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Trilinear Polars and Antiparallels

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Abstract. We study the triangle bounded by the antiparallels to the sidelines of a given triangle ABC through the intercepts of the trilinear polar of a point P other than the centroid G. We show that this triangle is perspective with the reference triangle, and also study the condition of concurrency of the antiparallels. Finally, we also study the configuration of induced GP-lines and obtain an interesting conjugation of finite points other than G.

1. Perspector of a triangle bounded by antiparallels

We use the barycentric coordinates with respect to triangle ABC throughout. Let $P = (u : v : w)$ be a finite point in the plane of ABC, distinct from its centroid G. The trilinear polar of P is the line

$$\mathcal{L} : \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,$$

which intersects the sidelines BC, CA, AB respectively at

$$P_a = (0 : v : -w), \quad P_b = (-u : 0 : w), \quad P_c = (u : -v : 0).$$

The lines through P_a, P_b, P_c antiparallel to the respective sidelines of ABC are

$$\begin{aligned}
\mathcal{L}_a &: \quad (b^2w - c^2v)x + (b^2 - c^2)wy + (b^2 - c^2)vz = 0,\\
\mathcal{L}_b &: \quad (c^2 - a^2)wx + (c^2u - a^2w)y + (c^2 - a^2)uz = 0,\\
\mathcal{L}_c &: \quad (a^2 - b^2)vx + (a^2 - b^2)uy + (a^2v - b^2u)z = 0.
\end{aligned}$$

They bound a triangle with vertices

$$\begin{aligned}
A' = \; & (-a^2(a^2(u^2 - vw) + b^2u(w - u) - c^2u(u - v))\\
& : (c^2 - a^2)(a^2v(w - u) + b^2u(v - w))\\
& : (a^2 - b^2)(c^2u(v - w) + a^2w(u - v))),\\
B' = \; & ((b^2 - c^2)(a^2v(w - u) + b^2u(v - w))\\
& : -b^2(b^2(v^2 - wu) + c^2v(u - v) - a^2v(v - w))\\
& : (a^2 - b^2)(b^2w(u - v) + c^2v(w - u))),\\
C' = \; & ((b^2 - c^2)(c^2u(v - w) + a^2w(u - v))\\
& : (c^2 - a^2)(b^2w(u - v) + c^2v(w - u))\\
& : -c^2(c^2(w^2 - uv) + a^2w(v - w) - b^2w(w - u))).
\end{aligned}$$

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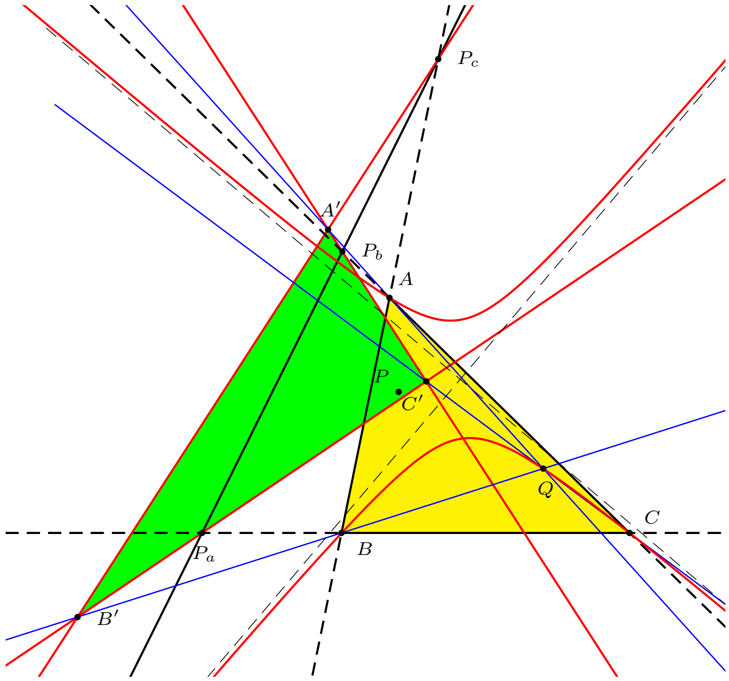

Figure 1. Perspector of triangle bounded by antiparallels

The lines AA', BB', CC' intersect at a point

$$Q = \left(\frac{b^2-c^2}{b^2w(u-v)+c^2v(w-u)} : \frac{c^2-a^2}{c^2u(v-w)+a^2w(u-v)} : \frac{a^2-b^2}{a^2v(w-u)+b^2u(v-w)}\right) \quad (1)$$

