Forum Geometricorum
Volume 9 (2009) 297–300.

FORUM GEOM
ISSN 1534-1178

Trilinear Polars of Brocardians

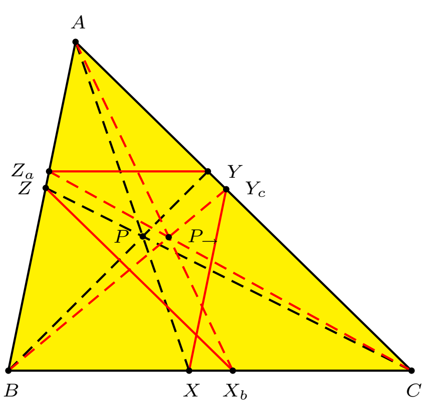
Francisco Javier García Capitán

Abstract. We study the trilinear polars of the Brocardians of a point, and investigate the condition for their orthogonality.

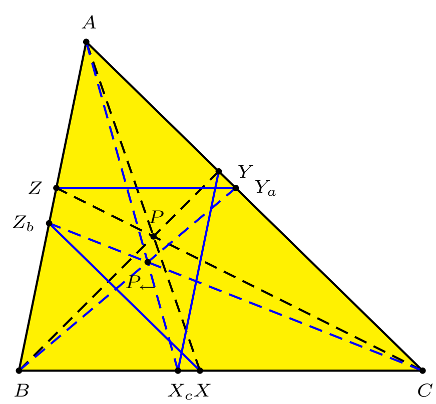
1. The Brocardians

Let P be a point not on any of the sidelines of triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$ and cevian triangle XYZ. Construct the parallels of $\begin{matrix}CA\\AB\\BC\end{matrix}$ through $\begin{matrix}Z\\X\\Y\end{matrix}$ to intersect $\begin{matrix}BC\\CA\\AB\end{matrix}$ at $\begin{matrix}X_b\\Y_c\\Z_a\end{matrix}$ (see Figure 1(a)). The triangle $X_bY_cZ_a$ is perspective with ABC at the point

$$P_{\rightarrow} := \left(\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right).$$



1(a) The Brocardian $P_{\rightarrow}$



1(b) The Brocardian $P_{\leftarrow}$

Likewise, the parallels of $\begin{matrix}AB\\BC\\CA\end{matrix}$ through $\begin{matrix}Y\\Z\\X\end{matrix}$ intersect $\begin{matrix}BC\\CA\\AB\end{matrix}$ at $\begin{matrix}X_c\\Y_a\\Z_b\end{matrix}$ such that triangle $X_cY_aZ_b$ is perspective with ABC at

$$P_{\leftarrow} := \left(\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right)$$

(see Figure 1(b)). The points $P_{\rightarrow}$ and $P_{\leftarrow}$ are called the Brocardians of P (see [2, §8.4]). For example, the Brocardians of the symmedian point are the Brocard points $\Omega = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$ and $\Omega' = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$.

Publication Date: December 18, 2009. Communicating Editor: Paul Yiu.

2. Noncollinearity of P and its Brocardians

The points P, $P_{\rightarrow}$ and $P_{\leftarrow}$ are never collinear since

$$\begin{vmatrix} u & v & w \\ \frac{1}{w} & \frac{1}{u} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{w} & \frac{1}{u} \end{vmatrix} = \frac{u^2+v^2+w^2-vw-wu-uv}{uvw} \neq 0.$$

It is well known that the Brocard points are equidistant from the symmedian point. It follows that the pedal of K on the line $\Omega\Omega'$ is the midpoint of the segment $\Omega\Omega'$, the triangle center $X_{39} = (a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2))$ in [1].

Now, for the Gergonne point $G_{\mathrm{e}} = \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$, the Brocardians are the points $G_{\mathrm{e}\rightarrow} = (a+b-c : b+c-a : c+a-b)$ and $G_{\mathrm{e}\leftarrow} = (c+a-b : a+b-c : b+c-a)$. The midpoint of $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$ is the incenter $I = (a : b : c)$. Indeed, I is the pedal of the Gergonne point on the line $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$

$$(b^2+c^2-a(b+c))x + (c^2+a^2-b(c+a))y + (a^2+b^2-c(a+b))z = 0.$$

3. Trilinear polars of the Brocardians

The trilinear polars of the Brocardians of P are the lines

$$\ell_{\rightarrow} \qquad wx + uy + vz = 0,$$

and

$$\ell_{\leftarrow} \qquad vx + wy + uz = 0.$$

These lines intersect at the point

$$Q = (u^2 - vw : v^2 - wu : w^2 - uv).$$

Since

$$(u^2-vw,\ v^2-wu,\ w^2-uv) = (u+v+w)(u,\ v,\ w) - (vw+wu+uv)(1,\ 1,\ 1),$$

the point Q divides the segment GP in the ratio

$$GQ : QP = (u+v+w)^2 : -3(vw+wu+uv).$$

The point Q is never an infinite point since

$$u^2+v^2+w^2-vw-wu-uv \neq 0.$$

It follows that the trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ are never parallel.

