ is the point X satisfying $AX : XA' = \sqrt{r} : \sqrt{r'}$. Let M be the midpoint of AX. The perpendiculars to OM at M, and to $\mathcal{L}$ at X intersect at the center O_1 of the required circle* (see Figure 2).

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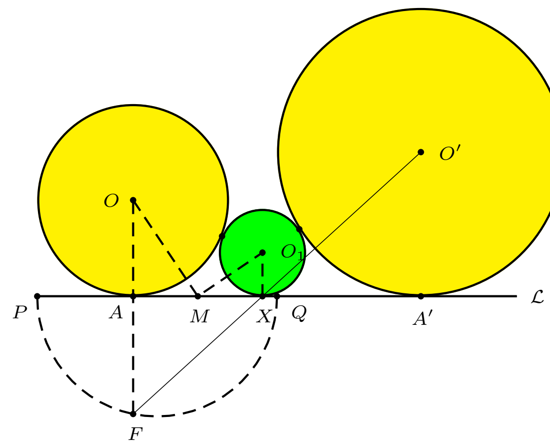

Figure 2

For a construction in the case when $\mathcal{L}$ is not necessarily tangent of (O) and (O'), see [1, Problem 471].

2. An application to the excircles of a triangle

We apply the above construction to the excircles of a triangle ABC. We adopt standard notations for a triangle, and work with homogeneous barycentric coordinates. The points of tangency of the excircles with the sidelines are as follows.

	BC	CA	AB
(I_a)	$A_a = (0 : s-b : s-c)$	$B_a = (-(s-b) : 0 : s)$	$C_a = (-(s-c) : s : 0)$
(I_b)	$A_b = (0 : -(s-a) : s)$	$B_b = (s-a : 0 : s-c)$	$C_b = (s : -(s-c) : 0)$
(I_c)	$A_c = (0 : s : -(s-a))$	$B_c = (s : 0 : -(s-c))$	$C_c = (s-a : s-b : 0)$

Consider the circle $O_1(X)$ tangent to the excircles $I_b(r_b)$ and $I_c(r_c)$, and to the line BC at a point X between A_c and A_b (see Figure 3).

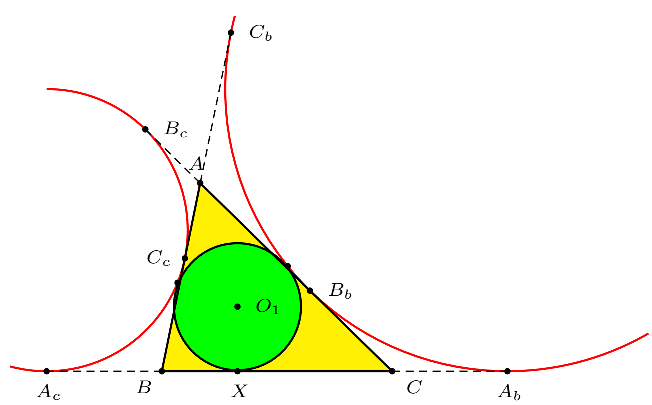

Figure 3

Lemma 2. *The point X has coordinates*

$$(0 : s\sqrt{s-c} - (s-a)\sqrt{s-b} : s\sqrt{s-b} - (s-a)\sqrt{s-c}).$$

Proof. If (O_1) is the circle tangent to (I_b), (I_c), and to BC at X between A_c and A_b, then $A_cX : XA_b = \sqrt{r_c} : \sqrt{r_b} = \sqrt{s-b} : \sqrt{s-c}$. Note that $A_cA_b = b+c$, so that

$$\begin{aligned} BX =& A_cX - A_cB \\ =& \frac{\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}} \cdot (b+c) - (s-a) \\ =& \frac{s\sqrt{s-b} - (s-a)\sqrt{s-c}}{\sqrt{s-b}+\sqrt{s-c}}. \end{aligned}$$

Similarly

$$XC = \frac{s\sqrt{s-c} - (s-a)\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}}.$$

It follows that the point X has coordinates given above. □

Similarly, there are circles $O_2(Y)$ and $O_3(Z)$ tangent to CA at Y and to AB at Z respectively, each also tangent to a pair of excircles. Their coordinates can be written down from those of X by cyclic permutations of a, b, c.

