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On n-Sections and Reciprocal Quadrilaterals

Eisso J. Atzema

Abstract. We introduce the notion of an n-section and reformulate a number of standard Euclidean results regarding angles in terms of 2-sections (with proof). Using 6-sections, we define the notion of reciprocal (complete) quadrangles and derive some properties of such quadrangles.

1. Introduction

While classical geometry is still admired as a model for mathematical reasoning, it is only fair to admit that following through an argument in Euclidean geometry in its full generality can be rather cumbersome. More often than not, a discussion of all manner of special cases is required. Specifically, Euclid's notion of an angle is highly unsatisfactory. With the rise of projective geometry in the 19th century, some of these issues (such as the role of points at infinity) were addressed. The need to resolve any of the difficulties connected with the notion of an angle was simply obviated by (largely) avoiding any direct appeal to the concept. By the end of the 19th century, as projective geometry and metric geometry aligned again and vectorial methods became commonplace, classical geometry saw the formal introduction of the notion of *orientation*. In the case of the concept of an angle, this led to the notion of a *directed (oriented, sensed)* angle. In France, the (elite) high school teacher and textbook author Louis Gérard was an early champion of this notion, as was Jacques Hadamard (1865-1963); in the USA, Roger Arthur Johnson (1890-1954) called for the use of such angles in classical geometry in two papers published in 1917.[1] Today, while the notion of a directed angle certainly has found its place in classical geometry research and teaching, it has by no means supplanted the traditional notion of an angle. Many college geometry textbooks still ignore the notion of orientation altogether.

In this paper we will use a notion very closely related to that of a directed angle. This notion was introduced by the Australian mathematician David Kennedy Picken (1879-1956) as the *complete angle* in 1922. Five years later and again in 1947, the New Zealand mathematician Henry George Forder (1889-1981) picked

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[1]See [5], as well as [6] and [7]; Johnson also consistently uses directed angles in his textbook [8].

up on the idea, preferring the term *cross*.[2] Essentially, where we can look upon an angle as the configuration of two rays departing from the same point, the cross is the configuration of two intersecting lines. Here, we will refer to a *cross* as a 2-section and consider it a special case of the more general notion of an n-section.

We first define these n-sections and establish ground rules for their manipulation. Using these rules, we derive a number of classical results on angles in terms of 2-sections. In the process, to overcome some of the difficulties that Picken and Forder ran into, we will also bring the theory of circular inversion into the mix. After that, we will focus on 6-sections formed by the six sides of a complete quadrangle. This will lead us to the introduction of the reciprocal to a complete quadrangle as first introduced by James Clerk Maxwell (1831-1879). We conclude this paper by studying some of the properties of reciprocal quadrangles.

2. The notion of an n-section

In the Euclidean plane, let $\{l\}$ denote the equivalence class of all lines parallel to the line l. We will refer to $\{l\}$ as the *direction* of l. Now consider the ordered set of directions of a set of lines $l_1, ..., l_n$ ($n \geq 2$). We refer to such a set as an n-section (of lines), which we will write as $\{\ell_1, ..., \ell_n\}$.[3]. Clearly, any n-section is an equivalence class of all lines $m_1, ..., m_n$ each parallel to the corresponding of $l_1, ..., l_n$. Therefore, we can think of any n-section as represented by n lines all meeting in one point. Also note that any n-section corresponds to a configuration of points on the line at infinity.

We say that two n-sections $\{l_1, ..., l_n\}$ and $\{m_1, ..., m_n\}$ are directly congruent if for any representation of the two sections by means of concurrent lines there is a rotation combined with a translation that maps each line of the one representation onto the corresponding line of the other. We write $\{l_1, ..., l_n\} \cong_D \{m_1, ..., m_n\}$. If in addition a reflection is required, $\{l_1, ..., l_n\}$ is said to be *inversely* congruent to $\{m_1, ..., m_n\}$, which we write as $\{l_1, ..., l_n\} \cong_I \{m_1, ..., m_n\}$.

Generally, no two n-sections can be both directly and inversely congruent to each other. Particularly, as a rule, an n-section is not inversely congruent to itself. A notable exception is formed by the 2-sections. Clearly, a 2-section formed by two parallel lines is inversely as well as directly congruent to itself. We will refer to such a 2-section as trivial. Any non-trivial 2-section that is inversely congruent to itself is called *perpendicular* and its two directions are said to be perpendicular to each other. We will just assume here that for every direction there always is exactly one direction perpendicular to it.[4]

No other n-sections can be both directly and inversely congruent, except for such n-sections which only consist of pairs of lines that either all parallel or are perpendicular. We will generally ignore such sections.

[2] See [3] (pp.120-121+151-154), [4], [16], and [17]. The term *cross* seems to have been coined by Edward Hope Neville in [14]. Forder may actually have also used crosses in his two geometry textbooks from 1930 and 1931, but we have not been able to locate copies of these.

[3] We adapt this notation from [15].

[4] A proof using SAS is fairly straightforward.

The following basic principles for the manipulation of n-sections apply. We would like to insist here that these principles are just working rules and not axioms (in particular they are not independent) and serve the purpose of providing a shorthand for frequent arguments more than anything else.

Principle 1 (Congruency)**.** *Two n-sections are congruent if and only if all corresponding sub-sections are congruent, where the congruencies are either all direct or all inverse.*

Principle 2 (Transfer)**.** *For any three directions $\{a\}$, $\{b\}$, and $\{c\}$, there is exactly one direction $\{d\}$ such that $\{a, b\} \cong_D \{c, d\}$.*

Principle 3 (Chain Rule)**.** *If $\{a, b\} \cong \{a', b'\}$ and $\{b, c\} \cong \{b', c'\}$ then $\{a, c\} \cong \{a', c'\}$, where the congruencies are either all direct or all inverse.*

Principle 4 (Rotation)**.** *Two n-sections $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$ are directly congruent if and only if all $\{a_i, b_i\}$ ($1 \leq i \leq n$) are directly congruent.*

Principle 5 (Reflection)**.** *Two n-sections $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$ are inversely congruent if and only if there is a direction $\{c\}$ such that $\{a_i, c\} \cong_I \{b_i, c\}$ for all $1 \leq i \leq n$.*

Most of the usual triangle similarity tests are still valid (up to orientation) if we replace the notion of an angle by that of a cross or 2-section, except for Side-Cross-Side (SCS). Since we cannot make any assumptions about the orientation on an arbitrary line, SCS is ambiguous in terms of sections in that a 2-section with a length on each of its legs, (generally) determines two non-congruent triangles. The only situation in which SCS holds true (up to orientation) is for perpendicular sections. Since we are in the Euclidean plane, the *Dilation Principle* applies to any 2-section as well: For any triangle $\triangle ABC$ with P on CA and Q on CB, $\triangle PQC$ is directly similar to $\triangle ABC$ if and only if $\overline{CP}/\overline{CA} = \overline{CQ}/\overline{CB}$, where $\overline{CA}$ and so on denote *directed* lengths.

Once again, note that we do not propose to use the n-sections to completely replace the notion of an angle. The notion of n-sections just provides a uniform way to discuss the large number of problems in geometry that are really about configurations of lines rather than configurations of rays. Starting from our definition of a perpendicular section, for instance, the basic principles suffice to give a formal proof that all perpendicular sections are congruent. In other words, they suffice to prove that all perpendicular lines are made equal. Essentially this proof streamlines the standard proof (first given by Hilbert). Let $\{a, a'\}$ be a perpendicular section and let $\{b\}$ be arbitrary direction. Now, let $\{b'\}$ be such that (i) $\{a, b\} \cong_D \{a', b'\}$ (BP 2). Then, since $\{a, a'\} \cong_D \{a', a\}$, also $\{a', b\} \cong_D \{a, b'\}$ or (ii) $\{b', a\} \cong_D \{b, a'\}$ (BP 3). Combining (i) and (ii), it follows that $\{b, b'\} \cong_D \{b', b\}$ (BP 3). In other words, $\{b'\}$ is perpendicular to $\{b\}$. Finally, by BP 4, $\{b, b'\} \cong_D \{a, a'\}$.

The same rules also naturally allow for the introduction of both angle bisectors to an angle and do not distinguish the two. Indeed, note that the “symmetry” direction $\{c\}$ in BP 5 is not unique. If $\{a_i, c\} \cong_I \{b_i, c\}$, then the same is true for the direction $\{c'\}$ perpendicular to $\{c\}$ by BP 3. Conversely, for any direction $\{d\}$ such that $\{a_i, d\} \cong_I \{b_i, d\}$, it follows that $\{d, c\} \cong_D \{c, d\}$ by BP 3. In other words, $\{c, d\}$ is a perpendicular section. We will refer to the perpendicular section $\{c, c'\}$ as the *symmetry section* of the inversely congruent n-sections $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$. In the case of the inversely congruent systems $\{a_1, a_2\}$ and $\{a_2, a_1\}$, we speak of the symmetry section of the 2-section. Obviously, the directions of the latter section are those of the angle bisectors of the angle formed by any two rays on any two lines representing $\{a_1, a_2\}$.

Using the notion of a symmetry section, we can now easily prove Thales' Theorem (as it is known in the Anglo-Saxon world).

Theorem 6 (Thales)**.** *For any three distinct points A, B and C, the line AC is perpendicular to BC if and only if C lies on the unique circle with diameter AB.*

Proof. Let O be the center of the circle with diameter AB. Since both $\triangle AOC$ and $\triangle BOC$ are isosceles, the two line of the symmetry section of $\{AB, OC\}$ are each perpendicular to one of AC and BC. Consequently, by BP 4 (Rotation), $\{AC, BC\}$ is congruent to the symmetry section, *i.e.*, AC and BC are perpendicular. Conversely, let A', B', C' be the midpoints of BC, CA, AB, respectively. Then, by dilation, $C'A'$ and $C'B'$ are parallel to CA and CB, respectively. It follows that $C'A'CB'$ is a rectangle and therefore $|B'A'| = |C'C|$, but by dilation $|B'A'| = |AC'| = |BC'|$, *i.e.*, C lies on the unique circle with diameter AB. □

3. Circular inversion

To allow further comparison of n-sections, we need the equivalent of a number of the circle theorems from Book III of Euclid's *Elements*. It is easy to see how to state any of these theorems in terms of 2-sections. As Picken remarks, however, really satisfactory proofs (in terms of 2-sections) are not so obvious and probably impossible if we do not want to use rays and angles at all. Be that as it may, we can still largely avoid directly using angles.[5] In this paper we will have recourse to the notion of *circular inversion*, which allows for reasonably smooth derivations. This transformation of (most of) the affine plane is defined with respect to a given circle with radius r and center O. For any point P of the plane other than O, its image under inversion with respect to O and the circle of radius r is defined as the unique point P' such that $\overline{OP} \cdot \overline{OP'} = r^2$ (where $\overline{OP}$ and so on denote *directed* lengths). Note that by construction circular inversion is a closed (and bijective) operation on the affine plane (excluding O). Also, if A' and B' are the images of A and B under a circular inversion with respect to a point O, then by construction $\triangle A'B'O$ is inversely similar to $\triangle ABO$. The following fundamental lemma applies.

Lemma 7. *Let O be the center of a circular inversion. Then, under this inversion (i) any circle not passing through O is mapped onto a circle not passing through O, (ii) any line not passing through O is mapped onto a circle passing through O and vice versa (with the point at infinity of the line corresponding to O), (iii) any*

[5] See [16], p.190 and [4], p.231. Forder is right to claim that the difficulty lies with the lack of an ordering for crosses and that directed angles need to be used at some point.

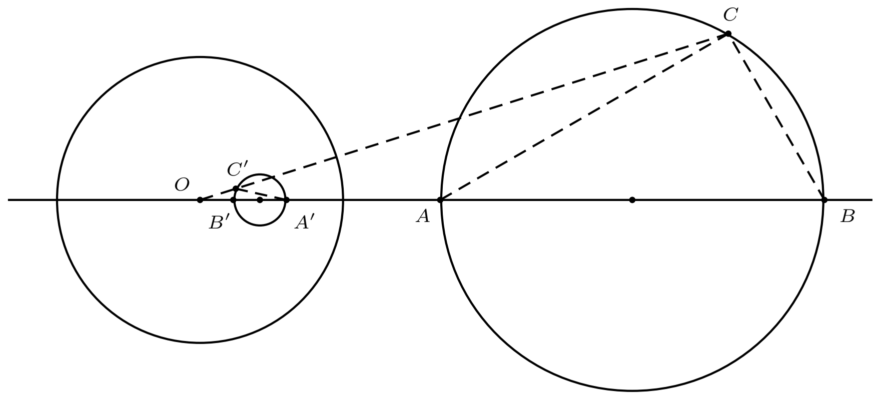

Figure 1. Circular Inversion of a Circle

line passing through O is mapped onto itself (with the point at infinity of the line again corresponding to O).

Proof. Starting with (i), draw the line connecting O with the center of the circle not passing through O and let the points of intersection of this line with the latter circle be A and B (see Figure 1). Then, by Theorem 6 (Thales), the lines AC and BC are perpendicular. Let A', B' and C' be the images of A, B, and C under the inversion. By the previous lemma $\{OA, AC, CO\}$ is indirectly congruent to $\{OC', C'A', A'O\}$. Likewise $\{OB, BC, CO\} \cong_I \{OC', C'B', B'O\}$. Since OA, OB, OA', and OB' coincide, it follows that $\{OB, BC, CA, CO\}$ is inversely congruent to $\{OC', C'B', C'A', B'O\}$. Therefore $\{BC, CA\}$ is indirectly congruent to $\{C'B', C'A'\}$. Consequently, $C'A'$ and $C'B'$ are perpendicular as well. This means that C' lies on the circle that has the segment $A'B'$ for a diameter. The second statement is proved in a similar way, while the third statement is immediate. □

We can now prove the following theorem, which is essentially a rewording in the language of sections of Propositions 21 and 22 from Book III of Euclid's *Elements* (with a trivial extension).

Theorem 8 (Equal Angle)**.** *For four points on either a circle or a straight line, let X, Y, Z, W be any permutation of A, B, C, D. Then, any 2-section $X\{Y, Z\}$ is directly congruent to the 2-section $W\{Y, Z\}$ and the sections are either trivial (in case the points are collinear) or non-trivial (in case the points are co-cyclic). Conversely, any four (distinct) points A, B, C, D for which there is a permutation X, Y, Z, W such that $X\{Y, Z\}$ and $W\{Y, Z\}$ are directly congruent either are co-cyclic (in case the sections are non-trivial) or collinear (in case the two sections are trivial).*

Proof. It suffices to prove both statements for one permutation of A, B, C, D. Assume that A, B, C, D are co-cyclic or collinear. Let B', C' and D' denote the images of B, C and D, respectively, under circular inversion with respect to A. Then, $\{DA, DC\} \cong_I \{C'A, D'C'\}$ and $\{BA, BC\} \cong_I \{C'A, B'C'\}$. Since by

Lemma 7, $B'C'$ coincides with $D'C'$, it follows that $\{DA, DC\} \cong_D \{BA, BC\}$. Conversely, assume that $\{DA, DC\} \cong_D \{BA, BC\}$. Then, $\{C'A, D'C'\} \cong_D \{C'A, B'C'\}$, *i.e.*, B', C' and D' are collinear. By Lemma 7 again, if the two 2-sections are non-trivial, A, B, C, D are co-cyclic. If not, the four points are collinear. $\square$

Corollary 9. *Let A, B, C, D be any four co-cyclic points with $E = AC \cap BD$. Then the product of directed lengths $\overline{AE} \cdot \overline{CE}$ equals the product $\overline{BE} \cdot \overline{DE}$.*

Proof. Let A' and B' be the images under inversion of A and B with respect to E (and a circle of radius r). Then $\triangle A'B'E$ and $\triangle CDE$ are directly similar with two legs in common. Therefore $\overline{CE}/\overline{A'E} = \overline{DE}/\overline{B'E}$ or $\overline{CE} \cdot \overline{AE}/r^2 = \overline{DE} \cdot \overline{BE}/r^2$. $\square$

For the sake of completeness, although we will not use it in this paper, we end with a sometimes quite useful reformulation of Propositions 20 and 32 from Book III of Euclid's *Elements*.