We show that Q is a point on the Jerabek hyperbola. The coordinates of Q in (1) can be rewritten as

$$Q = \left(\frac{a^2(b^2-c^2)}{\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2}} : \frac{b^2(c^2-a^2)}{\frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2}} : \frac{c^2(a^2-b^2)}{\frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2}}\right).$$

If we also write this in the form $\left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$, then

$$\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2} : \frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2} : \frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2} = (b^2-c^2)x : (c^2-a^2)y : (a^2-b^2)z.$$

$$\frac{\frac{1}{v}-\frac{1}{w}}{a^2}:\frac{\frac{1}{w}-\frac{1}{u}}{b^2}:\frac{\frac{1}{u}-\frac{1}{v}}{c^2}=-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z:\dots:\dots$$
$$\frac{1}{v}-\frac{1}{w}:\frac{1}{w}-\frac{1}{u}:\frac{1}{u}-\frac{1}{v}=a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z):\dots:\dots$$

Since $\left(\frac{1}{v}-\frac{1}{w}\right)+\left(\frac{1}{w}-\frac{1}{u}\right)+\left(\frac{1}{u}-\frac{1}{v}\right)=0$, we have

$$\begin{aligned} 0 = & \sum_{\text{cyclic}} a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z) \\ = & \sum_{\text{cyclic}} (-a^2(b^2-c^2)+b^2(b^2-c^2)+c^2(b^2-c^2))x \\ = & \sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x. \end{aligned}$$

This is the equation of the Euler line. It shows that the point Q lies on the Jerabek hyperbola. We summarize this in the following theorem, with a slight modification of (1).

Theorem 1. *Let $P=(u:v:w)$ be a point in the plane of triangle ABC, distinct from its centroid. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC at a point*

$$Q(P)=\left(\frac{b^2-c^2}{b^2\left(\frac{1}{u}-\frac{1}{v}\right)+c^2\left(\frac{1}{w}-\frac{1}{u}\right)}:\dots:\dots\right)$$

on the Jerabek hyperbola.

Here are some examples.

P	X_1	X_3	X_4	X_6	X_9	X_{23}	X_{24}	X_{69}
$Q(P)$	X_{65}	X_{64}	X_4	X_6	X_{1903}	X_{1177}	X_3	X_{66}
P	X_{468}	X_{847}	X_{193}	X_{93}	X_{284}	X_{943}	X_{1167}	X_{186}
$Q(P)$	X_{67}	X_{68}	X_{69}	X_{70}	X_{71}	X_{72}	X_{73}	X_{74}

Table 1. The perspector $Q(P)$

Note that for the orthocenter $X_4=H$ and $X_6=K$, we have $Q(H)=H$ and $Q(K)=K$. In fact, for $P=H$, the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ bound the orthic triangle. On the other hand, for $P=K$, these lines bound the tangential triangle, anticevian triangle of K. We prove that these are the only points satisfying $Q(P)=P$.

Proposition 2. *The perspector $Q(P)$ coincides with P if and only if P is the orthocenter or the symmedian point.*

Proof. The perspector R coincides with P if and only if the lines AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent, so are the triples BP, $\mathcal{L}_c$, $\mathcal{L}_a$ and CP, $\mathcal{L}_a$, $\mathcal{L}_b$. Now, AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if

$$\begin{vmatrix} 0 & w & -v \\ (c^2-a^2)w & (c^2u-a^2w) & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & (a^2v-b^2u) \end{vmatrix} = 0,$$

or

$$a^2(a^2-b^2)v^2w + a^2(c^2-a^2)vw^2 - b^2(c^2-a^2)w^2u - c^2(a^2-b^2)uv^2 = 0.$$

From the other two triples we obtain

$$-a^2(b^2-c^2)vw^2 + b^2(b^2-c^2)w^2u + b^2(a^2-b^2)wu^2 - c^2(a^2-b^2)u^2v = 0$$

and

$$-a^2(b^2-c^2)v^2w - b^2(c^2-a^2)wu^2 + c^2(c^2-a^2)u^2v + c^2(b^2-c^2)uv^2 = 0.$$