3. The triangle bounded by the polars of the vertices with respect to the excircles

Consider the triangle bounded by the polars of the vertices of ABC with respect to the corresponding excircles. The polar of A with respect to the excircle (I_a) is the line B_aC_a; similarly for the other two polars.

Lemma 3. *The polars of the vertices of ABC with respect to the corresponding excircles bound a triangle with vertices*

$$\begin{aligned} U =&(-a(b+c) : S_C : S_B), \\ V =&(S_C : -b(c+a) : S_A), \\ W =&(S_B : S_A : -c(a+b)). \end{aligned}$$

Proof. The polar of A with respect to the excircle (I_a) is the line B_aC_a, whose barycentric equation is

$$\begin{vmatrix} x & y & z \\ -(s-b) & 0 & s \\ -(s-c) & s & 0 \end{vmatrix} = 0,$$

or

$$sx + (s-c)y + (s-b)z = 0.$$

Similarly, the polars C_bA_b and A_cB_c have equations

$$\begin{aligned} (s-c)x + sy + (s-a)z =&0, \\ (s-b)x + (s-a)y + sz =&0. \end{aligned}$$

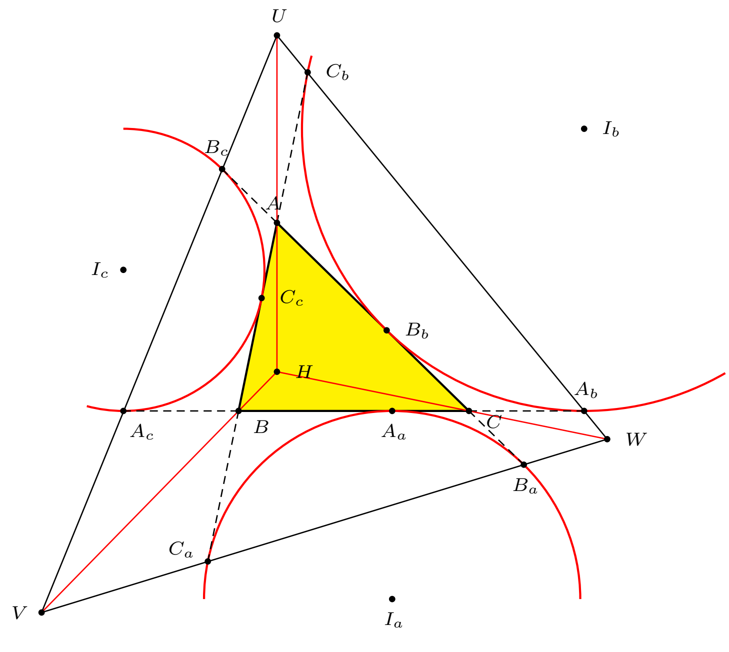

Figure 4

These intersect at the point

$$\begin{aligned} U =&(-a(2s-a) : ab - 2s(s-c) : ca - 2s(s-b)) \\ =&(-2a(b+c) : a^2+b^2-c^2 : c^2+a^2-b^2) \\ =&(-a(b+c) : S_C : S_B). \end{aligned}$$

The coordinates of V and W can be obtained from these by cyclic permutations of a, b, c. □

Corollary 4. *Triangles UVW and ABC are*
(a) *perspective at the orthocenter H,*
(b) *orthologic with centers H and I respectively.*

Proposition 5. *The triangle UVW has circumcenter H and circumradius $2R+r$.*

Proof. Since H, B, V are collinear, HV is perpendicular to CA. Similarly, HW is perpendicular to AB. Since VW makes equal angles with CA and AB, it makes equal angles with HV and HW. This means $HV = HW$. For the same reason, $HU = HV$, and H is the circumcenter of UVW.

