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Ten Concurrent Euler Lines

Nikolai Ivanov Beluhov

Dedicated to Svetlozar Doichev

Abstract. Let F_1 and F_2 denote the Fermat-Toricelli points of a given triangle ABC. We prove that the Euler lines of the 10 triangles with vertices chosen from A, B, C, F_1, F_2 (three at a time) are concurrent at the centroid of triangle ABC.

Given a (positively oriented) triangle ABC, construct externally on its sides three equilateral triangles BCT_a, CAT_b, and ABT_c with centers N_a, N_b and N_c respectively (see Figure 1). As is well known, triangle $N_aN_bN_c$ is equilateral. We call this the first Napoleon triangle of ABC.

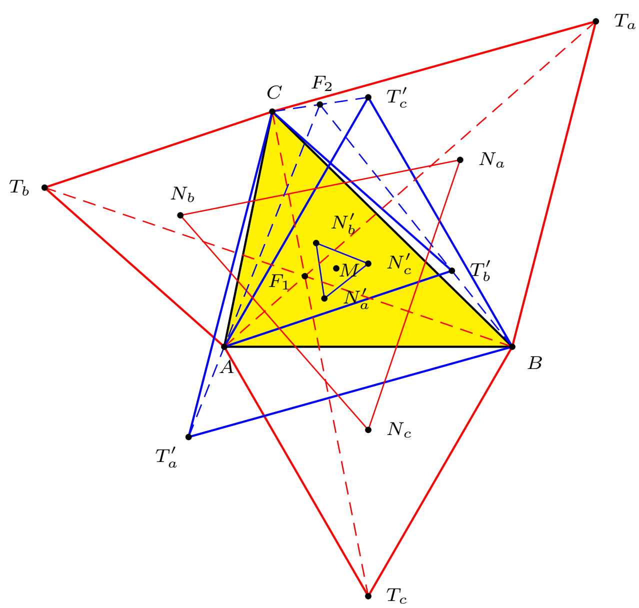

Figure 1.

The same construction performed internally gives equilateral triangles BCT'_a, CAT'_b and ABT'_c with centers N'_a, N'_b, and N'_c respectively, leading to the second Napoleon triangle $N'_aN'_bN'_c$. The centers of both Napoleon triangles coincide with the centroid M of triangle ABC.

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The lines AT_a, BT_b and CT_c make equal pairwise angles, and meet together with the circumcircles of triangles BCT_a, CAT_b, and ABT_c at the first Fermat-Toricelli point F_1. Denoting by $\angle XYZ$ the oriented angle $\angle(YX, YZ)$, we have $\angle AF_1B = \angle BF_1C = \angle CF_1A = 120°$. Analogously, the second Fermat-Toricelli point satisfies $\angle AF_2B = \angle BF_2C = \angle CF_2A = 60°$.

Clearly, the sides of the Napoleon triangles are the perpendicular bisectors of the segments joining their respective Fermat-Toricelli points with the vertices of triangle ABC (as these segments are the common chords of the circumcircles of the the equilateral triangles ABT_c, BCT_a, etc).

We prove the following interesting theorem.

Theorem *The Euler lines of the ten triangles with vertices from the set* $\{A,\ B,\ C,\ F_1,\ F_2\}$ *are concurrent at the centroid* M *of triangle* ABC.

Proof. We divide the ten triangles in three types:
(I): Triangle ABC by itself, for which the claim is trivial.
(II): The six triangles each with two vertices from the set $\{A, B, C\}$ and the remaining vertex one of the points F_1, F_2.
(III) The three triangles each with vertices F_1, F_2, and one from $\{A, B, C\}$.

For type (II), it is enough to consider triangle ABF_1. Let M_c be its centroid and M_C be the midpoint of the segment AB. Notice also that N_c is the circumcenter of triangle ABF_1 (see Figure 2).

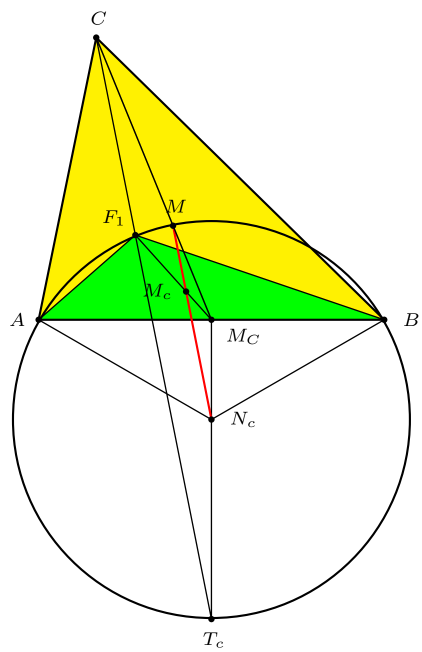

Figure 2.