Lemma 10 (Bow, String and Arrow)**.** *For any triangle $\triangle ABC$, let C' be the midpoint of AB and let O be the circumcenter of the triangle and let $T_{AB,C}$ denote the tangent line to the circumcircle of $\triangle ABC$ at C. Then $C\{B, A\}$ is directly congruent to (i) both $O\{C', A\}$ and $O\{B, C'\}$ and (ii) $\{BA, T_{CB,A}\}$ and $\{T_{CA,B}, AB\}$.*

Proof. It suffices to prove the first statements of (i) and (ii). Let A' be the midpoint of BC. Since OC' is perpendicular to AB and OA' is perpendicular to BC, it follows that $\square C'OA'B$ is cyclic and therefore that $A'\{B, C'\} \cong_D O\{B, C'\}$. But $A'C'$ is parallel to CA and therefore $A'\{B, C'\} \cong_D C\{B, A\}$ as well, which proves the first statement of (i). As for (ii), since BA is perpendicular to OC' and $T_{CB,A}$ is perpendicular to OA, it follows that $\{BA, T_{CB,A}\}$ is directly congruent to $O\{C', A\}$ by BP 4 (Rotation). Since $O\{C', A\}$ is directly congruent to $C\{B, A\}$, the first statement of (ii) follows. $\square$

The preceding results provide a workable framework for the application of n-sections to a great many problems in plane geometry involving configurations of circles and lines (as opposed to rays). The well-known group of circle theorems usually attributed to Steiner and Miquel as well as most theorems associated with the Wallace line are particularly amenable to the use of n-sections. Examples can be found in [16], [17], and [5].

4. 6-sections and complete quadrangles

So far we have essentially only used 2-sections and 3-sections. Note how any 3-section (with distinct directions) always corresponds to a unique class of directly similar triangles. Clearly, there is no such correspondence for 4-sections. To determine a quadrilateral, we need the direction of at least one of its diagonals as well. Therefore, it makes sense to consider the 6-sections and their connection to the so-called *complete quadrangles* $\boxtimes ABCD$, *i.e.*, all configurations of four points (with no three collinear) and the six lines passing through each two of them. Clearly any $\boxtimes ABCD$ defines a 6-section. Conversely, not every 6-section can be represented by the six sides of a complete quadrangle. In order to see under what condition a 6-section originates from a complete quadrangle, we need a little bit of projective geometry.

Any two n-sections are said to be *in perspective* or to form a *perspectivity* if for a representation of each of the sections by concurrent lines the points of intersection of the corresponding lines are collinear. Two sections are said to be *projective* if a representation by concurrent lines of the one section can be obtained from a similar representation of the other as a sequence of perspectivities. It can be shown that any two sections that are congruent are also projective. In the case of 2-sections and 3-sections all are actually projective. As for 4-sections, the projectivity of two sections is determined by their so-called *cross ratio*. Every 4-section $\{\ell_1, ..., \ell_4\}$ has an associated cross ratio $[\ell_1, ..., \ell_4]$. If $\underline{A}$ denotes the pencil of lines passing through A, represent the lines of any section by lines $\ell_i \in \underline{A}$. If ℓ_i has an equation $\mathcal{L}_i = 0$, we can write $\mathcal{L}_3$ as $\lambda_{31}\mathcal{L}_1 + \lambda_{32}\mathcal{L}_2$ and $\mathcal{L}_4$ as $\lambda_{41}\mathcal{L}_1 + \lambda_{42}\mathcal{L}_2$. We now (unambiguously) define the cross ratio $[\ell_1, \ldots, \ell_4]$ as the quotient $(\lambda_{31}/\lambda_{32}) : (\lambda_{41}/\lambda_{42})$. From this definition of a cross ratio it follows that its value does not change when the first pair of elements and the second pair are switched or when the elements within each pair are swapped. Note that for any two 3-sections $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ (with $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ each formed by three distinct directions), the cross ratio defines a bijective map φ between any two pencils $\underline{A}$ and $\underline{B}$, by choosing the l_i in $\underline{A}$ and the m_i in $\underline{B}$ and defining the image $\varphi(l)$ of any line $l \in \underline{A}$ as the line of $\underline{B}$ such that $[l_1, l_2; l_3, l]$ equals $[m_1, m_2; m_3, \varphi(l)]$. The map φ is called a projective map (of the pencil). It can be shown that any projective map can be obtained as a projectivity and vice versa. Therefore, two 4-sections are projective if and only if their corresponding cross ratios are equal. By the duality of projective geometry, all of the preceding applies to the points of a line instead of the lines of a pencil as well. Moreover, for any four points L_1, L_2, L_3, L_4 on a line ℓ and a point L_0 outside ℓ, the cross ratio $[L_1, L_2; L_3, L_4]$ is equal to $[L_0L_1, L_0L_2; L_0L_3, L_0L_4]$. By the latter property, we can associate any projective map defined by two sections of lines with a projective map from the line at infinity to itself.

The notion of a projective map can be extended to the projective plan where any such map φ maps any line to a straight line and the restriction of φ to a line and its image line is a projective map. Where a projective map between two lines is defined by two triples of (non-coinciding) points, a projective map between two planes requires two sets of four points, no three of which can be collinear. In other words, any two quadrilaterals define a projective map. Finally, we define an *involution* as a projective map which is its own inverse. In the case of an involution of a line or pencil, any two distinct pairs of elements (with the elements within each pair possibly coinciding) fully determine the map.

We can now formulate the following result.

Theorem 11. *An arbitrary* 6*-section* $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ *can be formed from the sides of a complete quadrangle* $\boxtimes ABCD$ *(such that* l_1*,* l_2 *and so on are pairs* *of opposite sides) if and only if the three pairs of opposite sides can be rearranged such that* $\{l_1, l_2\}$ *is non-trivial and* $[l_1, l_2; m_1, n_2]$ *equals* $[l_1, l_2; n_1, m_2]$.

Proof. Since any two quadrilaterals determine a projective map, every complete quadrangle is projective to the configuration of a rectangle and its diagonals. Therefore the diagonal points of a complete quadrangle are never collinear and every quadrangle in the affine plane has at least one pair of opposite sides which are not parallel. Without loss of generality, we may assume that $\{l_1, l_2\}$ corresponds to this pair of opposite sides. Let $\underline{A}$, $\underline{B}$ denote the pencils of lines through A and B respectively. Now define a map φ from $\underline{A}$ to $\underline{B}$ by assigning the line AX to BX for all X on a line ℓ not passing through A or B. It is easily verified that φ is a projectivity, which assigns AB to itself and the line of $\underline{A}$ parallel to L to the corresponding parallel line of $\underline{B}$. Therefore, if C and D are distinct points on ℓ, the cross ratio $[AB, CD; AC, AD]$ equals the cross ratio $[AB, CD, BC, BD]$. Conversely, for any 6-section $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ such that $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, we can choose A and B such that AB is parallel to l_1 and let D be the point of intersection of the line of $\underline{A}$ parallel to m_1 and the line of $\underline{B}$ parallel to n_1. Likewise, let C be the point of intersection of the line of $\underline{A}$ parallel to m_2 and the line through D parallel to l_2. Then, since $\{l_1, l_2\}$ is non-trivial, the line BC has to be parallel to n_2. $\square$

Note that the previous theorem is a projective version of Ceva's Theorem determining the concurrency of transversals in a triangle and the usual expression of that theorem can be readily derived from the condition above. We now have the following corollary.

Corollary 12. *For any complete quadrangle* $\boxtimes ABCD$*, there is an involution that pairs the points of intersection of its opposite sides with the line at infinity.*

Proof. Without loss of generality, we may assume that $\{AB, CD\}$ is non-trivial. Let $L_1 = AB \cap \ell_\infty$ and so on. Then $[L_1, L_2, M_1, N_2] = [L_1, L_2, N_1, M_2]$. Now let φ be the involution of ℓ_∞ determined by pairing L_1 with L_2 and M_1 with M_2. Then $[L_1, L_2, M_1, N_2]$ equals $[L_2, L_1, M_2, \varphi(N_2)]$. Since the former expression is also equal to $[L_2, L_1, M_2, N_1]$ (and L_1, L_2 and M_2 are distinct), it follows that $\varphi(N_2) = N_1$. In other words, the involution pairs N_1 and N_2 as well. $\square$

In case $\boxtimes ABCD$ is a trapezoid, the point on the line at infinity corresponding to the parallel sides is a fixed point of the involution; in case the complete quadrangle is a parallelogram, the two points corresponding to the two pairs of parallel sides both are fixed points.

In the language of classical projective geometry, we say that a 6-section formed by the sides of any complete quadrangle defines an involution of six lines pairing the opposite sides of the quadrangle. Note that this statement implies what is known as Desargues' Theorem, which states that any complete quadrangle defines an involution (of points) on any line not passing through any of its vertices that pairs the points of intersection of that line with the opposite sides of the quadrangle. For this reason, we will say that any 6-section satisfying the condition of Theorem 11 is *Desarguesian*.

Corollary 13. *Any Desarguesian* 6*-section is associated with two similarity classes of quadrilaterals (which may coincide).*

Proof. Let the 6-section be denoted by $\{l_1, l_2, m_1, m_2, n_1, n_2\}$. If the cross ratio $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, then the cross ratio $[l_2, l_1; m_2, n_1]$ also equals $[l_2, l_1; n_2, m_1]$. Whereas the quadrilateral constructed from the first equality contains a triangle formed by the lines l_2, m_2, n_2, while l_1, m_1, n_1 meet in one point, this is reversed for the quadrilateral formed from the second equality. Since $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ are not necessarily congruent, the two quadrilaterals will be different (but may coincide in some cases). □

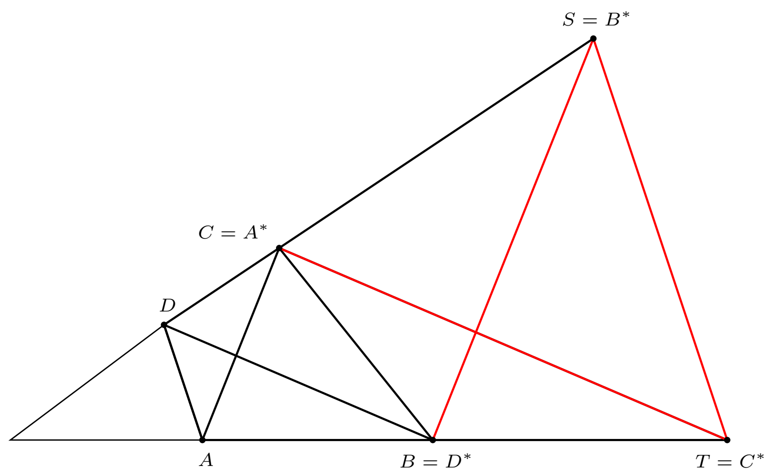

Figure 2. Constructing $\boxtimes A^*B^*C^*D^*$

If the complete quadrangle $\boxtimes ABCD$ is one of the two quadrangles forming a given 6-section, we can easily construct the other quadrangle $\boxtimes A^*B^*C^*D^*$. Indeed, let $\boxtimes ABCD$ be as in Figure 2. Then, draw the line through B parallel to AC, meeting CD in S. Likewise, draw the line through C parallel to BD meeting AB in T. Then, by construction ST is a parallel to AD and all the opposite sides of $\boxtimes ABCD$ are parallel to a pair of opposite sides of $\boxtimes BCST$. The two quadrangles, however, are generally not similar. Alternatively, we can consider the quadrangle formed by the circumcenters of the four circles circumscribing the four triangle formed by A, B, C, D. For this quadrangle, all three pairs of opposite sides are parallel to a pair of opposite sides of the original quadrangle. Again, it is easy to see that this quadrilateral is generally not similar to the original one. The latter construction was first systematically studied by Maxwell in [10] and [11], in which he referred to the quadrilateral of circle centers as a reciprocal figure. For this reason, we will refer to the two complete quadrangles associated with a Desarguesian 6-section as *reciprocal* quadrangles.

Relabeling the vertices of the preceding quadrangles as indicated in Figure 2, we will formally define two complete quadrilaterals $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ as directly/inversely reciprocal if and only if

$$\{AB, CD, AC, BD, DA, BC\} \cong \{C^*D^*, A^*B^*, B^*D^*, A^*C^*, B^*C^*, D^*A^*\},$$

where the congruence is either direct or inverse. From this definition, we immediately derive the following two corollaries.

Corollary 14. *A complete quadrangle is directly reciprocal to itself if and only if it is orthocentric.*

Proof. Since for any complete quadrangle directly reciprocal to itself all three 2-sections of opposite sides have to be both directly and inversely congruent, it follows that all opposite sides are perpendicular to each other. In other words, every vertex is the orthocenter of the triangle formed by the other three vertices, which is what orthocentric means. $\square$

Corollary 15. *A complete quadrangle is inversely reciprocal to itself if and only if it is cyclic.*

Proof. Let $\boxtimes ABCD$ denote the complete quadrangle. Then, if $\boxtimes ABCD$ is inversely reciprocal to itself, $\{AB, AC\}$ has to be inversely congruent to $\{CD, BD\}$ or $A\{B, C\} \cong_D D\{B, C\}$. But this means that $\boxtimes ABCD$ is cyclic. The converse readily follows. $\square$

Because of the preceding corollaries, when studying the relations between reciprocal quadrangles, we can often just assume that a complete quadrangle is neither orthocentric nor cyclic. Also, as a special case, note that if a complete quadrangle $\boxtimes ABCD$ has a pair of parallel opposite sides, then its reciprocal is directly congruent to $\boxtimes BADC$. For this reason, it is usually fine to assume that $\boxtimes ABCD$ does not have any parallel sides either.

Maxwell's application of his reciprocal figures to the study of statics contributed to the development of a heavily geometrical approach to that field (know as *graphostatics*) which ultimately made projective geometry a required course at many engineering schools until well into the 20th century. At the same time, the idea of "reciprocation" was largely ignored within the classical geometry community. This only changed in the 1890s, when (probably not entirely independently of Maxwell) Joseph Jean Baptiste Neuberg (1840-1826) reintroduced the concept of reciprocation under the name of *metapolarity*. This notion, however, seems to have been quickly eclipsed by the related notion of *orthology* that was introduced by Émile Michel Hyacinthe Lemoine (1840-1912) and others as a tool to study triangles. In this context, consider a triangle $\triangle ABC$ and a point P in the plane of the triangle. Now, construct a new triangle $\triangle A'B'C'$ such that each of its sides is perpendicular to the corresponding side of $\{CP, AP, BP\}$. In this new triangle, construct transversals each perpendicular to the corresponding line of $\triangle ABC$. Then, these three transversals will meet in a new point P'. The triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *orthologic* with poles P and P'. Clearly, for any two orthologic triangles $\triangle ABC$ and $\triangle A'B'C'$ with poles P and P', $\boxtimes ABCP$ and $\boxtimes A'B'C'P'$ are reciprocal quadrangles. Conversely, for any two reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, $\triangle ABC$ and $\triangle A^*B^*C^*$ are orthologic with poles D and D^* (up to a rotation), and similarly for the three other pairs of triangles contained in the two quadrangles. It is in the form of some variation of orthology that the notion of reciprocation is best known today.[6]

A nice illustration of the use of reciprocal quadrangles (or orthology, in this case) is the following problem from a recent International Math Olympiad Training Camp.[7]

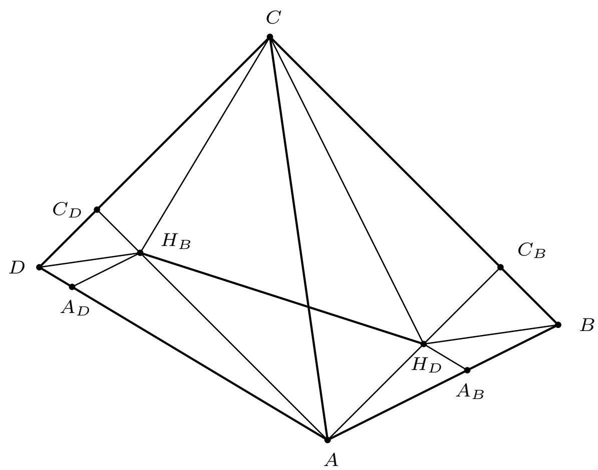

Figure 3. $\boxtimes ABCD$ with H_B and H_D

Problem (IMOTC 2005)**.** Let $ABCD$ be a quadrilateral, and H_D the orthocenter of triangle $\triangle ABC$. The parallels to the lines AD and CD through the point H_D meet the lines AB and BC at the points C_B and A_B, respectively. Prove that the perpendicular to the line C_BA_B through the point H_D passes through the orthocenter H_B of triangle $\triangle ACD$.

Solution. The proposition still has to be true if we switch the role of B and D. Now note that the complete quadrangles $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$ have five parallel corresponding sides. Therefore, they are similar. Moreover, five of the sides of the complete quadrangle $\boxtimes H_BCH_DA$ are perpendicular to the opposite of the corresponding sides of $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DB$. We conclude that $\boxtimes H_BCH_DA$ is directly reciprocal to $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$. Consequently, its sixth side H_BH_D is perpendicular to A_DC_D and A_BC_B.

5. Some relations between reciprocal quadrangles

In order to study the relations between reciprocal quadrangles, we note yet another way to generate a reciprocal to a given complete quadrangle. In fact,

[6]On metapolar quadrangles, see e.g. [12] and [13] or (more accessibly) Neuberg's notes to [18] (p.458). On orthology, see [9]. In 1827, well before Lemoine (and Maxwell), Steiner had also outlined the idea of orthology (see [19], p.287, Problem 54), but nobody seems to have picked up on the idea at the time. Around 1900, the Spanish mathematician Juan Jacobo Durán Loriga (1854-1911) extended the notion of orthology to that of *isogonology*, which concept was completely equivalent to reciprocation. Durán-Loriga's work, however, met with the same fate as Neuberg's metapolarity.