Applying the law of sines to triangle AUB_c, we have we have

$$AU = AB_c \cdot \frac{\sin\frac{180^\circ - C}{2}}{\sin\frac{C}{2}} = (s-b)\cot\frac{C}{2} = r_a.$$

The circumradius of UVW is $HU = HA + AU = 2R\cos A + r_a = 2R + r$, as a routine calculation shows. □

Proposition 6. *The triangle UVW and the intouch triangle DEF are homothetic at the point*

$$J = \left(\frac{b+c}{b+c-a} : \frac{c+a}{c+a-b} : \frac{a+b}{a+b-c}\right). \tag{1}$$

The ratio of homothety is $-\frac{2R+r}{r}$.

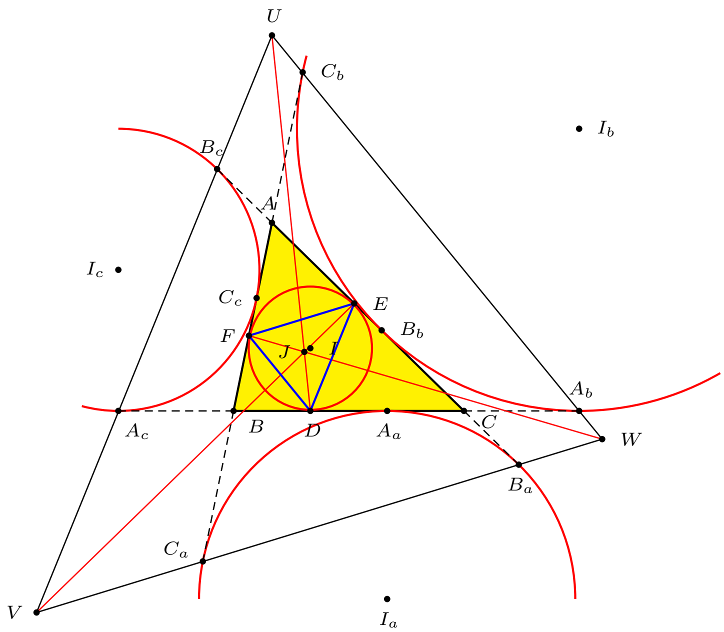

Figure 5

Proof. The homothety follows easily from the parallelism of VW and EF, and of WU, FD, and UV, DE. The homothetic center is the common point J of the lines DU, EV, and FW (see Figure 5). These lines have equations

$$\begin{aligned}
(b-c)(b+c-a)x + (b+c)(c+a-b)y - (b+c)(a+b-c)z &= 0,\\
-(c+a)(b+c-a)x + (c-a)(c+a-b)y + (c+a)(a+b-c)z &= 0,\\
(a+b)(b+c-a)x - (a+b)(c+a-b)y + (a-b)(a+b-c)z &= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(b+c-a)x : (c+a-b)y : (a+b-c)z\\
=&\begin{vmatrix} c-a & c+a \\ -(a+b) & a-b \end{vmatrix} : \begin{vmatrix} c+a & -(c+a) \\ a+b & a-b \end{vmatrix} : \begin{vmatrix} -(c+a) & c-a \\ a+b & -(a+b) \end{vmatrix}\\
=&2a(b+c) : 2a(c+a) : 2a(a+b)\\
=&b+c : c+a : a+b.
\end{aligned}$$

The coordinates of the homothetic center J are therefore as in (1) above.

Since the triangles UVF and DEF have circumcircles $H(2R + r)$ and $I(r)$, the ratio of homothety is $-\frac{2R+r}{r}$. The homothetic center J divides IH in the ratio $IJ : JH = r : 2R + r$. □

Remark. The triangle center J appears as X_{226} in Kimberling's list [2].