Now, the points C, F_1 and T_c are collinear, and the points M, M_c and N_c divide the segments M_CC, M_CF_1 and M_CT_c in the same ratio $1 : 2$. Therefore, they are collinear, and the Euler line of triangle ABF contains M.

For type (III), it is enough to consider triangle CF_1F_2. Let M_c and O_c be its centroid and circumcenter. Let also M_C and M_F be the midpoints of AB and F_1F_2. Notice that O_c is the intersection of N_aN_b and $N'_aN'_b$ as perpendicular bisectors of F_1C and F_2C. Let also P be the intersection of N_bN_c and $N'_cN'_a$, and F' be the reflection of F_1 in M_C (see Figure 3).

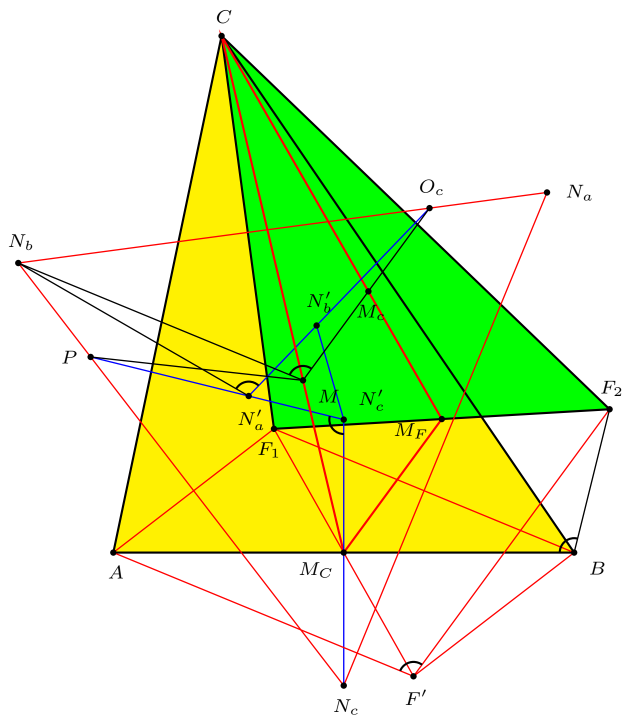

Figure 3.

The rotation of center M and angle 120° maps the lines N_aN_b and $N'_aN'_b$ into N_bN_c and $N'_cN'_a$ respectively. Therefore, it maps O_c to P, and $\angle O_cMP = 120^\circ$. Since $\angle O_cN'_aP = 120^\circ$, the four points O_c, M, N'_a, P are concyclic. The circle containing them also contains N_b since $\angle PN_bO_c = 60^\circ$. Therefore, $\angle O_cMN_b = \angle O_cN'_aN_b$.

The same rotation maps angle $O_cN'_aN_b$ onto angle PN'_cN_c, yielding $\angle O_cN'_aN_b = \angle PN'_cN_c$. Since $PN'_c \perp BF_2$ and $N_cN'_c \perp BA$, $\angle PN'_cN_c = \angle F_2BA$.

Since $\angle BF'A = \angle AF_1B = 120^\circ = 180^\circ - \angle AF_2B$, the quadrilateral AF_2BF' is also cyclic and $\angle F_2BA = \angle F_2F'A$. Thus, $\angle F_2F'A = \angle O_cMN_b$.

Now, $AF' \| F_1B \perp N_aN_c$ and $N_bM \perp N_aN_c$ yield $AF' \| N_bM$. This, together with $\angle F_2F'A = \angle O_cMN_b$, yields $F'F_2 \| MO_c$.

Notice now that the points M_c and M divide the segments CM_F and CM_C in ratio $1:2$, therefore $M_cM \| M_CM_F$. The same argument, applied to the segments F_1F_2 and F_1F' with ratio $1:1$, yields $M_CM_F \| F'F_2$.

In conclusion, we obtain $M_cM \| F'F_2 \| MO_c$. The collinearity of the points M_c, M and O_c follows. $\square$

References

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Nikolai Ivanov Beluhov: ”Bulgarsko opalchenie” str. 5, Stara Zagora 6000, Bulgaria
E-mail address: nbehulov@abv.bg