[7]See [2] and the references there.

let $\boxtimes ABCD$ be a complete quadrangle with diagonal points $E = AC \cap BD$, $F = BC \cap DA$, $G = AB \cap CD$, with A, B, C, D, and, F in the affine plane. Now let A^* be the image of D under circular inversion with respect to F (see Figure 4). Likewise let D^* be the image of A under the same inversion. Similarly B^* is the image of C and C^* is the image of B. Then, using the properties of inversion it is easily verified that $\boxtimes A^*B^*C^*D^*$ is inversely reciprocal to $\boxtimes ABCD$. We can use this construction to derive the following two lemmata.

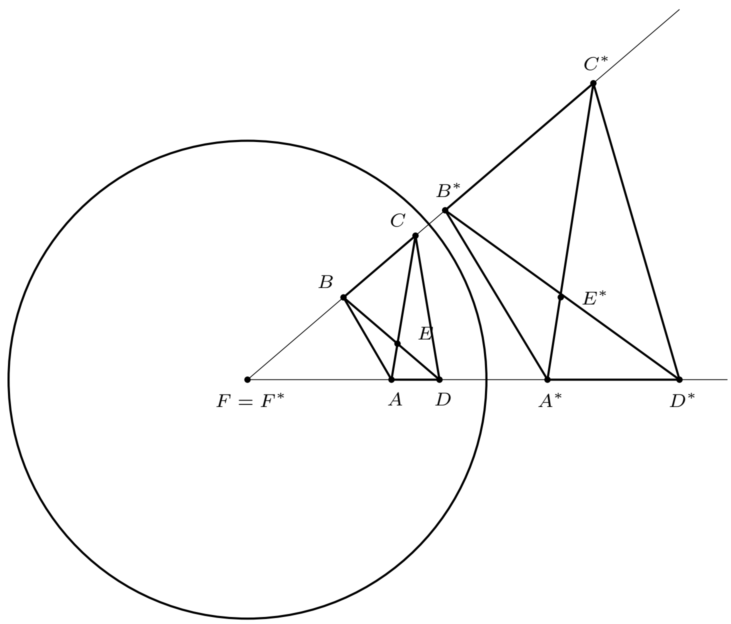

Figure 4. Constructing $\boxtimes A^*B^*C^*D^*$ by Inversion

Lemma 16 (Invariance of Ratios)**.** *Let $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ be a pair of (affine) reciprocal quadrangles and diagonal points E, F, G and E^*, F^*, G^*, respectively. Moreover, let X, Y, and Z be any collinear triple of two vertices and a diagonal point of $\boxtimes ABCD$ with X^*, Y^*, Z^* the corresponding triple of $\boxtimes A^*B^*C^*D^*$. Then*

$$\frac{\overline{XY}}{\overline{YZ}} = \frac{\overline{X^*Y^*}}{\overline{Y^*Z^*}},$$

where $\overline{XY}$ denotes the directed length of the line segment XY and so on.

Proof. The statement is trivial for any diagonal point on ℓ_∞. Without loss of generality, let us assume that the diagonal point F is in the affine plane. It now suffices to prove the statement for B, C and F. Under inversion with respect to F and a circle of radius r, we find that $\overline{B^*F^*} = r^2/\overline{CF}$ and $\overline{C^*F^*} = r^2/\overline{BF}$. The statement of the lemma now immediately follows. $\square$

Lemma 17 (Maxwell)**.** *Let* $\boxtimes ABCD$ *and* $\boxtimes A^*B^*C^*D^*$ *be a pair of reciprocal quadrangles. Then*

$$\frac{|AB||CD|}{|A^*B^*||C^*D^*|} = \frac{|AC||BD|}{|A^*C^*||B^*D^*|} = \frac{|AD||CB|}{|A^*D^*||C^*B^*|},$$

where $|AB|$ *denotes the absolute length of the segment* AB *and so on.*

Proof. Assume again that the point F is in the affine plane. Under inversion with respect to F and a circle of radius r, we find $|A^*D^*| = |r^2/|FD| - r^2/|FA|| = r^2|AD|/(|FA||FD|)$ and $|B^*C^*| = |r^2/|FC| - r^2/|FB|| = r^2|BC|/(|FB||FD|)$. Similarly, $|A^*B^*| = r^2|AB|/(|FA||FB|)$ and $|C^*D^*| = r^2|CD|/(|FC||FD|)$, while $|A^*C^*| = r^2|AC|/(|FA||FC|)$ and $|B^*D^*| = r^2|BD|/(|FB||FD|)$. Combining these expressions shows the equality of the three expressions. □

Note that for any three collinear points, the ratio $|AC|/|BC|$ equals the cross ratio $[A, B, C, I_{AB}]$, where I_{AB} denotes the point at infinity of the line AB. Now, for any pair of reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, let φ be the unique projective map sending A to A^* and so on. Then, φ maps the line AB to the line A^*B^* and $[A, B, G, I_{AB}] = [A^*, B^*, G^*, \varphi(I_{AB})]$ (where $G = AB \cap CD$). By Lemma 16, $[A, B, G, I_{AB}]$ also equals $[A^*, B^*, G^*, I_{A^*B^*}]$. Therefore, since G is distinct from A and B, φ maps I_{AB} to $I_{A^*B^*}$. Likewise, the points at infinity of BC and CA are mapped to the points at infinity of B^*C^* and C^*A^*, respectively. But then, φ must map the whole line at infinity onto itself. Therefore, any map defined by "reciprocation" of a complete quadrangle is an *affine* map. Conversely, any affine map can be modeled by a reciprocation of a complete quadrangle (which we may assume not to have any parallel sides). To see this, we first need another lemma.

Lemma 18. *For a given triangle* $\triangle ABC$ *and any non-trivial* 3*-section* $\{l, m, n\}$ *not inversely congruent to* $\{BC, CA, AB\}$ *there is exactly one point* D *in the plane of* $\triangle ABC$ *(and not on the sides of* $\triangle ABC$*) such that* $\{AD, BD, CD\}$ *is directly congruent to* $\{l, m, n\}$*. In case* $\{BC, CA, AB\} \cong_I \{l, m, n\}$*,* $\{AD, BD, CD\}$ *will be directly congruent to* $\{l, m, n\}$ *for any point* D *on the circumcircle of* $\triangle ABC$*.*

Proof. Without loss of generality, we may assume that l, m, and n are concurrent at a point Q. Let a point L be a fixed point on l and let M be a variable point on m. Now construct a triangle $\triangle LMN$ directly similar to $\triangle ABC$. Then, the locus of N as M moves along m is a straight line as N is obtained from M by a fixed dilation followed by a rotation over a fixed angle. Therefore, this locus will intersect n in exactly one point as long as $\{AC, AB\}$ is not directly congruent to $\{n, m\}$. The point D we are looking for now has the same position with respect to $\triangle ABC$ as has Q with respect to $\triangle LMN$. If the two 2-sections are directly congruent, we can repeat the process starting with M or N. This means that we cannot find a point D as stated in the lemma using the procedure above only if $\triangle ABC$ is inversely congruent to $\{l, m, n\}$. But if the latter is the case, we can take any point D on the circumcircle of $\triangle ABC$ by Cor. 15. □

As an aside, note that for $\{l, m, n\}$ directly congruent to either $\{AB, BC, CA\}$ or $\{CA, AB, BC\}$, this construction also guarantees the existence of the two so-called Brocard points Ω^+ and Ω^- of $\triangle ABC$. Moreover, it is easily checked that $\boxtimes ABC\Omega^+$ and $\boxtimes BCA\Omega^-$ are reciprocal quadrangles. This explains the congruence of the two Brocard angles. We are now ready to prove the following theorem.

Theorem 19. *A projective map of the plane is affine if and only if it can be obtained by reciprocation of a complete quadrangle $\boxtimes ABCD$ with no parallel sides. Any such map reverses orientation if $\boxtimes ABCD$ is convex and retains orientation when not. The map is Euclidean if and only if $\boxtimes ABCD$ is orthocentric (in which case the map retains orientation) or cyclic (in which case the map reverses orientation).*

Proof. We already proved the if-part above. For an affine map, consider a triangle $\triangle ABC$ and its image $\triangle A^*B^*C^*$. By the previous lemma there is at least one point D (not on the sides of $\triangle ABC$) such that $\{AD, BD, CD\}$ is directly congruent to $\{B^*C^*, C^*A^*, A^*B^*\}$. The reciprocation of $\boxtimes ABCD$ that A maps to $\triangle ABC$ maps to $\triangle A^*B^*C^*$, then, must be the affine map. The connection between convexity of $\boxtimes ABCD$ follows from the various constructions (and relabeling) of a reciprocal quadrangle. The last statement follows immediately. In case $\boxtimes ABCD$ has parallel opposite sides, note that the affine map (after a rotation aligning one pair of parallel sides with their images) induces a map on the line at infinity with either one or two fixed points (if not just a translation combined with a dilation), corresponding to a glide or a dilation in two different directions. This means that if we choose the sides of $\triangle ABC$ such that they are not parallel to the directions represented by the fixed points on the line at infinity, no opposite sides of $\boxtimes ABCD$ will be parallel. $\square$

Finally, note that if a complete quadrangle $\boxtimes ABCD$ is cyclic, then its reciprocal $\boxtimes A^*B^*C^*D^*$ is as well. Likewise, by Lemma 17, if for a complete quadrangle the product of the lengths of a pair of opposite sides equals that of the lengths of another pair, the same is true for the corresponding pairs of its reciprocal. More surprisingly perhaps, reciprocation also retains inscribability, *i.e.*, if $\square ABCD$ has an incircle, then so has $\square A^*B^*C^*D^*$. To see this, we can use the following generalization of a standard result.

Lemma 20 (Generalized Ptolemy)**.** *For any six points A, B, C, D, P, and Q in the (affine) plane*

$$\begin{aligned} &|\triangle PAB||\triangle QCD| + |\triangle PCD||\triangle QAB| \\ &+ |\triangle PAD||\triangle QBC| + |\triangle PBC||\triangle QAD| \\ &= |\triangle PAC||\triangle QBD| + |\triangle PBD||\triangle QAC|. \end{aligned}$$

Proof. We represent the points A, B, C, D, P, and Q by vectors $\vec{a} = (a_1, a_2, 1)$ and so on. Now consider the vectors $(\vec{a} \oplus \vec{a})^T$, ..., $(\vec{d} \oplus \vec{d})^T$, as well as the vectors $(\vec{p} \oplus i\vec{p})^T$ and $(i\vec{q} \oplus \vec{q})^T$. Then clearly, the 6×6-determinant formed by these six vectors equals zero. If we now evaluate this determinant as the sum of the signed product of every 3×3-determinant contained in the three first rows and its complementary 3×3-determinant in the three bottom rows, we obtain exactly the identity of the lemma. $\square$

Note that the imaginary numbers are necessary to ensure that no two of the products automatically cancel against each other. Also, note that this result really is about octahedrons in 3-space and can immediately be extended to their analogs in any dimension. Ptolemy's Theorem follows by letting P and Q coincide and assuming this point is on the circumcircle of $\boxtimes ABCD$.

Corollary 21. *For any complete quadrangle $\boxtimes ABCD$ and $E = AC \cap BD$ and a point P both in the (affine) plane of the quadrangle,*

$$\begin{aligned} &|\triangle PDA| \cdot |\triangle EBC| + |\triangle PBC| \cdot |\triangle EDA| \\ &= |\triangle PCD| \cdot |\triangle EAB| + |\triangle PAB| \cdot |\triangle ECD|, \end{aligned}$$

where E is the point of intersection of AC with BD.

Proof. Let Q coincide with E. $\square$

Now, let $\boxtimes ABCD$ be convex. Then $E = AC \cap BD$ is in the affine plane and we can obtain $\boxtimes A^*B^*C^*D^*$ by circular inversion with respect to E. Also, $\square A^*B^*C^*D^*$ is convex by Theorem 19. Therefore, the equality of $|A^*B^*| + |C^*D^*|$ and $|D^*A^*| + |B^*C^*|$ is both necessary and sufficient for the quadrangle to be inscribable. By the properties of inversion, this condition is equivalent to the condition

$$\frac{|AB|}{|EA||EB|} + \frac{|CD|}{|EC||ED|} = \frac{|DA|}{|ED||EA|} + \frac{|BC|}{|EB||EC|},$$

or

$$DA \cdot |\triangle EBC| + BC \cdot |\triangle EDA| = CD \cdot |\triangle EAB| + BA \cdot |\triangle ECD|.$$

If $\square ABCD$ is inscribable, this condition can also be written in the form

$$|\triangle IDA| \cdot |\triangle EBC| + |\triangle IBC| \cdot |\triangle EDA| = |\triangle ICD| \cdot |\triangle EAB| + |\triangle IAB| \cdot |\triangle ECD|.$$

But this equality is true by Cor. 21. We conclude that if $\square ABCD$ is inscribable, then so is $\square A^*B^*C^*D^*$.

Alternatively, we can use a curious result that received some on-line attention in recent years, but which is probably considerably older.

Theorem 22. *For any convex quadrilateral $\square ABCD$ with $E = AC \cap BD$, let I_{AB} be the incenter of $\triangle EAB$ and so on. Then $\boxtimes I_{AB}I_{BC}I_{CD}I_{DA}$ is cyclic if and only if $\square ABCD$ is inscribable.*

Proof. See [1] and the references there. The convexity requirement might not be necessary. $\square$

Let us assume again that $\square ABCD$ is inscribable. This means that the quadrangle is convex and that $E = AC \cap BD$ is in the affine plane. Also, note that $E = I_A I_C \cap I_B I_D$. Therefore, $\overline{EI_{AB}} \cdot \overline{EI_{CD}}$ equals $\overline{EI_{BC}} \cdot \overline{EI_{DA}}$ by Theorem 22. Now, let $\boxtimes A^*B^*C^*D^*$ be a reciprocal of $\boxtimes ABCD$ obtained by circular inversion with respect to E and a circle with radius r. As we assumed that $\boxtimes ABCD$ is convex, so is $\boxtimes A^*B^*C^*D^*$ by Theorem 19. Since $\triangle EA^*B^*$ is inversely similar to $\triangle ECD$ while $|A^*B^*| = r^2|BD|/(|EB||ED|)$, it follows that $\overline{EI_{A^*B^*}} = r^2\overline{EI_{CD}}/(|EC||ED|)$ and so on. Consequently, $\overline{EI_{A^*B^*}} \cdot \overline{EI_{C^*D^*}} = \overline{EI_{B^*C^*}} \cdot \overline{EI_{D^*A^*}}$. Therefore $\boxtimes I_{A^*B^*}I_{B^*C^*}I_{C^*D^*}I_{D^*A^*}$ is cyclic and $\square A^*B^*C^*D^*$ is inscribable by Theorem 22 again.

As a third proof, it is relatively straightforward to actually construct a reciprocal $\boxtimes A^*B^*C^*D^*$ with its sides tangent to the incircle of $\square ABCD$. More generally, this approach proves that the existence of any tangent circle to a quadrangle implies the existence of one for its reciprocal. This construction can actually be looked upon as a special case of yet another way to construct reciprocal quadrangles. The proof of the validity of this more general construction, however, seems to require a property of reciprocal quadrangles that we have not touched upon in this paper. We plan to discuss this property (and the specific construction of reciprocal quadrangles that follows from it) in a future paper.

6. Conclusions

In this paper we outlined how in many cases the concept of an angle can be replaced by the more rigorous notion of an n-section. Other than the increased rigor, one advantage of n-sections over angles is that reasoning with the former is somewhat more similar to the kind of reasoning one might see in other parts of mathematics, particularly in algebra. Although perhaps a little bit of an overstatement, Picken did have a point when he claimed that his paper did not have diagrams because they were "quite unnecessary."[8] Also, the formalism of n-sections provides a natural framework in which to study geometrical problems involving multiple lines and their respective inclinations. As such, it both provides a clearer description of known procedures and is bound to lead to questions that the use of the notion of angles would not naturally give rise to. As a case in point, we showed how the notion of n-section suggests both a natural description of the procedure involving orthologic triangles in the form of the notion of reciprocal quadrangles and give rise to the question what properties of a complete quadrangle are retained under the "reciprocation" of quadrangles.

At the same time, the fact that the "reciprocation" of quadrangles does not favor any of the vertices of the figures involved comes at a cost. Indeed, its use does not naturally give rise to certain types of questions that the use of orthologic triangles does lead to. For instance, it is hard to see how an exclusive emphasis on the notion of reciprocal quadrangles could ever lead to the study of antipedal triangles and similar constructions. In short, the notion of reciprocal quadrangles should be seen as a general notion underlying the use of orthologic triangles and not as a replacement of the latter.

[8] See [16], p.188.

