Forum Geometricorum
Volume 9 (2009) 281–282.

FORUM GEOM
ISSN 1534-1178

Pythagorean Triangles with Square of Perimeter Equal to an Integer Multiple of Area

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Abstract. We determine all primitive Pythagorean triangles with square on perimeter equal to an integer multiple of its area.

Complete solutions can be found for several special cases of the problem of solving $P^2 = nA$, where P is the perimeter and A is the area of an integer-sided triangle, and n is an integer. The general problem is considered in a recent paper [1]. We consider the case of right triangles. Let the sides be a, b, and c, where c is the hypotenuse. We require

$$n = \frac{2(a+b+c)^2}{ab}.$$

By the homogeneity of the problem, it is enough to consider primitive Pythagorean triangles. It is well known that there are positive integers p and q, relatively prime and of different parity, such that

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.$$

With these, $n = \frac{4p(p+q)}{q(p-q)} = \frac{4t(t+1)}{t-1}$, where $t = \frac{p}{q}$. Rewriting this as

$$4t^2 - (n-4)t + n = 0,$$

we obtain

$$t = \frac{(n-4) \pm d}{8},$$

where

$$d^2 = (n-4)^2 - 16n = (n-12)^2 - 128. \tag{1}$$

Since t is rational, d must be an integer (which we may assume positive). Equation (1) may be rewritten as

$$(n-12-d)(n-12+d) = 128 = 2^7.$$

Publication Date: November 16, 2009. Communicating Editor: Paul Yiu.

From this,

$$n - 12 - d = 2^k,$$
$$n - 12 + d = 2^{7-k},$$

for $k = 1, 2, 3$. We have

$$t = \frac{n-4+d}{8} = 2^{4-k} + 1 \quad \text{or} \quad t = \frac{n-4-d}{8} = \frac{2^k+8}{8}.$$

Since p and q are relatively prime integers of different parity, we exclude the cases when t is an odd integer. Thus, the primitive Pythagorean triangles solving $P^2 = nA$ are precisely those shown in the table below.

k	t	(p,q)	(a,b,c)	n	A	P
1	$\frac{5}{4}$	$(5,4)$	$(9,40,41)$	45	180	90
2	$\frac{3}{2}$	$(3,2)$	$(5,12,13)$	30	30	30
3	2	$(2,1)$	$(3,4,5)$	24	6	12

Among these three solutions, only in the case of $(3,4,5)$ can the square on the perimeter be tessellated by n copies of the triangle.

References

[1] A. J. MacLeod, On integer relations between the area and perimeter of Heron triangles, *Forum Geom.*, 9 (2009) 41–46.

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