Trilinear Polars and Antiparallels

Shao-Cheng Liu

Abstract. We study the triangle bounded by the antiparallels to the sidelines of a given triangle \(ABC\) through the intercepts of the trilinear polar of a point \(P\) other than the centroid \(G\). We show that this triangle is perspective with the reference triangle, and also study the condition of concurrency of the antiparallels. Finally, we also study the configuration of induced \(GP\)-lines and obtain an interesting conjugation of finite points other than \(G\).

1. Perspector of a triangle bounded by antiparallels

We use the barycentric coordinates with respect to triangle \(ABC\) throughout. Let \(P = (u : v : w)\) be a finite point in the plane of \(ABC\), distinct from its centroid \(G\). The trilinear polar of \(P\) is the line
\[
\mathcal{L} : \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,
\]
which intersects the sidelines \(BC, CA, AB\) respectively at
\[
P_a = (0 : v : -w), \quad P_b = (-u : 0 : w), \quad P_c = (u : -v : 0).
\]
The lines through \(P_a, P_b, P_c\) antiparallel to the respective sidelines of \(ABC\) are
\[
\mathcal{L}_a : (b^2 w - c^2 v)x + (b^2 - c^2)wy + (b^2 - c^2)vz = 0,
\]
\[
\mathcal{L}_b : (c^2 - a^2)wx + (c^2 u - a^2 w)y + (c^2 - a^2)uz = 0,
\]
\[
\mathcal{L}_c : (a^2 - b^2)vx + (a^2 - b^2)uy + (a^2 v - b^2 u)z = 0.
\]
They bound a triangle with vertices
\[
A' = (-a^2(a^2(u^2 - wv) + b^2 w(w - u) - c^2 u(u - v))
\]
\[
: (c^2 - a^2)(a^2 v(w - u) + b^2 u(v - w))
\]
\[
: (a^2 - b^2)(c^2 u(v - w) + a^2 w(u - v))),
\]
\[
B' = ((b^2 - c^2)(a^2 v(w - u) + b^2 u(v - w))
\]
\[
: -b^2(b^2(v^2 - wu) + c^2 v(u - v) - a^2 v(v - w))
\]
\[
: (a^2 - b^2)(b^2 w(u - v) + c^2 v(w - u))),
\]
\[
C' = ((b^2 - c^2)(c^2 u(v - w) + a^2 w(u - v))
\]
\[
: (c^2 - a^2)(b^2 w(u - v) + c^2 v(w - u))
\]
\[
: -c^2(c^2(w^2 - uv) + a^2 w(v - w) - b^2 w(w - u))).
\]
Figure 1. Perspector of triangle bounded by antiparallels

The lines $AA', BB', CC'$ intersect at a point

$$Q = \left( \frac{b^2 - c^2}{b^2w(u - v) + c^2v(w - u)} : \frac{c^2 - a^2}{c^2u(v - w) + a^2w(u - v)} : \frac{a^2 - b^2}{a^2v(w - u) + b^2u(v - w)} \right).$$ (1)

We show that $Q$ is a point on the Jerabek hyperbola. The coordinates of $Q$ in (1) can be rewritten as

$$Q = \left( \frac{a^2(b^2 - c^2)}{\frac{1}{c^2} + \frac{1}{a^2}} : \frac{b^2(c^2 - a^2)}{\frac{1}{a^2} + \frac{1}{c^2}} : \frac{c^2(a^2 - b^2)}{\frac{1}{b^2} + \frac{1}{a^2}} \right).$$

If we also write this in the form $\left( \frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z} \right)$, then

$$\frac{\frac{1}{b^2} - \frac{1}{a^2}}{c^2} + \frac{\frac{1}{a^2} - \frac{1}{c^2}}{b^2} + \frac{\frac{1}{c^2} - \frac{1}{b^2}}{a^2} = (b^2 - c^2)x : (c^2 - a^2)y : (a^2 - b^2)z.$$
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\[
\frac{1}{v} - \frac{1}{w} : \frac{1}{w} - \frac{1}{u} : \frac{1}{w} - \frac{1}{v} = -(b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z : \cdots : \cdots \\
\frac{1}{v} - \frac{1}{v} : \frac{1}{u} - \frac{1}{u} : \frac{1}{v} - \frac{1}{w} = a^2(-(b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z) : \cdots : \cdots 
\]

Since \((\frac{1}{v} - \frac{1}{w}) + (\frac{1}{w} - \frac{1}{u}) + (\frac{1}{u} - \frac{1}{v}) = 0\), we have

\[
0 = \sum_{\text{cyclic}} a^2(-(b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z) \\
= \sum_{\text{cyclic}} (-a^2(b^2 - c^2) + b^2(b^2 - c^2) + c^2(b^2 - c^2))x \\
= \sum_{\text{cyclic}} (b^2 - c^2)(b^2 + c^2 - a^2)x.
\]

This is the equation of the Euler line. It shows that the point \(Q\) lies on the Jerabek hyperbola. We summarize this in the following theorem, with a slight modification of (1).