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Eisso J. Atzema: Department of Mathematics, University of Maine, Orono, Maine 04469, USA
E-mail address: `atzema@math.umaine.edu`

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The Lost Daughters of Gergonne

Steve Butler

Abstract. Given a triangle center we can draw line segments from each vertex through the triangle center to the opposite side, this splits the triangle into six smaller triangles called daughters. Consider the following problem: Given a triangle S and a rule for finding a center find a triangle T, if possible, so that one of the daughters of T, when using the rule is S. We look at this problem for the incenter, median and Gergonne point.

1. Introduction

Joseph-Diaz Gergonne (1771–1859) was a famous French geometer who founded the *Annales de Gergonne*, the first purely mathematical journal. He served for a time in the army, was the chair of astronomy at the University of Montpellier, and to the best of our knowledge never misplaced a single daughter [3].

The "daughters" that we will be looking at come from triangle subdivision. Namely, for any well defined triangle center in the interior of the triangle one can draw line segments (or Cevians) connecting each vertex through the triangle center to the opposite edge. These line segments then subdivide the original triangle into six daughter triangles.

Given a triangle and a point it is easy to find the daughter triangles. We are interested in going the opposite direction.

Problem. Given a triangle S and a well defined rule for finding a triangle center; construct, if possible, a triangle T so that S is a daughter triangle of T for the given triangle center.

For instance suppose that we use the *incenter* as our triangle center (which can be found by taking the intersection of the angle bisectors). Then if we represent the angles of the triangle T by the triple (A, B, C) it easy to see that one daughter will have angles $\left(\frac{A}{2}, \frac{A}{2} + \frac{B}{2}, \frac{B}{2} + C\right)$, all the other daughter triangles are found by permuting A, B and C. Since this is a linear transformation this can be easily inverted. So if S has angles a, b and c then the possible candidates for T are $(2a, 2b-2a, c-b+a)$, along with any permutation of a, b and c. It is easy to show that if the triangle S is not equilateral or an isoceles triangle with largest angle $\geq 90^\circ$ then there is at least one non-degenerate T for S.

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We could also use the *centroid* which for a triangle in the complex plane with vertices at 0, Z and W is $\frac{Z+W}{3}$. In particular if $[0, Z, W]$ is the location of the vertices of T then $\left[0, \frac{Z}{2}, \frac{Z+W}{3}\right]$ is the location of the vertices of a daughter of T. This map is easily inverted, let S be $[0, z, w]$ then we can choose T to be $[0, 2z, 3w - 2z]$. So, for example, if S is an equilateral triangle then we should choose T to be a triangle similar to one with side lengths 2, $\sqrt{7}$ and $\sqrt{13}$.

In this note we will be focusing on the case when our triangle center is the *Gergonne point*, which is found by the intersection of the line segments connecting the vertices of the triangle to the point of tangency of the incircle on the opposite edges (see Figures 2-5 for examples).

Unlike centroids where every triangle is a possible daughter, or incenters where all but $(60°, 60°, 60°)$, $(45°, 45°, 90°)$ and obtuse isoceles are daughters, there are many triangles which cannot be a Gergonne daughter. We call such triangles *the lost daughters of Gergonne*.

To see this pictorially if we again represent triangles as triples (A, B, C) of the angles, then each "oriented" triangle (up to similarity) is represented by a point in P, where P is the intersection of the plane $A + B + C = 180°$ with the positive orthant (see [1, 2, 5, 6] for previous applications of P). Note that P is an equilateral triangle where the points on the edges are degenerate triangles with an angle of $0°$ and the vertices are $(180°, 0°, 0°)$, $(0°, 180°, 0°)$, $(0°, 0°, 180°)$; the center of the triangle is $(60°, 60°, 60°)$. In Figure 1 we have plotted the location of the possible Gergonne daughters in P, the large white regions are the lost daughters.

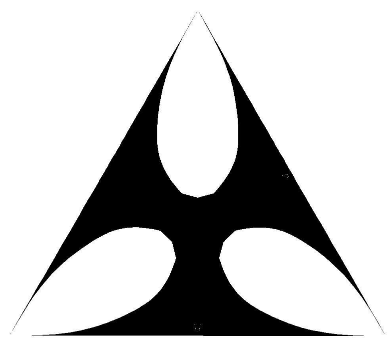

Figure 1. The possible Gergonne daughters in P.

2. Constructing T

We start by putting the triangle S into a standard position by putting one vertex at $(-1, 0)$ (with associated angle α), another vertex at $(0, 0)$ (with associated angle β) and the final vertex in the upper half plane. We now want to find (if possible) a triangle T which produces this Gergonne daughter in such a way that $(-1, 0)$ is a vertex and $(0, 0)$ is on an edge of T (see Figure 2). Since $(0, 0)$ will correspond to a point of tangency of the incircle we see that the incircle must be centered at $(0, t)$ with radius t for some positive t. Our method will be to solve for t in terms of α

and β. We will see that some values of α and β have no valid t, while others can have one or two.

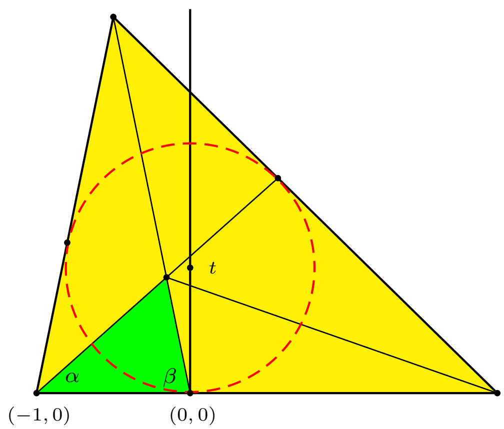

Figure 2. A triangle in standard position.

Since the point of tangency of the incircle to the edge opposite $(-1,0)$ must occur in the first quadrant, we immediately have that the angle α is acute and we will implicitly assume that in our calculations.

2.1. *The case* $\beta = 90°$. We begin by considering the special case $\beta = 90°$. In this setting it is easy to see that T must be an isosceles triangle of the form shown in Figure 3.

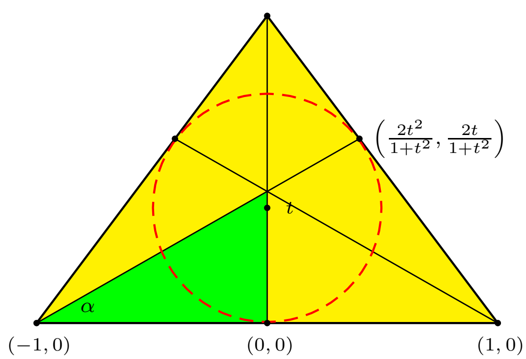

Figure 3. The $\beta = 90°$ case.

The important part of Figure 3 is the location of the point $\left(\frac{2t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. There are several ways to find this point. Ours will be to find the slope of the tangent line, then once this is found the point of tangency can easily be found. The key tool is the following lemma.

Lemma 1. *The slope* m *of the lines that pass through the point* (p,q) *and are tangent to the circle* $x^2 + (y-t)^2 = t^2$ *satisfy*

$$m^2 + \frac{2p(t-q)}{p^2-t^2}m + \frac{q^2-2qt}{p^2-t^2} = 0. \tag{1}$$

Proof. In order for the line $y = m(x - p) + q$ to be tangent to the circle $x^2 + (y - t)^2 = t^2$ the minimum distance between the line and $(0, t)$ must be t. Since the minimum distance between $(0, t)$ and the line $y = mx + (q - pm)$ is given by the formula $\frac{|t+pm-q|}{\sqrt{m^2+1}}$, we must have

$$t^2 = \left(\frac{|t + pm - q|}{\sqrt{m^2 + 1}}\right)^2.$$

Simplifying this relationship gives (1). $\square$

Applying this with $(p, q) = (1, 0)$ we have that the slopes must satisfy,

$$m^2 + \frac{2t}{1 - t^2}m = 0.$$

We already know the solution $m = 0$, so the slope of the tangent line is $\frac{-2t}{1-t^2}$. Some simple algebra now gives us the point of tangency. We also have that the top vertex is located at $\left(0, \frac{2t}{1-t^2}\right)$.

Using the newly found point we must have

$$\tan\alpha = \frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2} + 1} = \frac{2t}{1 + 3t^2},$$

which rearranges to

$$3(\tan\alpha)t^2 - 2t + \tan\alpha = 0, \text{ so that } t = \frac{1 \pm \sqrt{1 - 3\tan^2\alpha}}{3\tan\alpha}.$$

There are two restrictions. First, t must be real, and so we have $0 < \tan\alpha \le \frac{\sqrt{3}}{3}$, or $0 < \alpha \le 30°$. Second, $t < 1$ (if $t \ge 1$ then the triangle cannot close up), and so we need

$$\frac{1 + \sqrt{1 - 3\tan^2\alpha}}{3\tan\alpha} < 1 \text{ which reduces to } \tan\alpha > \frac{1}{2},$$

so for this root of t we need to have $\alpha > \arctan(1/2) \approx 26.565°$.

Theorem 2. *For $\beta = 90°$ and α given for a triangle S in standard position then*
(i) *if $\alpha > 30°$ there is no T which produces S;*
(ii) *if $\alpha = 30°$ then the T which produces S is an equilateral triangle;*
(iii) *if $\arctan\frac{1}{2} < \alpha < 30°$ then there are two triangles T which produce S, these correspond to the two roots $t = \frac{1\pm\sqrt{1-3\tan^2\alpha}}{3\tan\alpha}$;*
(iv) *if $\alpha \le \arctan\frac{1}{2}$ then there is one triangle T which produces S, this corresponds to the root $t = \frac{1-\sqrt{1-3\tan^2\alpha}}{3\tan\alpha}$.*

An example of the case when there can be two T is shown in Figure 4 for $\alpha = 29.85°$.

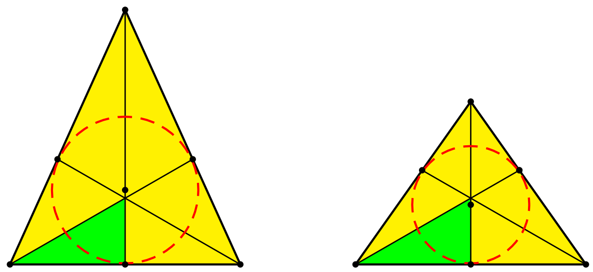

Figure 4. An example of a fixed S in standard position with two possible T.

2.2. *The case* $\beta \neq 90°$. Our approach is the same as in the previous case where we find the point of tangency opposite the vertex at $(-1, 0)$ and then use a slope condition to restrict t. The only difference now is that finding the point takes a few more steps.

To start we can apply Lemma 1 with $(p, q) = (-1, 0)$ and see that the slope of the line tangent to the circle is $\frac{2t}{1-t^2}$. The top vertex of T is then the intersection of the lines

$$y = \frac{2t}{1-t^2}(x+1) \text{ and } y = -(\tan\beta)x.$$

Solving for the point of intersection the top vertex is located at

$$(p^*, q^*) = \left(\frac{-2t}{2t + (1-t^2)\tan\beta}, \frac{2t\tan\beta}{2t + (1-t^2)\tan\beta}\right). \tag{2}$$

We can again apply Lemma 1 with (p^*, q^*) from (2), along with the fact that one of the two slopes is $\frac{2t}{1-t^2}$ to see that the slope of the edge opposite $(-1, 0)$ is

$$m^* = \frac{2\tan\beta(t\tan\beta - 2)}{t^2\tan^2\beta - 4t\tan\beta + 4 - \tan^2\beta}.$$

It now is a simple matter to check that the point of tangency is

$$(x^*, y^*) = \left(\frac{(m^*)^2 p^* + tm^* - q^* m^*}{(m^*)^2 + 1}, \frac{(m^*)^2 t - p^* m^* + q^*}{(m^*)^2 + 1}\right).$$

We can also find that the x-intercept of the line, which will correspond to the final vertex of the triangle, is located at $\big(t\tan\beta/(t\tan\beta - 2), 0\big)$.

So as before we must have

$$\begin{aligned}\tan\alpha &= \frac{y^*}{x^*+1} = \frac{(m^*)^2 t - p^* m^* + q^*}{(m^*)^2 p^* + tm^* - q^* m^* + (m^*)^2 + 1} \\ &= \frac{2t\tan^2\beta}{3t^2\tan^2\beta - 8t\tan\beta + \tan^2\beta + 4}.\end{aligned}$$

Which can be rearranged to give

$$(3\tan\alpha\tan^2\beta)t^2 - (2\tan^2\beta + 8\tan\alpha\tan\beta)t + (\tan\alpha\tan^2\beta + 4\tan\alpha) = 0.$$

Finally giving

$$t = \frac{\tan\beta + 4\tan\alpha \pm \sqrt{\tan^2\beta + 8\tan\alpha\tan\beta + 4\tan^2\alpha - 3\tan^2\alpha\tan^2\beta}}{3\tan\alpha\tan\beta}. \tag{3}$$

Theorem 3. *For $\beta \neq 90°$ and α given there are at most two triangles T which can produce S in standard position. These triangles T have vertices located at*

$$(-1,0), \left(\frac{-2t}{2t+(1-t^2)\tan\beta}, \frac{2t\tan\beta}{2t+(1-t^2)\tan\beta}\right), \textit{ and } \left(\frac{t\tan\beta}{t\tan\beta-2}, 0\right),$$

where t satisfies (3). *Further, we must have that t is positive and satisfies*

$$\frac{2}{\tan\beta} < t < \frac{1+\sec\beta}{\tan\beta}.$$

Proof. The only thing left to prove are the bounds. For the upper bound, we must have that the second vertex is in the top half plane and so we need

$$\frac{2t\tan\beta}{2t+(1-t^2)\tan\beta} > 0.$$

If $\tan\beta > 0$ then we need

$$2t + (1-t^2)\tan\beta > 0 \text{ or } (\tan\beta)t^2 - 2t - \tan\beta < 0.$$

This is an upward facing parabola with negative y-intercept and so we need that t is less than the largest root, i.e.,

$$t < \frac{2+\sqrt{4+4\tan^2\beta}}{2\tan\beta} = \frac{1+\sec\beta}{\tan\beta}.$$

The case for $\tan\beta < 0$ is handled similarly.

For the lower bound we must have that the x-coordinate of the third vertex is positive. If $\tan\beta < 0$ this is trivially satisfied. If $\tan\beta > 0$ then we need $t\tan\beta - 2 > 0$ giving the bound. □

As an example, if we let $\alpha = \beta = 45°$, then (3) gives $t = \frac{5\pm\sqrt{10}}{3} \approx 0.6125$, or 2.7207. But neither of these satisfy $2 < t < 1+\sqrt{2}$, so there is no T for this S. Combined with Theorem 2 this shows that $(45°, 45°, 90°)$ is a lost daughter of Gergonne.

On the other hand if we let $\alpha = \beta = 60°$ then (3) gives $t = \frac{\sqrt{3}}{3}, \frac{7\sqrt{3}}{9}$. The value $\frac{\sqrt{3}}{3}$ falls outside the range of allowable t, but the other one does fall in the range. The resulting triangle is shown in Figure 5 and has side lengths $\frac{19}{5}$, 8 and $\frac{49}{5}$.

3. Concluding comments

We now have a way given a triangle S to construct, if possible, a triangle T so that S is a Gergonne daughter of T. Using this it is possible to characterize triangles which are not Gergonne daughters. One can then look at what triangles are not Gergonne granddaughters (i.e., triangles which can be formed by repeating the subdivision rule on the daughters). Figure 6 shows the location of the Gergonne

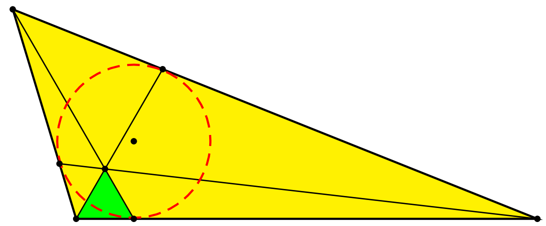

Figure 5. The unique triangle which has an equilateral triangle as a Gergonne daughter.

granddaughters in P. It can be shown the triangle in Figure 5 is not a Gergonne daughter, so that the equilateral triangle is not a Gergonne granddaughter.

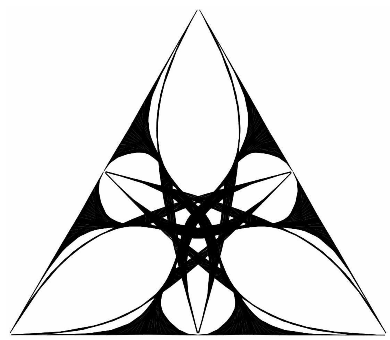

Figure 6. The possible Gergonne granddaughters in P.

One interesting problem is to find what triangles (up to similarity) can occur if we repeat the subdivision rule arbitrarily many times (see [2])? One example of this would be any triangle which is similar to one of its Gergonne daughters. Do any such triangles exist? (For the incenter there are only two such triangles, $(36°, 72°, 72°)$ and $(40°, 60°, 80°)$; for the centroid there is none.)