4. Perspectivity of XYZ and UVW

Theorem 7. *Triangles XYZ and UVW are perspective at a point with coordinates*

$$\left(\frac{S_B}{\sqrt{s-c}}+\frac{S_C}{\sqrt{s-b}}-\frac{a(b+c)}{\sqrt{s-a}}:\frac{S_C}{\sqrt{s-a}}+\frac{S_A}{\sqrt{s-c}}-\frac{b(c+a)}{\sqrt{s-b}}:\frac{S_A}{\sqrt{s-b}}+\frac{S_B}{\sqrt{s-a}}-\frac{c(a+b)}{\sqrt{s-c}}\right).$$

Proof. With the coordinates of X and U from Lemmas 1 and 2, the line XU has equation

$$\begin{vmatrix} x & y & z \\ -a(b+c) & S_C & S_B \\ 0 & s\sqrt{s-c}-(s-a)\sqrt{s-b} & s\sqrt{s-b}-(s-a)\sqrt{s-c} \end{vmatrix} = 0.$$

Since the coefficient of x is

$$\begin{aligned} &(s(S_B+S_C)-aS_B)\sqrt{s-b}-(s(S_B+S_C)-aS_C)\sqrt{s-c} \\ =&a((as-S_B)\sqrt{s-b}-(as-S_C)\sqrt{s-c}) \\ =&a(b+c)((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c}). \end{aligned}$$

From this, we easily simplify the above equation as

$$\begin{aligned} &((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c})x \\ &+(s\sqrt{s-b}-(s-a)\sqrt{s-c})y+((s-a)\sqrt{s-b}-s\sqrt{s-c})z=0. \end{aligned}$$

With $u = \sqrt{s-a}$, $v = \sqrt{s-b}$, and $w = \sqrt{s-c}$, we rewrite this as

$$-vw(v-w)x+(v(u^2+v^2+w^2)-u^2w)y+(u^2v-w(u^2+v^2+w^2))z=0. \quad (2)$$

Similarly the equations of the lines VY, WZ are

$$(v^2w-u(u^2+v^2+w^2))x-wu(w-u)y+(w(u^2+v^2+w^2)-uv^2)z=0, \quad (3)$$

$$(u(u^2+v^2+w^2)-vw^2)x+(w^2u-v(u^2+v^2+w^2))y-uv(u-v)z=0. \quad (4)$$

It is clear that the sum of the coefficients of x (respectively y and z) in (2), (3) and (4) is zero. The system of equations therefore has a nontrivial solution. Solving them, we obtain the coordinates of the common point of the lines XU, YV, ZW as