From the difference of the last two, we have, apart from a factor b^2-c^2,

$$u(b^2w^2-c^2v^2) + v(c^2u^2-a^2w^2) + w(a^2v^2-b^2u^2) = 0.$$

This shows that P lies on the Thomson cubic, the isogonal cubic with pivot the centroid G. The Thomson cubic is appears as K002 in Bernard Gibert's catalogue [2]. The same point, as a perspector, lies on the Jerabek hyperbola. Since the Thomson cubic is self-isogonal, its intersections with the Jerabek hyperbola are the isogonal conjugates of the intersections with the Euler line. From [2], P^* is either G, O or H. This means that P is K, H, or O. Table 1 eliminates the possiblility $P=O$, leaving H and K as the only points satisfying $Q(P)=P$. □

Proposition 3. *Let P be a point distinct from the centroid G, and Γ the circum-hyperbola containing G and P. If T traverses Γ, the antiparallels through the intercepts of the triliner polar of T bound a triangle perspective with ABC with the same perspector $Q(P)$ on the Jerabek hyperbola.*

Proof. The circum-hyperbola containing G and P is the isogonal transform of the line KP^*. If we write $P^*=(u:v:w)$, then a point T on Γ has coordinates $\left(\frac{a^2}{u+ta^2}:\frac{b^2}{v+tb^2}:\frac{c^2}{w+tc^2}\right)$ for some real number t. By Theorem 1, we have

$$\begin{aligned} Q(T) &= \left(\frac{b^2-c^2}{b^2\left(\frac{u+ta^2}{a^2}-\frac{v+tb^2}{b^2}\right)+c^2\left(\frac{w+tc^2}{c^2}-\frac{u+ta^2}{a^2}\right)}:\cdots:\cdots\right) \\ &= \left(\frac{b^2-c^2}{b^2\left(\frac{u}{a^2}-\frac{v}{b^2}\right)+c^2\left(\frac{w}{c^2}-\frac{u}{a^2}\right)}:\cdots:\cdots\right) \\ &= Q(P). \end{aligned}$$

□

2. Concurrency of antiparallels

Proposition 4. *The three lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if*

$$-2(a^2-b^2)(b^2-c^2)(c^2-a^2)uvw+\sum_{\text{cyclic}} b^2c^2u((c^2+a^2-b^2)v^2-(a^2+b^2-c^2)w^2)=0. \tag{2}$$

Proof. The three lines are concurrent if and only if

$$\begin{vmatrix} b^2w-c^2v & (b^2-c^2)w & (b^2-c^2)v \\ (c^2-a^2)w & c^2u-a^2w & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & a^2v-b^2u \end{vmatrix}=0.$$

□

For $P=X_{25}$ (the homothetic center of the orthic and tangential triangles), the trilinear polar is parallel to the Lemoine axis (the trilinear polar of K), and the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ concur at the symmedian point (see Figure 2).

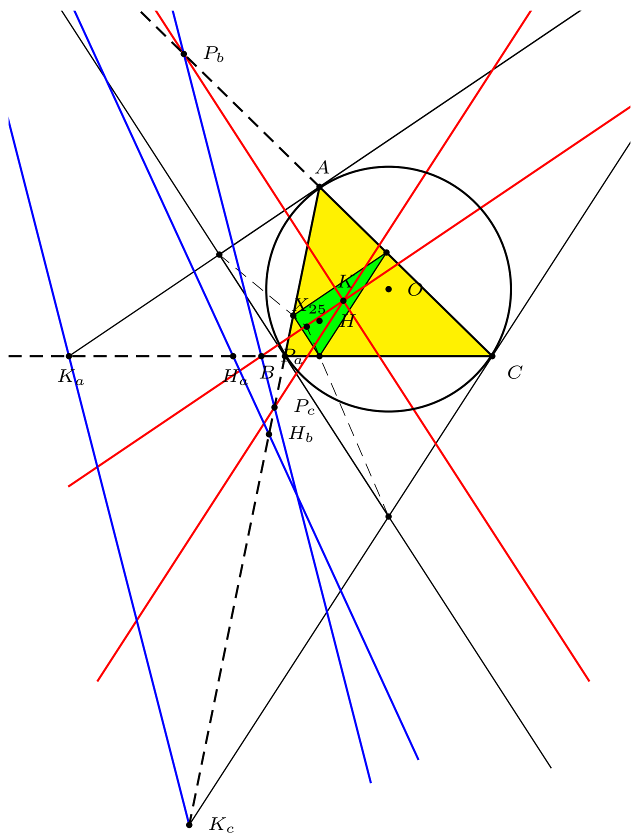

Figure 2. Antiparallels through the intercepts of X_{25}

The cubic defined by (2) can be parametrized as follows. If Q is the point $\left(\frac{a^2}{a^2(b^2+c^2-a^2)+t}:\cdots:\cdots\right)$ on the Jerabek hyperbola, then the antiparallels through the intercepts of the trilinear polar of