Besides the incenter, centroid and Gergonne point there are many other possible center points to consider (see the Encyclopedia of Triangle Centers [4] for a complete listing of well known center points, along with many others). One interesting point would be the Lemoine point, which can have up to *three* triangles T for a triangle S in standard position (as compared to 2 for the Gergonne point and 1 for the centroid).

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Steve Butler: Department of Mathematics, UCLA, Los Angeles, California 90095, USA
E-mail address: butler@math.ucla.edu

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Mappings Associated with Vertex Triangles

Clark Kimberling

Abstract. Methods of linear algebra are applied to triangle geometry. The vertex triangle of distinct circumcevian triangles is proved to be perspective to the reference triangle ABC, and similar results hold for three other classes of vertex triangles. Homogeneous coordinates of the perspectors define four mappings $\mathcal{M}_i$ on pairs of points (U, X). Many triangles homothetic to ABC are examined, and properties of the four mappings are presented. In particular, $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for $i = 1, 2, 3, 4$, and $\mathcal{M}_1(U, \mathcal{M}_1(U, X)) = X$; for this reason, $\mathcal{M}_1(U, X)$ is given the name *U-vertex conjugate of X*. In the introduction of this work, *point* is defined algebraically as a homogeneous function of three variables. Subsequent definitions and methods include symbolic substitutions, which are strictly algebraic rather than geometric, but which have far-reaching geometric implications.

1. Introduction

In [1], H. S. M. Coxeter proved a number of geometric results using methods of linear algebra and homogenous trilinear coordinates. However, the fundamental notions of triangle geometry, such as point and line in [1] are of the traditional geometric sort. In the present paper, we begin with an algebraic definition of point.

Suppose a, b, c are variables (or indeterminates) over the field of complex numbers and that x, y, z are homogeneous algebraic functions of (a, b, c) :

$$x = x(a, b, c), \quad y = y(a, b, c), \quad z = z(a, b, c),$$

all of the same degree of homogeneity and not all identically zero. Triples (x, y, z) and (x_1, y_1, z_1) are *equivalent* if $xy_1 = yx_1$ and $yz_1 = zy_1$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point.* Let

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1.$$

These three points define the *reference triangle* ABC. The set of all points is the *transfigured plane,* as in [6]. If we assign to a, b, c numerical values which are the sidelengths of a euclidean triangle, then $x : y : z$ are homogeneous coordinates (e.g., trilinear or barycentric) as in traditional geometry, and points as defined just above are then points in the plane of a euclidean triangle ABC.

Possibly the earliest treatment of triangle-related points as functions rather than two-dimensional points appears in [3]; in [3]–[9], points-as-functions methods

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lead to problems whose meanings and solutions are nongeometric but which have geometric consequences. Perhaps the most striking are symbolic substitutions [6]–[8], the latter typified by substituting bc, ca, ab for a, b, c respectively. To see the nongeometric character of this substitution, one can easily find values of a, b, c that are sidelengths of a euclidean triangle but bc, ca, ab are not – and yet, this substitution and others have deep geometric consequences, as they preserve collinearity, tangency, and algebraic degree of loci. (In §6, the symbolic substitution $(a, b, c) \to (bc, ca, ab)$ is again considered.)

Having started with an algebraic definition of "point" as in [3], we now use it as a basis for defining other *algebraic* objects. A *line* is a set of points $x : y : z$ such that $lx + my + nz = 0$ for some point $l : m : n$; in particular, the line of two points $p : q : r$ and $u : v : w$ is given by

$$\begin{vmatrix} x & y & z \\ p & q & r \\ u & v & w \end{vmatrix} = 0.$$

A *triangle* is a set of three points. Harmonic conjugacy, isogonal conjugacy, and classes of curves are likewise defined by algebraic equations that are familiar in the literature of geometry (e.g. [1], [5], [10], [12], and many nineteenth-century works), where they occur as consequences of geometric foundations, not as definitions The same is true for other relationships, such as concurrence of lines, collinearity of points, perspectivity of triangles, similarity, and homothety.

So far in this discussion, coordinates have been general homogeneous. In traditional triangle geometry, two specific systems of homogeneous coordinates are common: barycentric and trilinear. In order to define special points and curves, we shall use their traditional trilinear representations. Listed here are a few examples: the centroid of ABC is *defined* as the point $1/a : 1/b : 1/c$; the line $\mathcal{L}^\infty$ at infinity, as $ax + by + cz = 0$. The isogonal conjugate of a point $x : y : z$ satisfying $xyz \neq 0$ is defined as the point $1/x : 1/y : 1/z$ and denoted by X^{-1}, and the circumcircle Γ is defined by $ayz + bzx + cxy = 0$, this being the set of isogonal conjugates of points on $\mathcal{L}^\infty$. Of course, we may illustrate definitions and relationships by evaluating a, b, c numerically—and then all the algebraic objects become geometric objects. (On the other hand, if, for example, $(a, b, c) = (6, 2, 3)$, then the algebraic objects remain intact even though there is no triangle with sidelengths $6, 2, 3$.)

Next, we define four classes of triangles. Suppose $X = x : y : z$ is a point not on a sideline of ABC; *i.e.*, $xyz \neq 0$. Let

$$\begin{aligned} A_1 &= AX \cap BC = 0 : y : z \\ B_1 &= BX \cap CA = x : 0 : z \\ C_1 &= CX \cap AB = x : y : 0. \end{aligned}$$

The triangle $A_1B_1C_1$ is the *cevian triangle* of X. Let A_2 be the point, other than A, in which the line AX meets Γ. Define B_2 and C_2 cyclically. The triangle $A_2B_2C_2$ is the *circumcevian triangle* of X, as indicated in Figure 1.

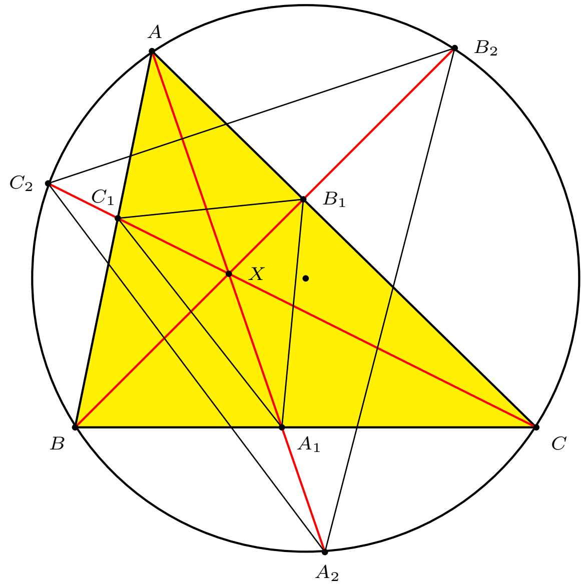

Figure 1.

Let A_3 be the $\{A, A_1\}$-harmonic conjugate of X (*i.e.*, $A_3 = -x : y : z$), and define B_3 and C_3 cyclically. Then $A_3B_3C_3$ is the *anticevian triangle* of X. Let

$$A' = BC \cap B_1C_1, \quad B' = CA \cap C_1A_1, \quad C' = AB \cap A_1B_1,$$

so that $A' = \{B, C\}$-harmonic conjugate of A_1 (*i.e.*, $A_1 = 0 : y : -z$), etc. The lines AA', BB', CC' are the *anticevians* of X, and the points

$$A_4 = AA' \cap \Gamma, \qquad B_4 = BB' \cap \Gamma, \qquad C_4 = CC' \cap \Gamma,$$

as in Figure 2, are the vertices of the *circum-anticevian triangle,* $A_4B_4C_4$, of X.

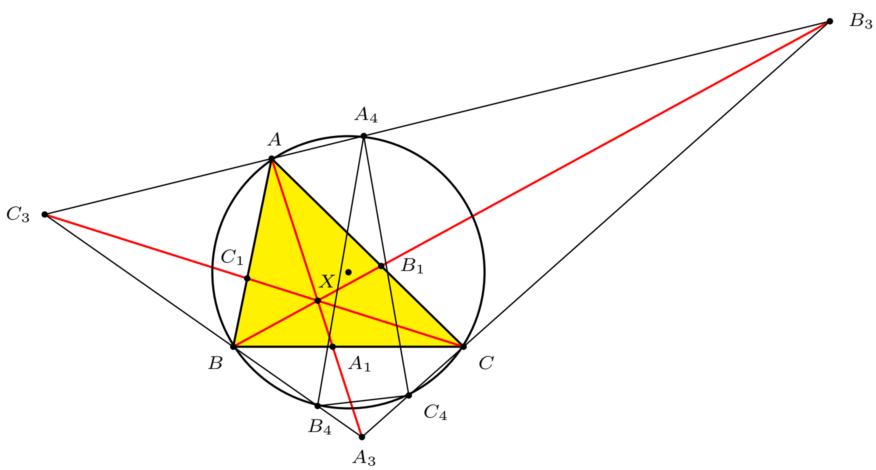

Figure 2.

With these four classes of triangles in mind, suppose $T = DEF$ and $T' = D'E'F'$ are triangles. The *vertex triangle* of T and T' is formed by the lines DD', EE', FF' as in Figure 3. Note that T and T' are perspective if and only if their vertex triangle is a single point.

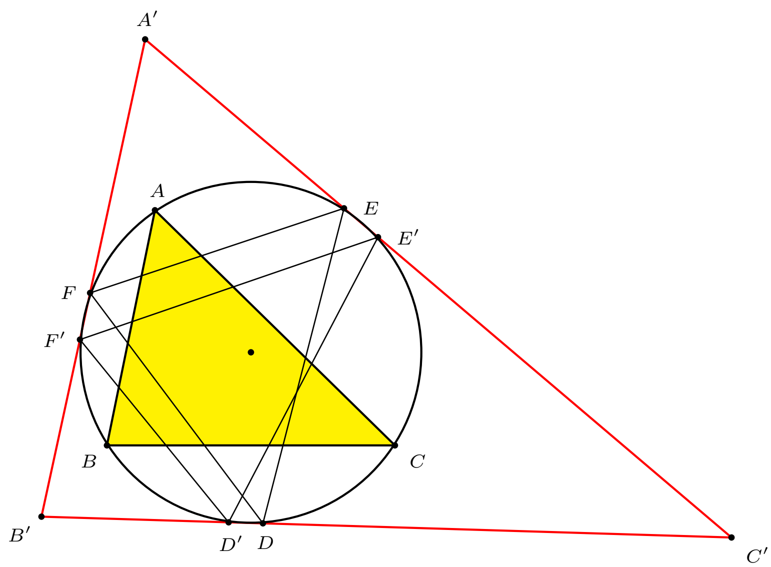

Figure 3.

2. The first mapping $\mathcal{M}_1$

Theorem 1. *The vertex triangle of distinct circumcevian triangles is perspective to ABC.*

Proof. Let $A'B'C'$ be the circumcevian triangle of $X = x : y : z$, and let $A''B''C''$ be the circumcevian triangle of $U = u : v : w$. The former can be represented as a matrix (e.g. [5, p.201]), as follows:

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$
$$= \begin{pmatrix} -ayz & (cy+bz)y & (bz+cy)z \\ (cx+az)x & -bzx & (az+cx)z \\ (bx+ay)x & (ay+bx)y & -cxy \end{pmatrix},$$

and likewise for $A''B''C''$ using vertices $u_i : v_i : w_i$ in place of $x_i : y_i : z_i$. Lines $A'A'', B'B'', C'C''$ are given by equations $x_i\alpha + y_i\beta + z_i\gamma = 0$ for $i = 4, 5, 6$, where

$$\begin{pmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{pmatrix} = \begin{pmatrix} y_1w_1 - z_1v_1 & z_1u_1 - x_1w_1 & x_1v_1 - y_1u_1 \\ y_2w_2 - z_2v_2 & z_2u_2 - x_2w_2 & x_2v_2 - y_2u_2 \\ y_3w_3 - z_3v_3 & z_3u_3 - x_3w_3 & x_3v_3 - y_3u_3 \end{pmatrix},$$

so that the vertex triangle is given by

$$\begin{pmatrix} A''' \\ B''' \\ C''' \end{pmatrix} = \begin{pmatrix} x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \\ x_9 & y_9 & z_9 \end{pmatrix}$$
$$= \begin{pmatrix} y_5z_6 - z_5y_6 & z_5x_6 - x_5z_6 & x_5y_6 - y_5x_6 \\ y_6z_4 - z_6y_4 & z_6x_4 - x_6z_4 & x_6y_4 - y_6x_4 \\ y_4z_5 - z_4y_5 & z_4x_5 - x_4z_5 & x_4y_5 - y_4x_5 \end{pmatrix}. \tag{1}$$

The line AA''' thus has equation $0\alpha + z_7\beta - y_7\gamma = 0$, and equations for the lines BB''' and CC''' are obtained cyclically. The three lines concur if

$$\begin{vmatrix} 0 & z_7 & -y_7 \\ -z_8 & 0 & x_8 \\ y_9 & -x_9 & 0 \end{vmatrix} = 0,$$

and this is found (by computer) to be true. □

In connection with Theorem 1, the perspector is the point $P = x_8x_9 : x_8y_9 : z_8x_9$. After canceling long common factors, we obtain

$$\begin{aligned} P = {} & a/(a^2vwyz - ux(bw + cv)(bz + cy)) \\ & : b/(b^2wuzx - vy(cu + aw)(cx + az)) \\ & : c/(c^2uvxy - wz(av + bu)(ay + bx)). \end{aligned} \tag{2}$$

The right-hand side of (2) defines the first mapping, $\mathcal{M}_1(U, X)$. If U and X are triangle centers (defined algebraically, for example, in [3], [5], [11]), then so is $\mathcal{M}_1(U, X)$. It can be easily shown that $\mathcal{M}_1(U, X)$ is an involution; that is, $\mathcal{M}_1(\mathcal{M}_1(U, X)) = X$. In view of this property, we call $\mathcal{M}_1(U, X)$ the U*-vertex conjugate* of X. For example, the incenter-vertex conjugate of the circumcenter is the isogonal conjugate of the Bevan point; *i.e.*, $\mathcal{M}_1(X_1, X_3) = X_{84}$. The indexing of named triangle centers, such as X_{84}, is given in the *Encyclopedia of Triangle Centers* [9].

Vertex-conjugacy shares the *iso-* property with another kind of conjugacy called isoconjugacy; viz., the U-vertex-conjugate of X is the same as the X-vertex-conjugate of U. (The U-isoconjugate of X is the point $vwyz : wuzx : uvxy$; see the Glossary at [9].)

Other properties of $\mathcal{M}_1$ are given in §9 and in Gibert's work [2] on cubics associated with vertex conjugates.

3. The second mapping $\mathcal{M}_2$

Theorem 2. *The vertex triangle of distinct circum-anticevian triangles is perspective to* ABC.

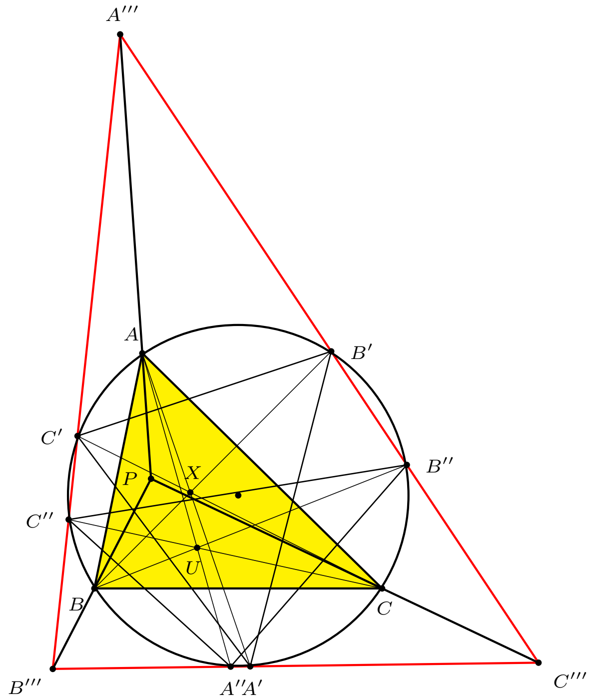

Figure 4.

Proof. The method of §2 applies, starting with

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

$$= \begin{pmatrix} ayz & (cy-bz)y & (bz-cy)z \\ (cx-az)x & bzx & (az-cx)z \\ (bx-ay)x & (ay-bx)y & cxy \end{pmatrix},$$

and likewise for $A''B''C''$. □

The perspector is given by

$$P = p : q : r = \frac{a}{f(a,b,c,x,y,z)} : \frac{b}{f(b,c,a,y,z,a)} : \frac{c}{f(c,a,b,z,a,b))}, \quad (3)$$

where

$$f(a,b,c,s,y,z) = a^2vwyz - xu(bw-cv)(bz-cy),$$

and we define $\mathcal{M}_2(U,X) = P$ as in (3).