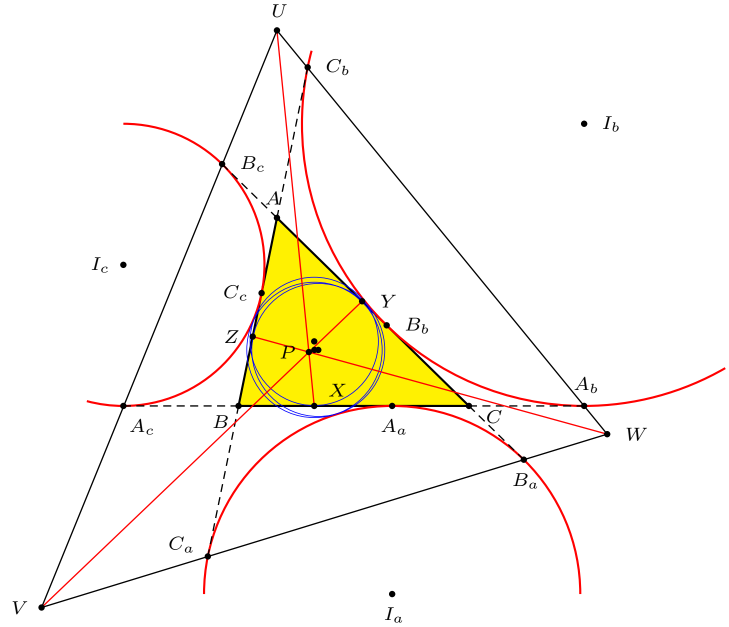

Figure 6

$$
\begin{aligned}
& x : y : z \\
=& uv(v^2(u^2+v^2+w^2) - w^2u^2) + wu(w^2(u^2+v^2+w^2) - u^2v^2) \\
& - vw(v^2+w^2)(2u^2+v^2+w^2) \\
:& vw(w^2(u^2+v^2+w^2) - u^2v^2) + uv(u^2((u^2+v^2+w^2) - v^2w^2) \\
& - wu(w^2+u^2)(u^2+2v^2+w^2) \\
:& wu(u^2(u^2+v^2+w^2) - v^2w^2) + vw(v^2((u^2+v^2+w^2) - w^2u^2) \\
& - uv(u^2+v^2)(u^2+v^2+2w^2)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{(s-b)s-(s-c)(s-a)}{w} + \frac{(s-c)s-(s-a)(s-b)}{v} - \frac{a(b+c)}{u} \\
:& \frac{(s-c)s-(s-a)(s-b)}{u} + \frac{(s-a)s-(s-b)(s-c)}{w} - \frac{b(c+a)}{v} \\
:& \frac{(s-a)s-(s-b)(s-c)}{v} + \frac{(s-b)s-(s-c)(s-a)}{u} - \frac{c(a+b)}{w} \\
=& \frac{S_B}{w} + \frac{S_C}{v} - \frac{a(b+c)}{u} : \frac{S_C}{u} + \frac{S_A}{w} - \frac{b(c+a)}{v} : \frac{S_A}{v} + \frac{S_B}{u} - \frac{c(a+b)}{w}.
\end{aligned}
$$

□

The triangle center constructed in Theorem 3 above does not appear in [2].

5. Another construction

Given three circles $O_i(r_i)$, $i = 1, 2, 3$, on one side of a line $\mathcal{L}$, tangent to the line, we construct a circle $O(r)$, tangent to each of these three circles externally.

For $i = 1, 2, 3$, let the circle $O_i(r_i)$ touch $\mathcal{L}$ at S_i and $O(r)$ at T_i. If the line S_1T_1 meets the circle (O) again at T, then the tangent to (O) at T is a line $\mathcal{L}'$ parallel to $\mathcal{L}$. Hence, T, T_2, S_2 are collinear; so are T, T_3, S_3. Since the line T_2T_3 is antiparallel to $\mathcal{L}'$ with respect to the lines TT_2 and TT_3, it is also antiparallel to $\mathcal{L}$ with respect to the lines TS_2 and TS_3, and the points T_2, T_3, S_3, S_2 are concyclic. From $TT_2 \cdot TS_2 = TT_3 \cdot TS_3$, we conclude that the point T lies on the radical axis of the circles $O_2(r_2)$ and $O_3(r_3)$, which is the perpendicular from the midpoint of S_2S_3 to the line O_2O_3. For the same reason, it also lies on the radical axis of the circles $O_3(r_3)$ and $O_1(r_1)$, which is the perpendicular from the midpoint of S_1S_3 to the line O_1O_3. Hence T is the radical center of the three given circles $O_i(r_i)$, $i = 1, 2, 3$, and the circle $T_1T_2T_3$ is the image of the line $\mathcal{L}$ under the inversion with center T and power $TT_1 \cdot TS_1$. From this, the required circle (O) can be constructed as follows.