$$P_0(Q) = \left(\frac{a^2(b^2c^2+t)}{(b^2+c^2-a^2)(a^2(b^2+c^2-a^2)+t)} : \cdots : \cdots \right)$$

are concurrent at Q. On the other hand, given $P = (u : v : w)$, the antiparallels through the intercepts of the trilinear polars of

$$P_0 = \left(\frac{u(v-w)}{a^2(b^2+c^2-a^2)(b^2w(u-v)+c^2v(w-u))} : \cdots : \cdots \right)$$

are concurrent at $Q(P)$. Here are some examples.

P	X_1	X_3	X_4	X_6	X_{24}
Q	X_{65}	X_{64}	X_4	X_6	X_3
$P_0(Q)$	X_{278}	X_{1073}	X_{2052}	X_{25}	X_{1993}

Table 2. $P_0(Q)$ for Q on the Jerabek hyperbola

$$\begin{aligned}
P_0(X_{66}) &= \left(\frac{1}{a^2(b^4+c^4-a^4)} : \cdots : \cdots \right), \\
P_0(X_{69}) &= (b^2c^2(b^2+c^2-3a^2) : \cdots : \cdots), \\
P_0(X_{71}) &= (a^2(b+c-a)(a(bc+ca+ab)-(b^3+c^3)) : \cdots : \cdots), \\
P_0(X_{72}) &= (a^3-a^2(b+c)-a(b+c)^2+(b+c)(b^2+c^2) : \cdots : \cdots).
\end{aligned}$$

3. Triple of induced GP-lines

Let P be a point in the plane of triangle ABC, distinct from the centroid G, with trilinear polar intersecting BC, CA, AB respectively at P_a, P_b, P_c. Let the antiparallel to BC through P_a intersect CA and AB at B_a and C_a respectively; similarly define C_b, A_b, and A_c, B_c. These are the points

$$\begin{aligned}
B_a &= ((b^2-c^2)v : 0 : c^2v-b^2w), & C_a &= ((b^2-c^2)w : c^2v-b^2w : 0); \\
A_b &= (0 : (c^2-a^2)u : a^2w-c^2u), & C_b &= (a^2w-c^2u : (c^2-a^2)w : 0); \\
A_c &= (0 : b^2u-a^2v : (a^2-b^2)u), & B_c &= (b^2u-a^2v : 0 : (a^2-b^2)v).
\end{aligned}$$

The triangles AB_aC_a, A_bBC_b, A_cB_cC are all similar to ABC. For every point T with reference to ABC, we can speak of the corresponding points in these triangles with the same homogeneous barycentric coordinates. Thus, the P-points in these triangles are

$$\begin{aligned}
P_A &= (b^2c^2(u+v+w)(v-w)-c^4v^2+b^4w^2 : b^2w(c^2v-b^2w) : c^2v(c^2v-b^2w)), \\
P_B &= (a^2w(a^2w-c^2u) : c^2a^2(u+v+w)(w-u)-a^4w^2+c^4u^2 : c^2u(a^2w-c^2u)), \\
P_C &= (a^2v(b^2u-a^2v) : b^2u(b^2u-a^2v) : a^2b^2(u+v+w)(u-v)-b^4u^2+a^4v^2).
\end{aligned}$$

On the other hand, the centroids of these triangles are the points

$$\begin{aligned}
G_A =& \ (2b^2c^2(v-w) - c^4v + b^4w : b^2(c^2v - b^2w) : c^2(c^2v - b^2w)),\\
G_B =& \ (a^2(a^2w - c^2u) : 2c^2a^2(w-u) - a^4w + c^4u : c^2(a^2w - c^2u)),\\
G_C =& \ (a^2(b^2u - a^2v) : b^2(b^2u - a^2v) : 2a^2b^2(u-v) - b^4u + a^4v).
\end{aligned}$$

We call G_AP_A, G_BP_B, G_CP_C the triple of GP-lines induced by antiparallels through the intercepts of the trilinear polar of P, or simply the triple of induced GP-lines.