As this mapping is not involutory, we wish to solve the equation $P = \mathcal{M}_2(U,X)$ for X. The system to be solved, and the solution, are given by

$$\begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix} \begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} a/p \\ b/q \\ c/r \end{pmatrix}$$

and

$$\begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix}^{-1} \begin{pmatrix} a/p \\ b/q \\ c/r \end{pmatrix},$$

where

$$\begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix} = \begin{pmatrix} a^2vw & -bu(bw-cv) & cu(bw-cv) \\ av(cu-aw) & b^2wu & -cv(cu-aw) \\ -aw(av-bu) & bw(av-bu) & k_3 = c^2uv \end{pmatrix}.$$

Again, long factors cancel, leaving

$$X = x : y : z = g(a,b,c) : g(b,c,a) : g(c,a,b),$$

where

$$g(a,b,c) = \frac{a}{a^3qrv^2w^2 - a^2G_2 + aG_1 + G_0},$$

where

$$\begin{aligned} G_0 &= u^2p\,(bw-cv)^2\,(br+cq)\,, \\ G_1 &= uvwp\,(br-cq)\,(cv-bw)\,, \\ G_2 &= uvwrq\,(bw+cv)\,. \end{aligned}$$

4. The third mapping $\mathcal{M}_3$

Given a point $X = x : y : z$, we introduce a triangle $A'B'C'$ as follows:

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} ayz & (cy+bz)y & (bz+cy)z \\ (cx+az)x & bzx & (az+cx)z \\ (bx+ay)x & (ay+bx)y & cxy \end{pmatrix},$$

and likewise for $A''B''C''$ in terms of $u : v : w$. The method of §2 shows that ABC is perspective to the vertex triangle of $A'B'C'$ and $A''B''C''$. The perspector is given by

$$\begin{aligned} \mathcal{M}_3(U,X) = {} & a/(a^2vwyz + xu(bw+cv)(bz+cy)) \\ & : b/(b^2wuzx + yv(cu+aw)(cx+az)) \\ & : c/(c^2uvxy + zw(av+bu)(ay+bx)). \end{aligned} \tag{4}$$

A formula for inversion is found as in §3: if $\mathcal{M}_3(U,X) = P = p : q : r$, then X is the point $g(a,b,c) : g(b,c,a) : g(c,a,b)$, where

$$g(a,b,c) = \frac{a}{a^3qrv^2w^2 + a^2G_2 - aG_1 - G_0},$$

where

$$\begin{aligned} G_0 &= u^2p\left(b^2w^2 - c^2v^2\right)(br-cq)\,, \\ G_1 &= uvwp\,(br+cq)\,(bw+cv)\,, \\ G_2 &= uvwrq\,(bw+cv)\,. \end{aligned}$$

Geometrically, A' is the $\{A, \widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cocevian and circumcevian triangles of X, respectively. (The vertices of the cocevian triangle of X are $\widehat{A} = 0 : z : -y$, $\widehat{B} = z : 0 : x$, $\widehat{C} = y : -x : 0$. The point $\widehat{A}$ is the $\{B, C\}$-harmonic conjugate of the A-vertex of the cevian triangle of U. The triangle $\widehat{A}\widehat{B}\widehat{C}$ is degenerate, as its vertices are collinear.)

5. The fourth mapping $\mathcal{M}_4$

For given $X = x : y : z$, define a triangle $A'B'C'$ by

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} -ayz & (cy - bz)y & (bz - cy)z \\ (cx - az)x & -bzx & (az - cx)z \\ (bx - ay)x & (ay - bx)y & -cxy \end{pmatrix}.$$

Again, ABC is perspective to the vertex triangle of $A'B'C'$ and the triangle $A''B''C''$ similarly defined from U. The perspector is given by

$$\begin{aligned} \mathcal{M}_4(U, X) = & \, a/(a^2vwyz + xu(bw - cv)(bz - cy)) \\ & : b/(b^2wuzx + yv(cu - aw)(cx - az)) \\ & : c/(c^2uvxy + zw(av - bu)(ay - bx)). \end{aligned} \tag{5}$$

A formula for inversion is found as for §3: if $\mathcal{M}_4(U, X) = P = p : q : r$, then X is the point $g(a, b, c) : g(b, c, a) : g(c, a, b)$, where

$$g(a, b, c) = \frac{a}{(avw - buw - cuv)\,(a^2qrvw + up\,(cq - br)\,(bw - cv))}.$$

Geometrically, A' is the $\{A, \widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cevian and circum-anticevian triangles of X, respectively.

6. Summary and extensions

To summarize §§2–5, vertex triangles associated with circumcevian and circum-anticevian triangles are perspective to the reference triangle ABC, and all four perspectors, given by (2-5), are representable by the following form for first trilinear coordinate:

$$\frac{a}{\frac{a^2}{ux} \pm \left(\frac{b}{v} \pm \frac{c}{w}\right)\left(\frac{b}{y} \pm \frac{c}{z}\right)}; \tag{6}$$

here, the three $\pm$ signs are limited to $- + +$, $- - -$, $+ + +$, and $+ - -$, which correspond in order to the four mappings $\mathcal{M}_i(U, X)$.

Regarding each perspector $P = \mathcal{M}_i(U, X)$, formulas for the inverse mapping of X, for given U, have been given, and in the case of the first mapping, the transformation is involutory. The representation (6) shows that $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for each i, which is to say that $\mathcal{M}_i(U, X)$ can be viewed as a commutative binary operation. There are many interesting examples regarding the four mappings; some of them are given in §9.

For all four configurations, define $\mathcal{M}_i(U, U)$ by putting $x : y : z = u : v : w$ in (2)–(5), and note that (6) gives the perspector in these cases. In Figure 3, taking $X = U$ corresponds to moving E', F', G' to E, F, G so that in the limit the lines $B'C', C'A', A'B'$ are tangent to Γ. It would be of interest to know the set of triangle centers X for which there exists a triangle center U such that $\mathcal{M}_i(U,U) = X$.

The symbolic substitution

$$(a,b,c) \to (bc,ca,ab) \tag{7}$$

transforms the transfigured plane onto itself, as (7) maps each point $X = x : y : z = x(a,b,c) : y(a,b,c) : z(a,b,c)$ to the point

$$X' = x' : y' : z' = x(bc,ca,ab) : y(bc,ca,ab) : z(ca,ab,bc).$$

The circumcircle, as the locus of X satisfying $ayz + bzx + cxy = 0$, maps to the Steiner circumellipse [10], which is the locus of $x' : y' : z'$ satisfying

$$bcy'z' + caz'x' + abx'y' = 0.$$

Circumcevian triangles map to perspective triangles as in Theorem 1, and perspectors are given by applying (7) to (2). The substitution (7) likewise applies to the developments in §§3–5. Analogous (geometric) results spring from other (non-geometric) symbolic substitutions, such as $(a,b,c) \to (b+c, c+a, a+b)$ and $(a,b,c) \to (a^2,b^2,c^2)$.

7. Homothetic triangles

We return now to the vertex-triangles introduced in §§2–5 and establish that if U and X are a pair of isogonal conjugates, then their vertex triangle is homothetic to ABC.

Theorem 3. *Suppose X is a point not on a sideline of triangle ABC. Then the vertex triangle of the circumcevian triangles of X and X^{-1} is homothetic to ABC, and likewise for the pairs of vertex triangles in §§3–5.*

Proof. In accord with the definition of isogonal conjugate, trilinears for $U = X^{-1}$ are given by $u = yz$, $v = zx$, $w = xy$, so that

$$u : v : w = x^{-1} : y^{-1} : z^{-1}.$$

In the notation of §2, the vertex triangle (1) is given by its A-vertex $x_7 : y_7 : z_7$, where

$$\begin{aligned} x_7 =& \, abc(x^2+y^2)(x^2+z^2) + (a^2(bz+cy) + bc(by+cz))x^3 \\ & + a(a^2+b^2+c^2)x^2yz + (bcyz(bz+cy) + a^2yz(by+cz))x, \\ y_7 =& -bxz(ab(x^2+y^2) + (a^2+b^2-c^2)xy), \\ z_7 =& -cxy(ac(x^2+z^2) + (a^2-b^2+c^2)xz). \end{aligned}$$

Writing out the coordinates $x_8 : y_8 : z_8$ and $x_9 : y_9 : z_9$, we then evaluate the determinant for parallelism of sideline BC and the A-side of the vertex triangle:

$$\begin{vmatrix} a & b & c \\ 1 & 0 & 0 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} = 0.$$

Likewise, the other two pairs of sides are parallel, so that the vertex triangle of the two circumcevian triangles is now proved homothetic to ABC.

The same procedure shows that the vertex triangles in §§3–5, when $U = X^{-1}$, are all homothetic to ABC (hence homothetic to one another, as well as the medial triangle, the anticomplementary triangle, and the Euler triangle). $\square$

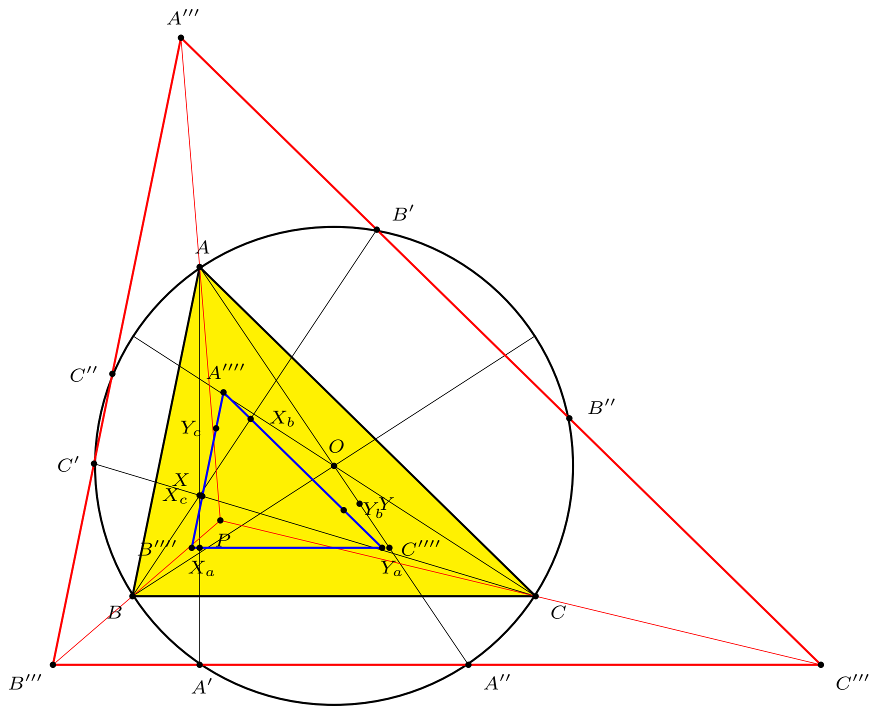

Figure 5.

Substituting into (6) gives a compact expression for the four classes of homothetic centers (*i.e.*, perspectors), given by the following first trilinear:

$$\frac{ayz}{(a^2 \pm (b^2 + c^2))yz \pm bc(y^2 + z^2)},$$

from which it is clear that the homothetic centers for X and X^{-1} are identical.

8. More Homotheties

Let $\mathcal{C}(X)$ denote the circumcevian triangle of a point X, and let O denote the circumcenter, as in Figure 6.

Theorem 4. *Suppose U is a point not on a sideline of triangle ABC. The vertex triangle of $\mathcal{C}(U)$ and $\mathcal{C}(O)$ is homothetic to the pedal triangle of U^{-1}.*

Proof. The vertex triangle $A'''B'''C'''$ of $C(U)$ and $C(O)$ is given by (1), using $U = u : v : w$ and

$$x = a(b^2 + c^2 - a^2), \quad y = b(c^2 + a^2 - b^2), \quad z = c(a^2 + b^2 - c^2).$$

Trilinears for A''' as initially computed include many factors. Canceling those common to all three trilinears leaves

$$x_7 = 4abc(avw + bwu + cuv) + u^2(a + b + c)(-a + b + c)(a - b + c)(a + b - c),$$
$$y_7 = uv(a^2 - b^2 + c^2)(b^2 - a^2 + c^2) - 2cw(a^2 + b^2 - c^2)(av + bu),$$
$$z_7 = uw(a^2 - c^2 + b^2)(c^2 - a^2 + b^2) - 2bv(a^2 + c^2 - b^2)(aw + cu),$$

and x_8, y_8, z_8 and x_9, y_9, z_9 are obtained from x_7, y_7, z_7 by cyclic permutations of a, b, c and u, v, w.

Since $U^{-1} = vw : wu : uv$, the vertices of the pedal triangle of U^{-1} are given ([5, p.186]) by

$$\begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix} = \begin{pmatrix} 0 & w(u + vc_1) & v(u + wb_1) \\ w(v + uc_1) & 0 & u(v + wa_1) \\ v(w + ub_1) & u(w + va_1) & 0 \end{pmatrix},$$

where

$$(a_1, b_1, c_1) = (a(b^2 + c^2 - a^2), b(c^2 + a^2 - b^2), c(a^2 + b^2 - c^2)).$$

Side $B'''C'''$ of the vertex triangle is parallel to the corresponding sideline of the pedal triangle if the determinant

$$\begin{vmatrix} a & b & c \\ g_2h_3 - h_2g_3 & h_2f_3 - f_2h_3 & f_2g_3 - g_2f_3 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} \tag{8}$$

equals 0. It is helpful to factor the polynomials in row 3 and cancel common factors. That and putting $f_1 = g_2 = h_3 = 0$ lead to the following determinant which is a factor of (8):

$$\begin{vmatrix} a & b & c \\ -h_2g_3 & h_2f_3 & f_2g_3 \\ 2a(bw + cv) & w(a^2 + b^2 - c^2) & v(a^2 - b^2 + c^2) \end{vmatrix}.$$

This determinant indeed equals 0. The parallelism of the other matching pairs of sides now follows cyclically. □

9. Properties of the four mappings

This section consists of properties of the mappings $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ introduced in §§2–5. Proofs are readily given by use of well known formulas. In several cases, a computer is needed because of very lengthy trilinears. Throughout, it is assumed that neither U nor X lies on a sideline of ABC.

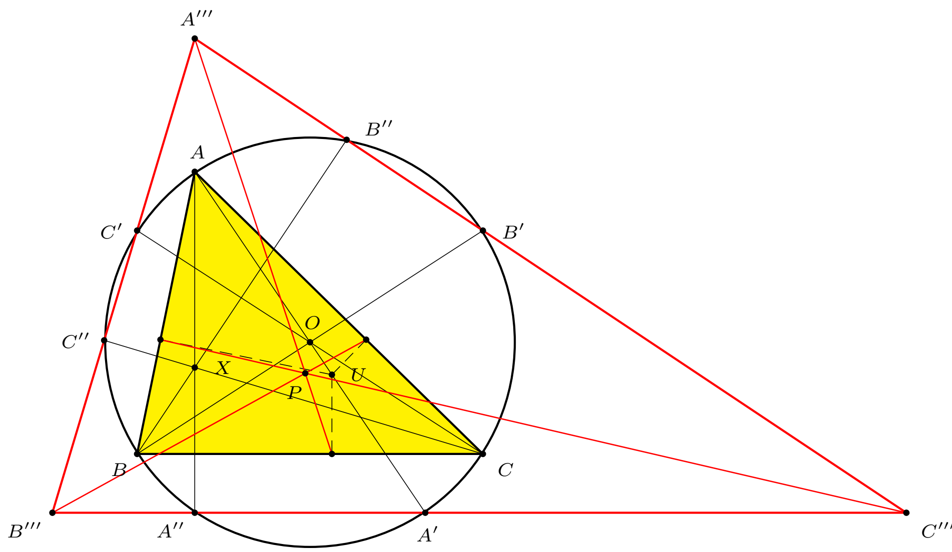

Figure 6.

1a. If $U \in \Gamma$, then $\mathcal{M}_1(U, X) = U$.

1b. If $X \notin \Gamma$, then

$$\mathcal{M}_1(X, X) = \frac{a}{ayz - bzx - cxy} : \frac{b}{bzx - cxy - ayz} : \frac{c}{cxy - ayz - bzx}.$$

If $U = \mathcal{M}_1(X, X)$, then

$$X = \frac{avw}{bw + cv} : \frac{bwu}{cu + aw} : \frac{cuv}{av + bu}.$$

1c. If X is the 1st Saragossa point of U, then $\mathcal{M}_1(U, X) = X$. (The 1st Saragossa point is the point

$$\frac{a}{bzx + cxy} : \frac{b}{cxy + ayz} : \frac{c}{ayz + bzx},$$

discussed at [9] just before X_{1166}.

1d. Suppose U is on the line at infinity, and let U^* be the isogonal conjugate of the antipode of the isogonal conjugate of U. Let L be the line X_3U^*. Then $\mathcal{M}_1(U, U^*) = X_3$, and if $X \in L$, then $\mathcal{M}_1(U, X)$ is the inverse-in-Γ of X.