**Theorem 1.** Let \(P = (u : v : w)\) be a point in the plane of triangle \(ABC\), distinct from its centroid. The antiparallels through the intercepts of the trilinear polar of \(P\) bound a triangle perspective with \(ABC\) at a point

\[
Q(P) = \left( \frac{b^2 - c^2}{b^2 \left( \frac{1}{u} - \frac{1}{v} \right) + c^2 \left( \frac{1}{w} - \frac{1}{u} \right)} : \cdots : \cdots \right)
\]

on the Jerabek hyperbola.

Here are some examples.

<table>
<thead>
<tr>
<th>(P)</th>
<th>(X_1)</th>
<th>(X_3)</th>
<th>(X_4)</th>
<th>(X_6)</th>
<th>(X_9)</th>
<th>(X_{23})</th>
<th>(X_{24})</th>
<th>(X_{69})</th>
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<tr>
<td>(Q(P))</td>
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<td>(Q(P))</td>
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<td>(X_{68})</td>
<td>(X_{69})</td>
<td>(X_{70})</td>
<td>(X_{71})</td>
<td>(X_{72})</td>
<td>(X_{73})</td>
<td>(X_{74})</td>
</tr>
</tbody>
</table>

Table 1. The perspector \(Q(P)\)

Note that for the orthocenter \(X_4 = H\) and \(X_6 = K\), we have \(Q(H) = H\) and \(Q(K) = K\). In fact, for \(P = H\), the lines \(L_a, L_b, L_c\) bound the orthic triangle. On the other hand, for \(P = K\), these lines bound the tangential triangle, anticevian triangle of \(K\). We prove that these are the only points satisfying \(Q(P) = P\).

**Proposition 2.** The perspector \(Q(P)\) coincides with \(P\) if and only if \(P\) is the orthocenter or the symmedian point.
Proof. The perspector \( R \) coincides with \( P \) if and only if the lines \( AP, L_b, L_c \) are concurrent, so are the triples \( BP, L_c, L_a \) and \( CP, L_a, L_b \). Now, \( AP, L_b, L_c \) are concurrent if and only if

\[
\begin{vmatrix}
0 & w & -v \\
(c^2 - a^2)w & (c^2 - a^2)u & (c^2 - a^2)u \\
(a^2 - b^2)v & (a^2 - b^2)u & (a^2 - b^2)u
\end{vmatrix} = 0,
\]
or

\[
a^2(a^2 - b^2)v^2w + a^2(c^2 - a^2)v^2w^2 - b^2(c^2 - a^2)w^2u - c^2(a^2 - b^2)uv^2 = 0.
\]

From the other two triples we obtain

\[
-a^2(b^2 - c^2)vw^2 + b^2(b^2 - c^2)w^2u + b^2(a^2 - b^2)wu^2 - c^2(a^2 - b^2)w^2v = 0
\]

and

\[
-a^2(b^2 - c^2)v^2w - b^2(c^2 - a^2)w^2u + c^2(c^2 - a^2)u^2v + c^2(b^2 - c^2)uv^2 = 0.
\]

From the difference of the last two, we have, apart from a factor \( b^2 - c^2 \),

\[
u(b^2w^2 - c^2v^2) + v(c^2u^2 - a^2w^2) + w(a^2v^2 - b^2u^2) = 0.
\]

This shows that \( P \) lies on the Thomson cubic, the isogonal cubic with pivot the centroid \( G \). The Thomson cubic is appears as \( K002 \) in Bernard Gibert’s catalogue [2]. The same point, as a perspector, lies on the Jerabek hyperbola. Since the Thomson cubic is self-isogonal, its intersections with the Jerabek hyperbola are the isogonal conjugates of the intersections with the Euler line. From [2], \( P^* \) is either \( G, O \) or \( H \). This means that \( P \) is \( K, H, \) or \( O \). Table 1 eliminates the possibility \( P = O \), leaving \( H \) and \( K \) as the only points satisfying \( Q(P) = P \). \( \square \)

Proposition 3. Let \( P \) be a point distinct from the centroid \( G \), and \( \Gamma \) the circum-hyperbola containing \( G \) and \( P \). If \( T \) traverses \( \Gamma \), the antiparallels through the intercepts of the trilinear polar of \( T \) bound a triangle perspective with \( ABC \) with the same perspector \( Q(P) \) on the Jerabek hyperbola.