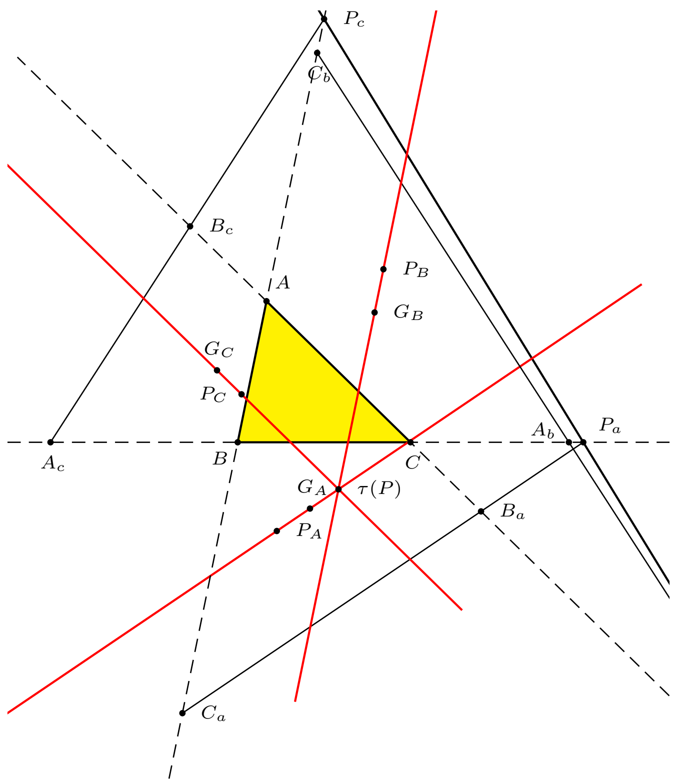

Figure 3. Triple of induced GP-lines

Theorem 5. *The triple of induced GP-lines are concurrent at*

$$\begin{aligned}
\tau(P) = (&-a^2(u^2 - v^2 + vw - w^2) + b^2u(u+v-2w) + c^2u(w+u-2v)\\
&: a^2v(u+v-2w) - b^2(v^2 - w^2 + wu - u^2) + c^2v(v+w-2u)\\
&: a^2w(w+u-2v) + b^2w(v+w-2u) - c^2(w^2 - u^2 + uv - v^2)).
\end{aligned}$$

Proof. The equations of the lines G_AP_A, G_BP_B, G_CP_C are

$$(c^2v - b^2w)x + (c^2(w+u-v) - b^2w)y - (b^2(u+v-w) - c^2v)z = 0,$$
$$-(c^2(v+w-u) - a^2w)x + (a^2w - c^2u)y + (a^2(u+v-w) - c^2u)z = 0,$$
$$(b^2(v+w-u) - a^2v)x - (a^2(w+u-v) - b^2u)y + (b^2u - a^2v)z = 0.$$

These three lines intersect at $\tau(P)$ given above. □

Remark. If T traverses the line GP, then $\tau(T)$ traverses the line $G\tau(P)$.

Note that the equations of induced GP-lines are invariant under the permutation $(x, y, z) \leftrightarrow (u, v, w)$, *i.e.*, these can be rewritten as

$$(c^2y - b^2z)u + (c^2(z+x-y) - b^2z)v - (b^2(x+y-z) - c^2y)w = 0,$$
$$-(c^2(y+z-x) - a^2z)u + (a^2z - c^2x)v + (a^2(x+y-z) - c^2x)w = 0,$$
$$(b^2(y+z-x) - a^2y)u - (a^2(z+x-y) - b^2x)v + (b^2x - a^2y)w = 0.$$

This means that the mapping τ is a conjugation of the finite points other than the centroid G.

Corollary 6. *The triple of induced GP-lines concur at Q if and only if the triple of induced GQ-lines concur at P.*

We conclude with a list of pairs of triangle centers conjugate under τ.

$X_1,\ X_{1054}$	$X_3,\ X_{110}$	$X_4,\ X_{125}$	$X_6,\ X_{111}$	X_{23}, X_{182}	X_{69}, X_{126}
X_{98}, X_{1316}	X_{100}, X_{1083}	$X_{184},\ X_{186}$	$X_{187},\ X_{353}$	$X_{352},\ X_{574}$	

Table 3. Pairs conjugate under τ

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