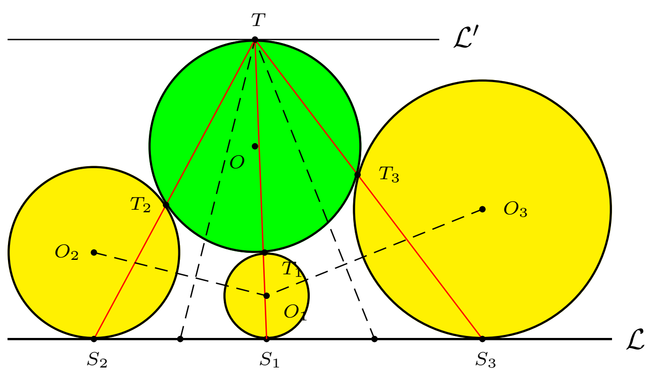

Figure 7

Construction 8. *Construct the perpendicular from the midpoint of S_1S_2 to O_1O_2, and from the midpoint of S_1S_3 to O_1O_3. Let T be the intersection of these two perpendiculars. For $i = 1, 2, 3$, let T_i be the intersection of the line TS_i with the circle (O_i). The required circle (O) is the one through T_1, T_2, T_3* (see Figure 7).

6. Circles tangent to the incircle and two excircles

We apply Construction 2 to obtain the circle tangent to the incircle (I) and the excircles (I_b) and (I_c). Let the incircle (I) touch the sides BC, CA, AB at D, E, F respectively.

Proposition 9. *The radical center of (I), (I_b), (I_c) is the point*

$$J_a = (b + c : c - a : b - a).$$

This is also the midpoint of the segment DU.

Proof. The radical axis of (I) and (I_b) is the line joining the midpoints of the segments DA_b and FC_b. These midpoints have coordinates $(0 : a-c : a+c)$ and $(c+a : c-a : 0)$. This line has equation

$$-(c-a)x + (c+a)y + (c-a)z = 0.$$

Similarly, the radical axis of (I) and (I_c) is the line

$$(a-b)x - (a-b)y + (a+b)z = 0.$$

The radical center J_a of the three circles is the intersection of these two radical axes. Its coordinates are as given above.

By Construction 2, J_a is the intersection of the lines through the midpoints of A_bD and A_cD perpendicular to II_b and II_c respectively. As such, it is the midpoint of DU. □

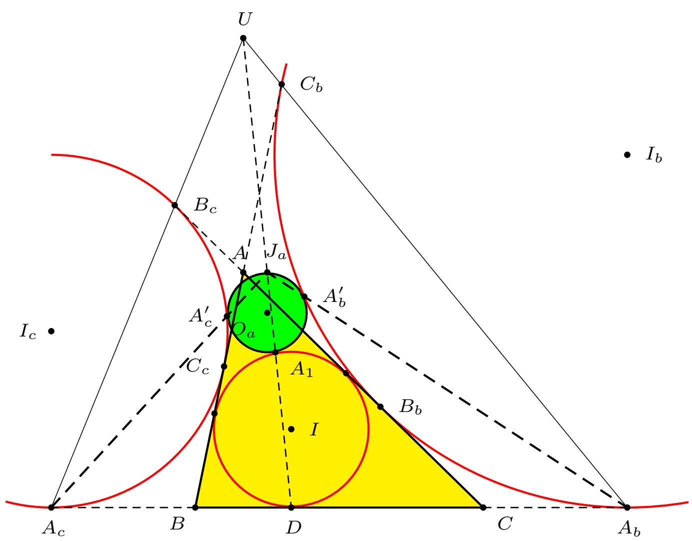

Figure 8.

The lines J_aD, J_aA_b and J_aA_c intersect the circles (I), (I_b) and (I_c) respectively again at

$$\begin{aligned}
A_1 =&((b+c)^2(s-b)(s-c) : c^2(s-a)(s-c) : b^2(s-a)(s-b)),\\
A'_b =&((b+c)^2 s(s-c) : -(ab-c(s-a))^2 : b^2 s(s-a)),\\
A'_c =&((b+c)^2 s(s-b) : c^2 s(s-a) : -(ca-b(s-a))^2).
\end{aligned}$$

The circle through these points is the one tangent to (I), (I_b), and (I_c) (see Figure 8). It has radius $\frac{a}{b+c} \cdot \frac{(s-a)^2+r_a^2}{4r_a}$.