1e. $\mathcal{M}_1$ maps the Darboux cubic to itself. (See [2] for a discussion of cubics associated with $\mathcal{M}_1$.

2a. $\mathcal{M}_2(X_6, X) = X$.

2b. $\mathcal{M}_2(X, X) = X$-Ceva conjugate of X_6.

2c. Let L be the line UX_6 and let L' be the line UU^c, where $U^c = \mathcal{M}_2(U, U)$. If $X \in L$, then $\mathcal{M}_2(U, X) \in L'$.

3a. $\mathcal{M}_3(X_6, X) = X$.

3b. If $X \in \Gamma$ and X is not on a sideline of ABC, then $\mathcal{M}_3(X, X)$ is the cevapoint X and X_6. (The cevapoint [9, Glossary] of points $P = p : q : r$ and $U = u : v : w$ is defined by trilinears

$$(pv + qu)(pw + ru) : (qw + rv)(qu + pv) : (ru + pw)(rv + qw).)$$

3c. If $U \in \Gamma$, then

$$\mathcal{M}_3(U, X) = \frac{u}{ayz - bzx - cxy} : \frac{v}{bzx - cxy - ayz} : \frac{w}{cxy - ayz - bzx},$$

which is the trilinear product $U \cdot \widehat{X}$, where $\widehat{X}$ is the X_2-isoconjugate of the X-Ceva conjugate of X_6.

4a. $\mathcal{M}_4(X_6, X) = X$.

4b. Suppose P is on the line at infinity (so that P^{-1} is on Γ). Let X be the cevapoint of X_6 and P. Then $\mathcal{M}_4(X_{251}, X) = P^{-1}$.

4c. Let $X^* = X \cdot \widehat{X}$, where $\widehat{X}$ is as in 3c. Then $\mathcal{M}_4(X, X^*) = X_6$.

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Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, U.S.A.

E-mail address: `ck6@evansville.edu`

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On Integer Relations Between the Area and Perimeter of Heron Triangles

Allan J. MacLeod

Abstract. We discuss the relationship $P^2 = nA$ for a triangle with integer sides, with perimeter P and area A, where n is an integer. We show that the problem reduces to finding rational points of infinite order in a family of elliptic curves. The geometry of the curves plays a crucial role in finding real triangles.

1. Introduction

In a recent paper, Markov [2] discusses the problem of solving $A = mP$, where A is the area and P is the perimeter of an integer-sided triangle, and m is an integer. This relation forces A to be integral and so the triangle is always a Heron triangle.

In many ways, this is not a proper question to ask, since this relation is not scale-invariant. Doubling the sides to a similar triangle changes the area/perimeter ratio by a factor of 2. Basically, we have unbalanced dimensions - area is measured in square-units, perimeter in units but m is a dimensionless quantity.

It would seem much better to look for relations between A and P^2, which is the purpose of this paper. Another argument in favour of this is that the recent paper of Baloglou and Helfgott [1], on perimeters and areas, has the main equations (1) to (8) all balanced in terms of units.

We assume the triangle has sides (a, b, c) with $P = a + b + c$ and $s = \frac{P}{2}$. Then the area is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

so that it is easy to see that $A < P^2/4$. Thus, to look for an integer link, we should study $P^2 = nA$ with $n > 4$.

It is easy to show that this bound on n can be increased quite significantly. We have

$$\frac{P^4}{A^2} = 16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} \tag{1}$$

and we can, without loss of generality, assume $a = 1$. Then the ratio in equation (1) is minimised when $b = 1, c = 1$. This is obvious from symmetry, but can be easily proven by finding derivatives. Thus $\frac{P^4}{A^2} \geq 432$ and so $P^2 \geq 12\sqrt{3}A$ for all triangles, so we need only consider $n \geq 21$.

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As an early example of a solution, the $(3,4,5)$ triangle has $P^2 = 144$ and $A = 6$ so $n = 24$.

To proceed, we consider the equation

$$16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} = n^2. \tag{2}$$

2. Elliptic Curve Formulation

Firstly, it is clear that we can let a, b, c be rational numbers, since a rational-sided solution is easily scaled up to an integer one.

From equation (2), we have

$$(n^2+16)a^3 + (48-n^2)(b+c)a^2 - (b^2(n^2-48) - 2bc(n^2+48) + c^2(n^2-48))a$$
$$+ (b+c)(b^2(n^2+16) + 2bc(16-n^2) + c^2(n^2+16)) = 0.$$

This cubic is very difficult to deal with directly, but a considerable simplification occurs if we use $c = P - a - b$, giving

$$4n^2(P-2b)a^2 - 4n^2(2b^2-3bP+P^2)a + P(4b^2n^2 - 4bn^2P + P^2(n^2+16)) = 0. \tag{3}$$

For this quadratic to have rational roots, we must have the discriminant being a rational square. This means that there must be rational solutions of

$$d^2 = 4n^2b^4 - 4n^2Pb^3 + n^2P^2b^2 + 32P^3b - 16P^4$$

and, if we define $y = \frac{2nd}{P^2}$ and $x = \frac{2nb}{P}$, we have

$$y^2 = x^4 - 2nx^3 + n^2x^2 + 64nx - 64n^2. \tag{4}$$

A quartic in this form is birationally equivalent to an elliptic curve, see Mordell [3]. Using standard transformations and some algebraic manipulation, we find the equivalent curves are

$$E_n : v^2 = u^3 + n^2u^2 + 128n^2u + 4096n^2 = u^3 + n^2(u+64)^2 \tag{5}$$

with the backward transformation

$$\frac{b}{P} = \frac{n(u-64)+v}{4nu}. \tag{6}$$

Thus, from a suitable point (u, v) on E_n, we can find b and P from this relation. To find a and c, we use the quadratic for a, but written as

$$a^2 - (P-b)a + \frac{P(16P^2 + n^2(P-2b)^2)}{4n^2(P-2b)} = 0. \tag{7}$$

The sum of the roots of this quadratic is $P - b = a + c$, so the two roots give a and c.

But, we should be very careful to note that the analysis based on equation (2) is just about relations between numbers, which could be negative. Even if they are all positive, they may not form a real-life triangle - they do not satisfy the triangle inequalities. Thus we need extra conditions to give solutions, namely $0 < a, b, c < \frac{P}{2}$.

3. Properties of E_n

The curves E_n are clearly symmetric about the u-axis. If the right-hand-side cubic has 1 real root R, then the curve has a single infinite component for $u \geq R$. If, however, there are 3 real roots $R_1 < R_2 < R_3$, then E_n consists of an infinite component for $u \geq R_3$ and a closed component for $R_1 \leq u \leq R_2$, usually called the "egg".

Investigating with the standard formulae for cubic roots, we find 3 real roots if $n^2 > 432$ and 1 real root if $n^2 < 432$. Since we assume $n \geq 21$, there must be 3 real roots and so 2 components. Descartes' rule of signs shows that all roots are negative.

It is clear that $u = -64$ does not give a point on the curve, but $u = -172$ gives $v^2 = 16(729n^2 - 318028)$ which is positive if $N \geq 21$. Thus we have $R_1 < -172 < R_2 < -64 < R_3 < 0$.

The theory of rational points on elliptic curves is an enormously developed one. The rational points form a finitely-generated Abelian group with the addition operation being the standard secant/tangent method. This group of points is isomorphic to the group $T \oplus \mathbb{Z}^r$, where T is one of $\mathbb{Z}_m,\ m = 1, 2, \ldots, 10, 12$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_m, m = 1, 2, 3, 4$, and r is the rank of the curve. T is known as the torsion-subgroup and consists of those points of finite order on the curve, including the point-at-infinity which is the identity of the group. Note that the form of E_n ensures that torsion points have integer coordinates by the Nagell-Lutz theorem, see Silverman and Tate [5].

We easily see the two points $(0, \pm 64n)$ and since they are points of inflexion of the curve, they have order 3. Thus T is one of $\mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_9, \mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Some of these possibilities would require a point of order 2 which correspond to integer zeroes of the cubic. Numerical investigations show that only $n = 27$, for $n \leq 499$, has an integer zero (at $u = -576$). Further investigations show $\mathbb{Z}_3$ as being the only torsion subgroup to appear, for $n \leq 499$, apart from $\mathbb{Z}_6$ for $n = -27$. Thus we conjecture that, apart from $n = 27$, the group of rational points is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}^r$. The points of order 3 give $\frac{b}{P}$ undefined so we would need $r \geq 1$ to possibly have triangle solutions.

For $n = 27$, we find the point $H = (-144, 1296)$ of order 6, which gives the isosceles triangle $(5, 5, 8)$ with $P = 18$ and $A = 12$. In fact, all multiples of H lead to this solution or $\frac{b}{P}$ undefined.

4. Rank Calculations

There is, currently, no known guaranteed method to determine the rank r. We can estimate r very well, computationally, if we assume the Birch and Swinnerton-Dyer conjecture [6]. We performed the calculations using some home-grown software, with the Pari-gp package for the multiple-precision calculations. The results for $21 \leq n \leq 99$ are shown in Table 1.

TABLE 1
Ranks of $E_n, n = 21, \ldots, 99$

n	0	1	2	3	4	5	6	7	8	9
20+	–	1	0	0	1	0	1	0	1	0
30+	1	2	0	1	0	1	1	1	0	1
40+	0	0	2	2	0	1	0	2	0	0
50+	1	1	1	0	0	1	1	0	2	0
60+	1	1	1	2	1	0	1	1	0	0
70+	0	1	0	0	2	1	1	1	0	2
80+	0	1	0	1	0	1	1	0	2	0
90+	1	2	0	1	1	1	1	0	1	1

We can see that the curve from the $(3, 4, 5)$ triangle with $n = 24$ has rank 1, but the $(5, 5, 8)$ triangle has a curve with rank 0, showing that this is the only triangle giving $n = 27$.

For those curves with rank 1, a by-product of the Birch and Swinnerton-Dyer computations is an estimate for the height of the generator of the rational points of infinite order. The height essentially gives an idea of the sizes of the numerators and denominators of the u coordinates. The largest height encountered was 10.25 for $n = 83$ (the height normalisation used is the one used by Silverman [4]).

All the heights computed are small enough that we could compute the generators fairly easily, again using some simple software. From the generators, we derive the list of triangles in Table 2.

5. Geometry of $0 < \frac{b}{P} < \frac{1}{2}$

The sharp-eyed reader will have noticed that several values of n, which have positive rank in Table 1, do not give a triangle in Table 2, despite generators being found To explain this, we need to consider the geometric implications on (u, v) from the bounds $0 < \frac{b}{P} < \frac{1}{2}$, or

$$0 < \frac{n(u-64)+v}{4nu} < \frac{1}{2} \tag{8}$$

Consider first $u > 0$. Then $\frac{b}{P} > 0$ when $v > 64n - nu$. The line $v = 64n - nu$, only meets E_n at $u = 0$, and the negative gradient shows that $\frac{b}{P} > 0$ when we take points on the upper part of the curve. To have $\frac{b}{P} < \frac{1}{2}$, we need $v < nu + 64n$. The line $v = nu + 64n$ has an intersection of multiplicity 3 at $u = 0$, so never meets E_n again. Thus $v < nu + 64n$ only on the lower part of E_n for $u > 0$. Thus, we cannot have $0 < \frac{b}{P} < \frac{1}{2}$ for any points with $u > 0$.

Now consider $u < 0$. Then, for $\frac{b}{P} > 0$ we need $v < 64n - nu$. The negative gradient and single intersection show that this holds for all points on E_n with $u < 0$. For $\frac{b}{P} < \frac{1}{2}$, we need $v > nu + 64n$. This line goes through $(0, 64n)$ on the curve and crosses the u-axis when $u = -64$, which we saw earlier lies strictly between the egg and the infinite component. Since $(0, 64n)$ is the only intersection we must have the line above the infinite component of E_n when $u < 0$ but below the egg.

TABLE 2
Triangles for $21 \leq n \leq 99$

n	a	b	c
21	15	14	13
28	35	34	15
31	85	62	39
35	97	78	35
39	37	26	15
43	56498	31695	29197
47	4747	3563	1560
51	149	85	72
55	157	143	30
58	85	60	29
62	598052	343383	275935
66	65	34	33
75	74	51	25
77	1435	2283	902
81	26	25	3
88	979	740	261
93	2325	2290	221
98	2307410	2444091	255319

n	a	b	c
24	5	4	3
30	13	12	5
33	30	25	11
36	17	10	9
42	20	15	7
45	41	40	9
50	1018	707	375
52	5790	4675	1547
56	41	28	15
60	29	25	6
63	371	250	135
74	740	723	91
76	47575	43074	7163
79	1027	1158	185
85	250	221	39
91	1625	909	742
95	24093	29582	6175
99	97	90	11

Thus, $0 < \frac{b}{P} < \frac{1}{2}$ only on the egg where $u < -64$. The results in Table 2 come from generators satisfying this condition.

It might be thought that forming integer multiples of generators and possibly adding the torsion points could resolve this. This is not the case, due to the closed nature of the egg. If a line meets the egg and is not a tangent to the egg, then it enters the egg and must exit the egg. Thus any line has a double intersection with the egg.

So, if we add a point on the infinite component to either torsion point, also on the infinite component, we must have the third intersection on the infinite component. Similarly doubling a point on the infinite component must lead to a point on the infinite component. So, if no generator lies on the egg, there will never be a point on the egg, and so no real-life triangle will exist.

We can generate other triangles for a value of n by taking multiples of the generator. Using the same arguments as before, it is clear that a generator G on the egg has $2G$ on the infinite component but $3G$ must lie on the egg. So, for $n = 24$, the curve $E_{24} : v^2 = u^3 + 576u^2 + 73728u + 2359296$ has $G = (-384, 1536)$, hence $2G = (768, -29184)$ and $3G = \left(-\frac{2240}{9}, \frac{55808}{27}\right)$, which leads to the triangle $(287, 468, 505)$ where $P = 1260$ and $A = 66150$.

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Allan J. MacLeod: Department of Mathematics and Statistics, University of the West of Scotland, High St., Paisley, Scotland. PA1 2BE.
E-mail address: `allan.macleod@uws.ac.uk`

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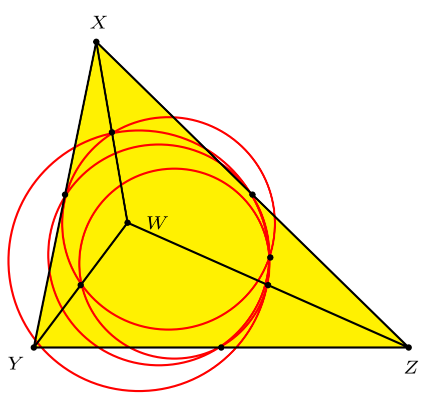
The Feuerbach Point and Reflections of the Euler Line

Jan Vonk

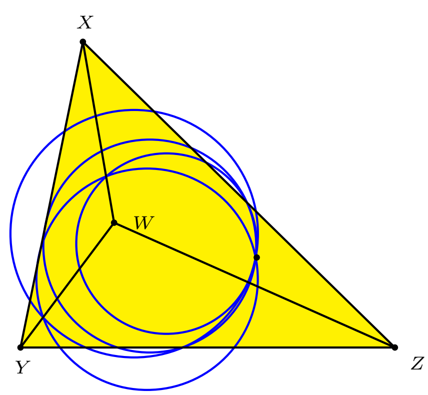
Abstract. We investigate some results related to the Feuerbach point, and use a theorem of Hatzipolakis to give synthetic proofs of the facts that the reflections of OI in the sidelines of the intouch and medial triangle all concur at the Feuerbach point. Finally we give some results on certain reflections of the Feuerbach point.

1. Poncelet point

We begin with a review of the Poncelet point of a quadruple of points W, X, Y, Z. This is the point of concurrency of
(i) the nine-point circles of triangles WXY, WXZ, XYZ, WYZ,
(ii) the four pedal circles of W, X, Y, Z with respect to XYZ, WYZ, WXZ, WXY respectively.



Nine-point circles



Pedal circles

Figure 1.

Basic properties of the Poncelet point can be found in [4]. Let I be the incenter of triangle ABC. The Poncelet point of I, A, B, C is the famous Feuerbach point F_{e}, as we show in Theorem 1 below. In fact, we can find a lot more circles passing through F_{e}, using the properties mentioned in [4].

Theorem 1. *The nine-point circles of triangles AIB, AIC, BIC are concurrent at the Feuerbach point F_{e} of triangle ABC.*

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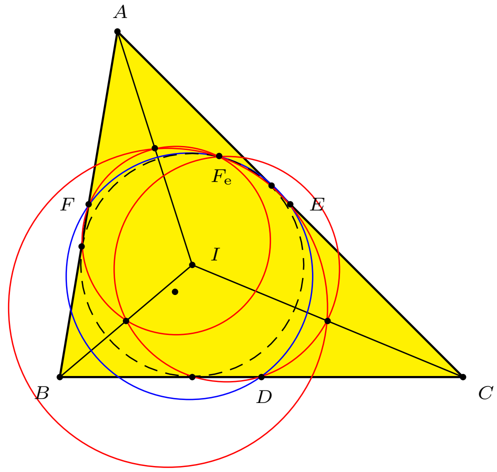

Figure 2.