4. Orthogonality of trilinear polars of Brocardians

The trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ have infinite points $(u-v : v-w : w-u)$ and $(w-u : u-v : v-w)$ respectively. They are orthogonal if and only if

$$S_A(u-v)(w-u) + S_B(v-w)(u-v) + S_C(w-u)(v-w) = 0 \qquad (1)$$

(see [2, §4.5]). Now, (1) defines a conic with center $G = (1 : 1 : 1)$ (see [2, §10.7.2]). Since the conic contains G, it is necessarily degenerate. Solving for the infinite points of the conic, we obtain the condition that the conic consists of a pair of real lines if and only if

$$S_{AA} + S_{BB} + S_{CC} - 2S_{BC} - 2S_{CA} - 2S_{AB} \geq 0.$$

Equivalently,

$$5(a^4 + b^4 + c^4) - 6(b^2c^2 + c^2a^2 + a^2b^2) \geq 0. \tag{2}$$

Here is a characterization of triangles satisfying condition (2). Given two points B and C with $BC = a$, we set up a Cartesian coordinates system such that $B = \left(-\frac{a}{2},\ 0\right)$ and $C = \left(\frac{a}{2},\ 0\right)$. If $A = (x, y)$, then

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \ b^2,$$
$$\left(x + \frac{a}{2}\right)^2 + y^2 = \ c^2.$$

With these, condition (2) becomes

$$(4x^2 + 4y^2 - 8ay + 3a^2)(4x^2 + 4y^2 + 8ay + 3a^2) \geq 0.$$

This is the exterior of the two circles, centers $(0,\ \pm a)$, radii $\frac{a}{2}$. Here is a simple example. If we require $C = \frac{\pi}{2}$, then $S_C = 0$ and the degenerate conic (1) is the union of the two lines $v - w = 0$ and $S_A(z - x) + S_B(y - z) = 0$. These are the C-median and the line GK_c, K_c being the C-trace of the symmedian point K. Figure 2 illustrates the trilinear polars of the Brocardians of a point P on GK_c

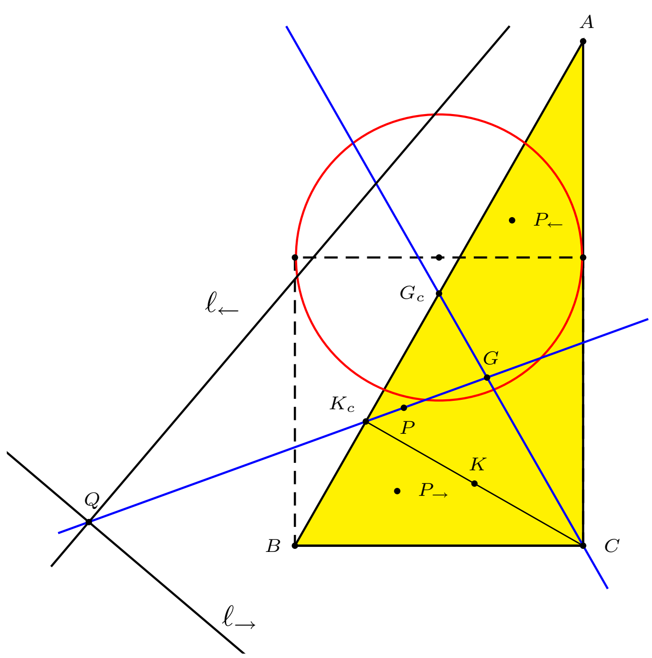


Figure 2.

On the other hand, for points A on the circumferences of the two circles, the triangle ABC has exactly one real line through the centroid G such that for every P on the line, the trilinear polars of the Brocardians intersect orthogonally (on the same line). It is enough to consider A on the circle $4(x^2+y^2)-8ay+3a^2=0$, with coordinates $\left(\frac{a}{2}\cos\theta,\ a+\frac{a}{2}\sin\theta\right)$. The center of triangle ABC is the point $G=\left(\frac{a}{6}\cos\theta,\ \frac{a}{6}(2+\sin\theta)\right)$. The line in question connects G to the fixed point $M=\left(0,\ \frac{a}{2}\right)$:

$$(1-\sin\theta)x+\cos\theta\left(y-\frac{a}{2}\right)=0.$$

The trilinear polars of the Brocardians of an arbitrary point P on this line are symmetric with respect to GM, and intersect orthogonally (see Figure 3).

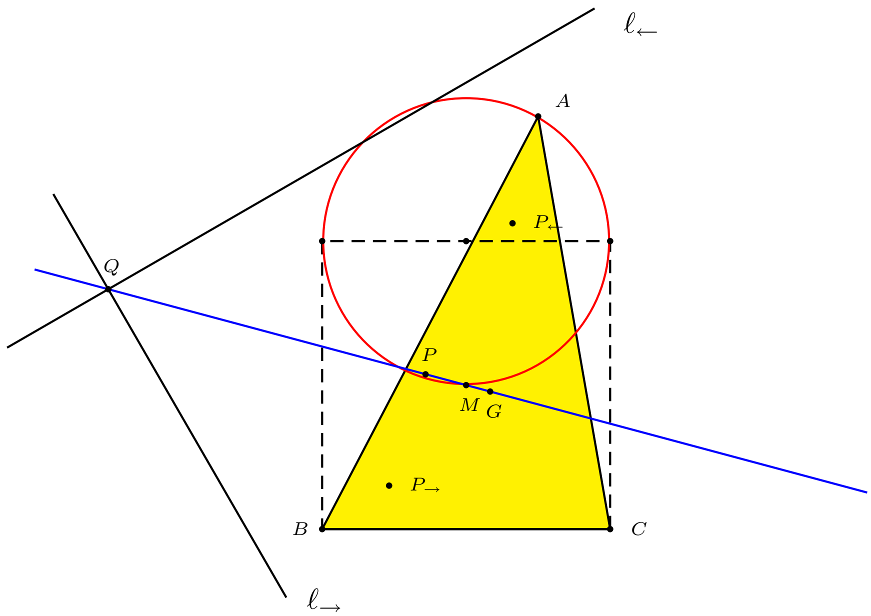


Figure 3.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: `garciacapitan@gmail.com`