In the same way, we have a circle (O_b) tangent to (I), (I_c), (I_a) respectively at B_1, B'_c, B'_a, and passing through the radical center J_b of these three circles, and another circle (O_c) tangent to (I), (I_a), (I_b) respectively at C_1, C'_a, C'_b, passing through the radical center J_c of the circles. J_b and J_c are respectively the midpoints of the segments EV and FW. The coordinates of J_b, J_c, B_1, C_1 are as follows.

$$\begin{aligned} J_b =&(c-b : c+a : a-b),\\ J_c =&(b-c : a-c : a+b);\\ B_1 =&(a^2(s-b)(s-c) : (c+a)^2(s-c)(s-a) : c^2(s-a)(s-b)),\\ C_1 =&(a^2(s-b)(s-c) : b^2(s-c)(s-a) : (a+b)^2(s-a)(s-b)). \end{aligned}$$

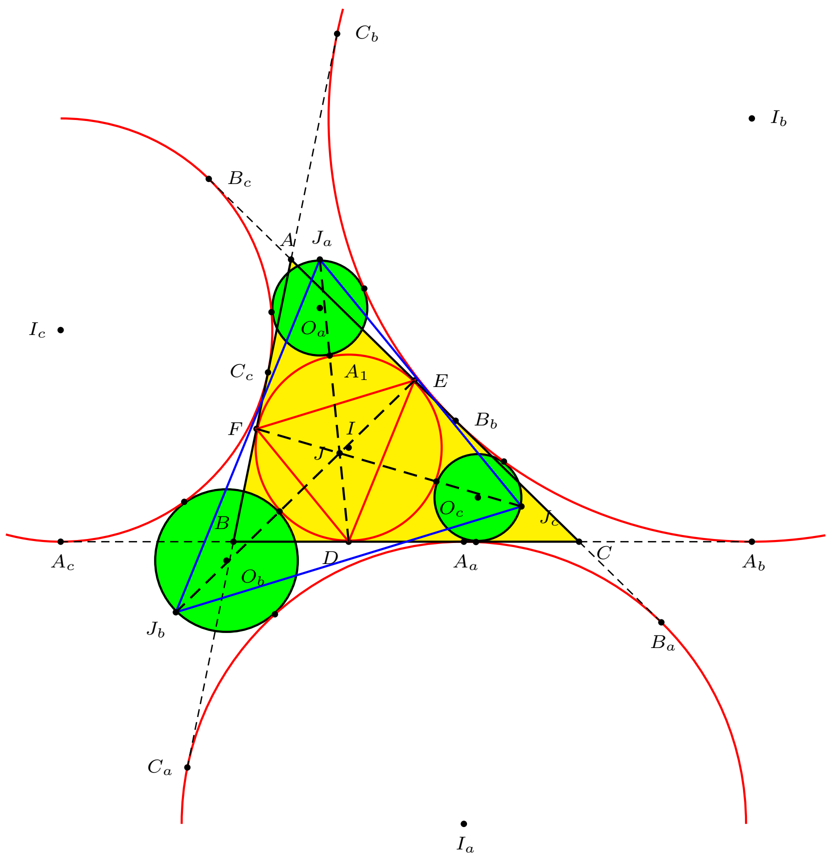

Figure 9.

Proposition 10. *The triangle $J_aJ_bJ_c$ is the image of the intouch triangle under the homothety* $\mathsf{h}(J, -\frac{R}{r})$.