Proof. The circum-hyperbola containing \( G \) and \( P \) is the isogonal transform of the line \( KP^* \). If we write \( P^* = (u : v : w) \), then a point \( T \) on \( \Gamma \) has coordinates

\[
\left(\frac{u^2}{u+ta} : \frac{v^2}{v+tb} : \frac{w^2}{w+tc}\right)
\]

for some real number \( t \). By Theorem 1, we have

\[
Q(T) = \left(\frac{b^2 - c^2}{b^2\left(\frac{u+ta^2}{a^2} - \frac{v+tb^2}{b^2}\right)} + c^2\left(\frac{w+tc^2}{c^2} - \frac{u+ta^2}{a^2}\right) : \cdots : \cdots\right)
\]

\[
= \left(\frac{b^2 - c^2}{b^2\left(\frac{a}{a} - \frac{b}{b}\right)} + c^2\left(\frac{c}{c} - \frac{a}{a}\right) : \cdots : \cdots\right)
\]

\[
= Q(P).
\]

\( \square \)
2. Concurrency of antiparallels

**Proposition 4.** The three lines $L_a$, $L_b$, $L_c$ are concurrent if and only if

$$-2(a^2-b^2)(b^2-c^2)(c^2-a^2)uvw + \sum_{cyclic} b^2c^2u((c^2+a^2-b^2)v^2-(a^2+b^2-c^2)w^2) = 0.$$  

(2)

*Proof.* The three lines are concurrent if and only if

$$\begin{vmatrix} b^2w - c^2v & (b^2 - c^2)w & (b^2 - c^2)v \\ (c^2 - a^2)w & c^2u - a^2w & (c^2 - a^2)u \\ (a^2 - b^2)v & (a^2 - b^2)u & a^2v - b^2u \end{vmatrix} = 0.$$  

□

For $P = X_{25}$ (the homothetic center of the orthic and tangential triangles), the trilinear polar is parallel to the Lemoine axis (the trilinear polar of $K$), and the lines $L_a$, $L_b$, $L_c$ concur at the symmedian point (see Figure 2).

![Figure 2. Antiparallels through the intercepts of $X_{25}$](image)

The cubic defined by (2) can be parametrized as follows. If $Q$ is the point \(\left(\frac{a^2}{a^2+(b^2+c^2-a^2)+t} : \cdots : \cdots\right)\) on the Jerabek hyperbola, then the antiparallels through
the intercepts of the trilinear polar of
\[ P_0(Q) = \left( \frac{a^2(b^2c^2 + t)}{(b^2 + c^2 - a^2)(a^2(b^2 + c^2 - a^2) + t)} : \cdots : \cdots \right) \]
are concurrent at \( Q \). On the other hand, given \( P = (u : v : w) \), the antiparallels through the intercepts of the trilinear polar of
\[ P_0 = \left( \frac{u(v-w)}{a^2(b^2 + c^2 - a^2)(b^2w(u-v) + c^2v(w-u))} : \cdots : \cdots \right) \]
are concurrent at \( Q(P) \). Here are some examples.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( X_1 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( X_6 )</th>
<th>( X_{24} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0(Q) )</td>
<td>( X_{278} )</td>
<td>( X_{1073} )</td>
<td>( X_{2052} )</td>
<td>( X_{25} )</td>
<td>( X_{1993} )</td>
</tr>
</tbody>
</table>

Table 2. \( P_0(Q) \) for \( Q \) on the Jerabek hyperbola

\[ P_0(X_{66}) = \left( \frac{1}{a^2(b^4 + c^4 - a^4)} : \cdots : \cdots \right), \]
\[ P_0(X_{69}) = (b^2c^2(b^2 + c^2 - 3a^2) : \cdots : \cdots), \]
\[ P_0(X_{71}) = (a^2(b + c - a)(a(bc + ca + ab) - (b^3 + c^3)) : \cdots : \cdots), \]
\[ P_0(X_{72}) = (a^3 - a^2(b + c) - a(b + c)^2 + (b + c)(b^2 + c^2) : \cdots : \cdots). \]

3. Triple of induced GP-lines

Let \( P \) be a point in the plane of triangle \( ABC \), distinct from the centroid \( G \), with trilinear polar intersecting \( BC, CA, AB \) respectively at \( P_a, P_b, P_c \). Let the antiparallel to \( BC \) through \( P_a \) intersect \( CA \) and \( AB \) at \( B_a \) and \( C_a \) respectively; similarly define \( C_b, A_b, A_c, B_c \). These are the points

\[ B_a = ((b^2 - c^2)v : 0 : c^2v - b^2w), \quad C_a = ((b^2 - c^2)w : c^2v - b^2w : 0); \]
\[ A_b = (0 : (c^2 - a^2)u : a^2w - c^2u), \quad C_b = (a^2w - c^2u : (c^2 - a^2)w : 0); \]
\[ A_c = (0 : b^2u - a^2v : (a^2 - b^2)u), \quad B_c = (b^2u - a^2v : 0 : (a^2 - b^2)v). \]