Proof. Since UVW and DEF are homothetic at J, and J_a, J_b, J_c are the midpoints of DU, EV, FW respectively, it is clear that $J_aJ_bJ_c$ and DEF are also homothetic at the same J. Note that $J_bJ_c = \frac{1}{2}(VW - EF)$. The circumradius of $J_aJ_bJ_c$ is $\frac{1}{2}((2R + r) - r) = R$. The ratio of homothety of $J_aJ_bJ_c$ and DEF is $\frac{-R}{r}$. □

Corollary 11. *J is the radical center of the circles (O_a), (O_b), (O_c).*

Proof. Note that $JJ_a \cdot JA_1 = \frac{R}{r} \cdot DJ \cdot JA_1$. This is $\frac{R}{r}$ times the power of J with respect to the incircle. The same is true for $JJ_b \cdot JB_1$ and $JJ_c \cdot JC_1$. This shows that J is the radical center of the circles (O_a), (O_b), (O_c). □

Since the incircle (I) is the inner Apollonius circle and the circumcircle (O_i), $i = 1, 2, 3$, it follows that $J_aJ_bJ_c$ is the outer Apollonius circle to the same three circles (see Figure 10). The center O' of the circle $J_aJ_bJ_c$ is the midpoint between the circumcenters of DEF and UVW, namely, the midpoint of IH. It is the triangle center X_{946} in [2].

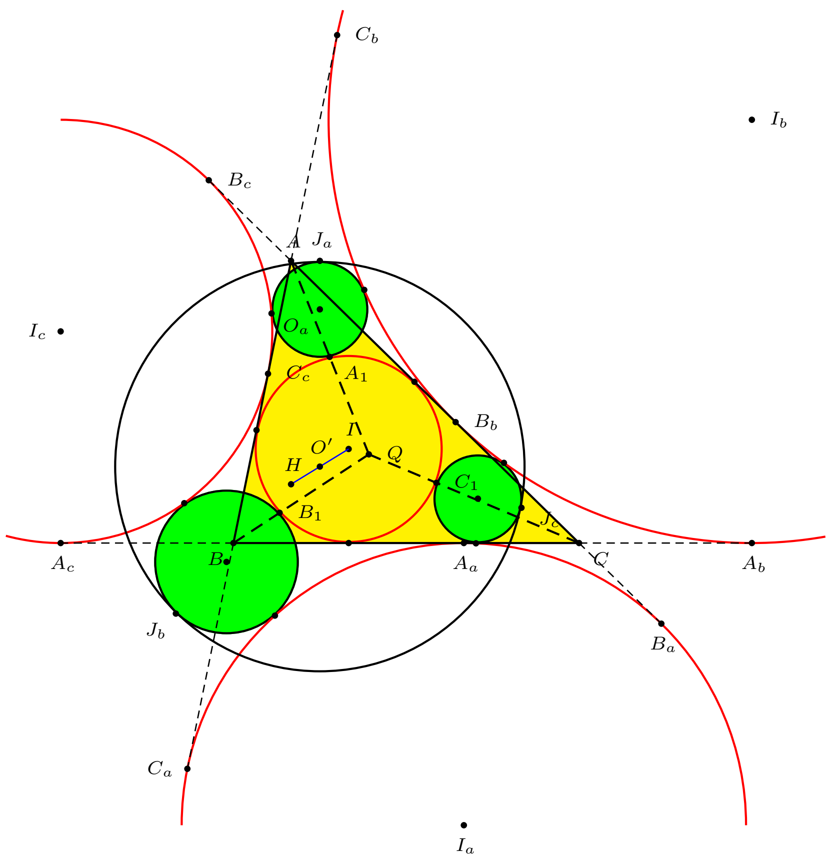

Figure 10.

Proposition 12. *$A_1B_1C_1$ is perspective with ABC at the point*

$$Q = \left(\frac{1}{a^2(s-a)} : \frac{1}{b^2(s-b)} : \frac{1}{c^2(s-c)} \right),$$

which is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle.

This is clear from the coordinates of A_1, B_1, C_1. The perspector Q is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle. It is not in the current listing in [2].

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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