The triangles \( AB_aC_a, A_bBC_b, A_cB_c \) are all similar to \( ABC \). For every point \( T \) with reference to \( ABC \), we can speak of the corresponding points in these triangles with the same homogeneous barycentric coordinates. Thus, the \( P \)-points in these triangles are

\[ P_A = (b^2c^2(u + v + w)(v-w) - c^4v^2 + b^4w^2 : b^2w(c^2v - b^2w) : c^2v(c^2v - b^2w)), \]
\[ P_B = (a^2w(a^2w - c^2u) : c^2a^2(u + v + w)(w-u) - a^4u^2 + c^4u^2 : c^2u(a^2w - c^2u)), \]
\[ P_C = (a^2v(b^2u - a^2v) : b^2u(b^2u - a^2v) : a^2b^2(u + v + w)(u-v) - b^4u^2 + a^4v^2). \]
On the other hand, the centroids of these triangles are the points

\[
G_A = (2b^2c^2(v - w) - c^4v + b^4w : b^2(c^2v - b^2w) : c^2(c^2v - b^2w)),
\]
\[
G_B = (a^2(a^2w - c^2u) : 2c^2a^2(w - u) - a^4w + c^4u : c^2(a^2w - c^2u)),
\]
\[
G_C = (a^2(b^2u - a^2v) : b^2(b^2u - a^2v) : 2a^2b^2(u - v) - b^4u + a^4v).
\]

We call \(G_A P_A, G_B P_B, G_C P_C\) the triple of \(GP\)-lines induced by antiparallels through the intercepts of the trilinear polar of \(P\), or simply the triple of induced \(GP\)-lines.

**Figure 3. Triple of induced \(GP\)-lines**

**Theorem 5.** The triple of induced \(GP\)-lines are concurrent at

\[
\tau(P) = \left(\frac{-a^2(u^2 - v^2 + vw - w^2) + b^2u(u + v - 2w) + c^2u(w + u - 2v)}{a^2v(u + v - 2w) - b^2(v^2 - w^2 + wu - u^2) + c^2v(v + w - 2u)} : \frac{a^2w(w + u - 2v) + b^2w(v + w - 2u) - c^2(w^2 - u^2 + uv - v^2)}{a^2w(w + u - 2v) + b^2w(v + w - 2u) - c^2(w^2 - u^2 + uv - v^2)} \right).
\]
Proof. The equations of the lines $G_A P_A, G_B P_B, G_C P_C$ are

\begin{align*}
(c^2 v - b^2 w)x + (c^2(w + u - v) - b^2 w)y - (b^2(u + v - w) - c^2 v)z &= 0, \\
-(c^2(v + w - u) - a^2 w)x + (a^2 w - c^2 u)y + (a^2(u + v - w) - c^2 u)z &= 0, \\
(b^2(v + w - u) - a^2 v)x - (a^2(w + u - v) - b^2 u)y + (b^2 u - a^2 v)z &= 0.
\end{align*}

These three lines intersect at $\tau(P)$ given above. \hfill \Box

Remark. If $T$ traverses the line $GP$, then $\tau(T)$ traverses the line $G\tau(P)$.

Note that the equations of induced $GP$-lines are invariant under the permutation $(x, y, z) \leftrightarrow (u, v, w)$, i.e., these can be rewritten as

\begin{align*}
(c^2 y - b^2 z)u + (c^2(z + x - y) - b^2 z)v - (b^2(x + y - z) - c^2 y)w &= 0, \\
-(c^2(y + z - x) - a^2 z)u + (a^2 z - c^2 x)v + (a^2(x + y - z) - c^2 x)w &= 0, \\
(b^2(y + z - x) - a^2 y)u - (a^2(z + x - y) - b^2 x)v + (b^2 x - a^2 y)w &= 0.
\end{align*}

This means that the mapping $\tau$ is a conjugation of the finite points other than the centroid $G$.

Corollary 6. The triple of induced $GP$-lines concur at $Q$ if and only if the triple of induced $GQ$-lines concur at $P$.

We conclude with a list of pairs of triangle centers conjugate under $\tau$.

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
$X_1$, $X_{1054}$ & $X_3$, $X_{110}$ \\
$X_98$, $X_{1316}$ & $X_{100}$, $X_{1083}$ \\
$X_4$, $X_{125}$ & $X_6$, $X_{111}$ \\
$X_{23}$, $X_{182}$ & $X_{69}$, $X_{126}$ \\
$X_{184}$, $X_{186}$ & $X_{187}$, $X_{353}$ \\
$X_{352}$, $X_{574}$ & \\
\hline
\end{tabular}
\caption{Pairs conjugate under $\tau$}
\end{table}

References


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