Proof. The Poncelet point of A, B, C, I must lie on the pedal circle of I with respect to triangle ABC, and on the nine-point circle of triangle ABC (see Figure 1). Since these two circles have only the Feuerbach point F_e in common, it must be the Poncelet point of A, B, C, I. □

A second theorem, conjectured by Antreas Hatzipolakis, involves three curious triangles which turn out to have some very surprising and beautiful properties. We begin with an important lemma, appearing in [9] as Lemma 2 with a synthetic proof. The midpoints of BC, AC, AB are labeled D, E, F.

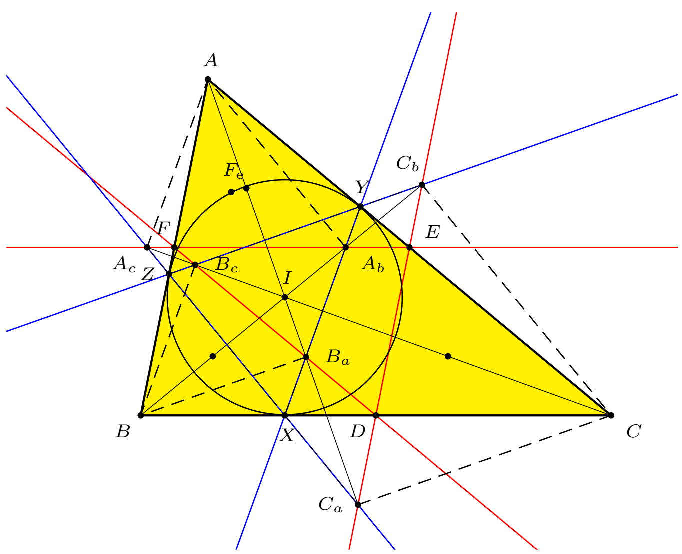

Figure 3.

We shall adopt the notations of [9]. Given a triangle ABC, let D, E, F be the midpoints of the sides BC, CA, AB, and X, Y, Z the points of tangency of the incircle with these sides. Let A_b and A_c be the orthogonal projections of A on the bisectors BI and CI respectively. Similarly define B_c, B_a, C_a, C_b (see Figure 3).

Lemma 2. (a) *A_b and A_c lie on EF.*
(b) *A_b lies on XY, A_c lies on XZ.*
Similar statements are true for B_a, B_c and C_a, C_b.

We are now ready for the second theorem, stated in [6]. An elementary proof was given by Khoa Lu Nguyen in [7]. We give a different proof, relying on the Kariya theorem (see [5]), which states that if X', Y', Z' are three points on IX, IY, IZ with $\frac{IX'}{IX} = \frac{IY'}{IY} = \frac{IZ'}{IZ} = k$, then the lines AX', BY', CZ' are concurrent. For $k = -2$, this point of concurrency is known to be X_{80}, the reflection of I in $F_{\rm e}$.

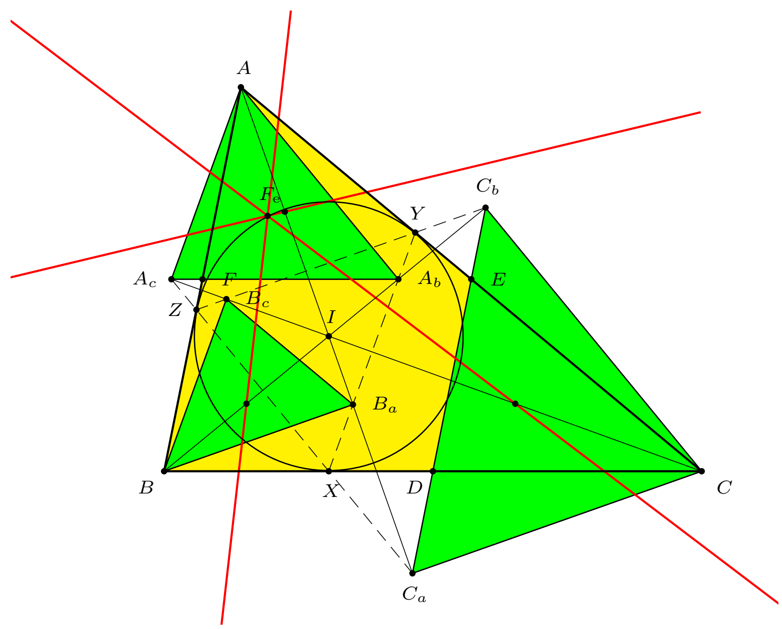

Figure 4.

Theorem 3 (Hatzipolakis)**.** *The Euler lines of triangles AA_bA_c, BB_aB_c, CC_aC_b are concurrent at $F_{\rm e}$* (see Figure 4).

Proof. If X' is the antipode of X in the incircle, O_a the midpoint of A and I, H_a the orthocenter of triangle AA_bA_c, then clearly H_aO_a is the Euler line of triangle AA_bA_c. Also, $\angle A_bAA_c = \pi - \angle A_cIA_b = \frac{B+C}{2}$. Because AI is a diameter of the circumcircle of triangle AA_bA_c, it follows that $AH_a = AI \cdot \cos\frac{B+C}{2} = AI \cdot \sin\frac{A}{2} = r$, where r is the inradius of triangle ABC. Clearly, $IX' = r$, and it follows from Lemma 1 that $AH_a \parallel IX'$. Triangles AH_aO_a and $IX'O_a$ are congruent, and X' is the reflection of H_a in O_a. Hence X' lies on the Euler line of triangle AA_bA_c.

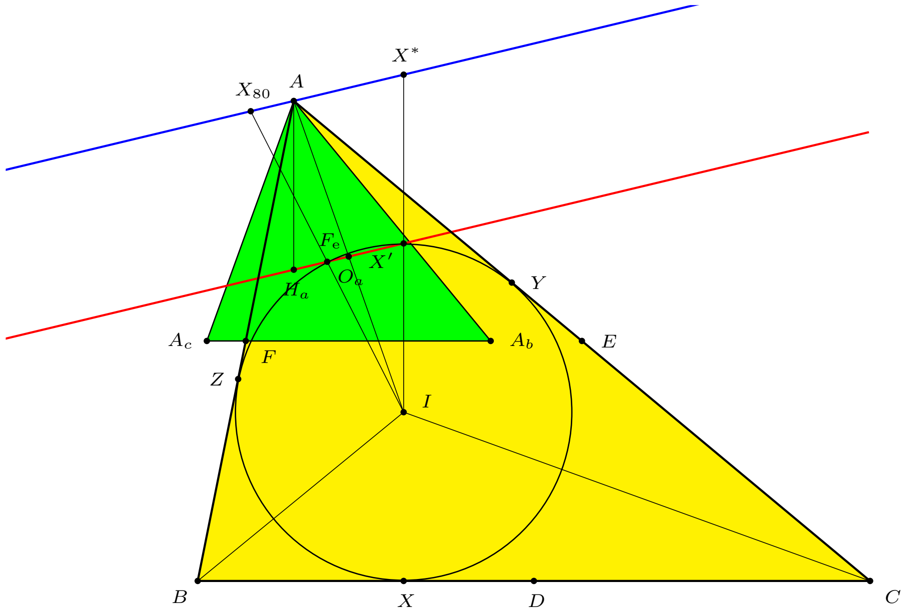

Figure 5.

If X^* is the reflection of I in X', we know by the Kariya theorem that A, X^*, and X_{80} are collinear. Now the homothety $\mathsf{h}(I, \frac{1}{2})$ takes A to O_a, X^* to X', and X_{80} to the Feuerbach point F_e. $\square$

We establish one more theorem on the Feuerbach point. An equivalent formulation was posed as a problem in [10].

Theorem 4. *If X'', Y'', Z'' are the reflections of X, Y, Z in AI, BI, CI, then the lines DX'', EY'', FZ'' concur at the Feuerbach point F_e* (see Figure 6).

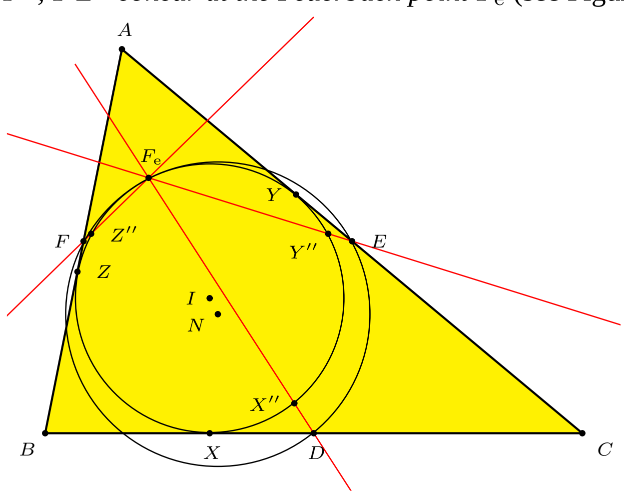

Figure 6.

Proof. We show that the line DX'' contains the Feuerbach point $F_{\rm e}$. The same reasoning will apply to EY'' and FZ'' as well.

Clearly, X'' lies on the incircle. If we call N the nine-point center of triangle ABC, then the theorem will follow from $IX'' \parallel ND$ since $F_{\rm e}$ is the external center of similitude of the incircle and nine-point circle of triangle ABC. Now, because $IX \parallel AH$, and because O and H are isogonal conjugates, $IX'' \parallel AO$. Furthermore, the homothety $\mathsf{h}(G, -2)$ takes D to A and N to O. This proves that $ND \parallel AO$. It follows that $IX'' \parallel ND$. □

2. The Euler reflection point

The following theorem was stated by Paul Yiu in [11], and proved by barycentric calculation in [8]. We give a synthetic proof of this result.

Theorem 5. *The reflections of OI in the sidelines of the intouch triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 7).

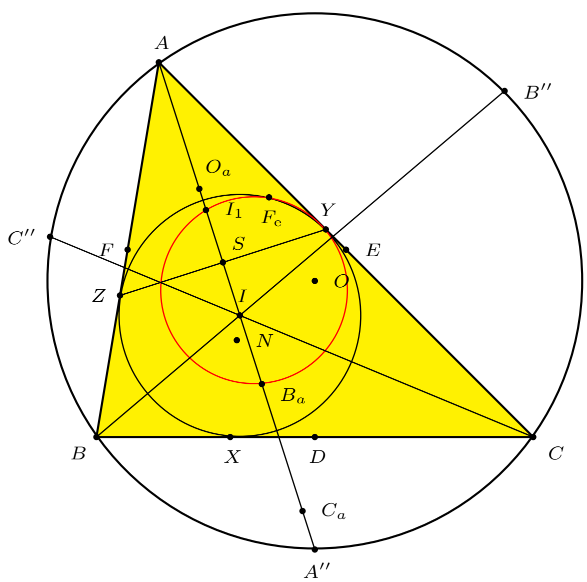

Figure 7.

Proof. Let us call I_1 the reflection of I in YZ. By Theorem 1, the nine-point circle of triangle AIC, which clearly passes through Y, O_a, C_a, also passes through $F_{\rm e}$. If S is the intersection of YZ and AI, then clearly A is the inverse of S with respect to the incircle. Because $2 \cdot IO_a = IA$ and $2 \cdot IS = II_1$, it follows that O_a is the inverse of I_1 with respect to the incircle. Because C_a lies on XZ, its polar line must pass through B and be perpendicular to AI. This shows that B_a is the inverse of C_a with respect to the incircle.

Now invert the nine-point circle of triangle AIC with respect to the incircle of triangle ABC. This circle can never pass through I since $\angle AIC > \frac{\pi}{2}$, so the image is a circle. This shows that $YI_1F_eB_a$ is a cyclic quadrilateral, so it follows that $\angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X$.

If we call $A''B''C''$ the circumcevian triangle of I, then we notice that $\angle AA_bA_c = \angle AIA_c = \angle A''IC$. Now, it is well known that $A''C = A''I$, so it follows that $\angle AA_bA_c = \angle ICA'' = \angle C''B''A''$. Similar arguments show that triangle AA_bA_c and triangle $A''B''C''$ are inversely similar.

As we have pointed out before as a consequence of Lemma 2, AH_a and IX' are parallel. By Theorem 3, F_eX' is the Euler line of triangle AA_bA_c. Therefor, $\angle F_eX'X = \angle O_aX'X = \angle O_aH_aA$. We know that triangle AA_bA_c is inversely similar to triangle $A''B''C''$. Since O and I are the circumcenter and orthocenter of triangle $A''B''C''$, it follows that $\angle O_aH_aA = \angle A''IO = \angle OIA$.

We conclude that $\angle F_eI_1S = \angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X = \angle AIO = \angle SIO$. This shows that the reflection of OI in EF passes through F_e. Similar arguments for the reflections of OI in XY and XZ complete the proof. □

A very similar result is stated in the following theorem. We give a synthetic proof, similar to the proof of the last theorem in many ways. First, we will need another lemma.

Lemma 6. *The vertices of the polar triangle of DEF with respect to the incircle are the orthocenters of triangles BIC, AIC, AIB. Furthermore, they are the reflections of the excenters in the respective midpoints of the sides.*

This triangle is the main subject of [9], in which a synthetic proof can be found.

Theorem 7. *The reflections of OI in the sidelines of the medial triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 8).

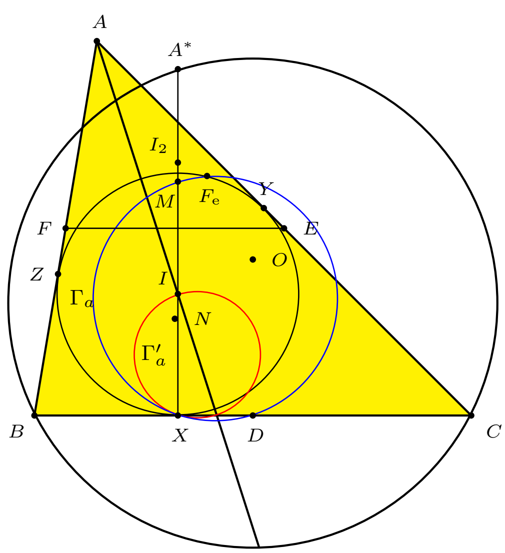

Figure 8.

Proof. Call I_2 the reflection of I in EF, and A^* the orthocenter of triangle BIC. The midpoint of I and A^* is called M. Using Lemma 6, we know that EF is the polar line of A^* with respect to the incircle. A similar argument as the one we gave in the proof of Theorem 5 shows that I_2 is the inverse of M with respect to the incircle.

Clearly, $F_{\rm e}$, M, X, D all lie on the nine-point circle of triangle BIC. Call this circle Γ_a and call Γ'_a the circumcircle of triangle IXX''. Clearly, the center of Γ'_a is on AI. Because I is the orthocenter of triangle BA^*C, we have that the reflection of I in D is the antipode of A^* in the circumcircle of A^*BC. Call this point L'. Consider the homothety $\mathsf{h}(I, 2)$, MD is mapped, and hence is parallel, to A^*L'. We know that A^* is the reflection in D of the A-excenter of triangle ABC (see [9]), so A^*L' is also parallel to AI. It follows that AI and MD are parallel.

If we call T the intersection of AI and BC, then it is clear that T lies on Γ'_a. Because IT and MD are parallel diameters of two circles, there exists a homothety centered at X which maps Γ'_a to Γ_a. Because X lies on both circles, we now conclude that X is the point of tangency of Γ_a and Γ'_a. Inverting these two circles in the incircle, we see that XX'' is tangent to the circumcircle of $XF_{\rm e}I_2$.

Finally, $\angle MIO = \angle AIO + \angle MIA = \angle F_{\rm e}X'X + \angle IMD = \angle F_{\rm e}XD + \angle XF_{\rm e}D = \angle F_{\rm e}XD + \angle DXX'' = \angle F_{\rm e}XX'' = \angle F_{\rm e}I_2X$, where the last equation follows from the alternate segment theorem. This proves that $I_2F_{\rm e}$ is the reflection of OI in EF. Similar arguments for DF and DE prove the theorem. □

The following theorem gives new evidence for the strong correlation between the nature of the Feuerbach point and the Euler reflection point.

Theorem 8. *The three reflections of H_aO_a in the sidelines of triangle AA_bA_c and the line OI are concurrent at the reflection E_a of $F_{\rm e}$ in A_bA_c. Similar theorems hold for triangles BB_aB_c, CC_aC_b* (see Figure 9).

Proof. The 3 reflections of H_aO_a in the sidelines of triangle AA_bA_c are concurrent at the Euler reflection point of triangle AA_bA_c. We will first show that this point is the reflection of $F_{\rm e}$ in A_bA_c.

The circle with diameter XH_a clearly passes through A_b, A_c by definition of A_b, A_c. It also passes through $F_{\rm e}$, since $H_aF_{\rm e} = X'F_{\rm e} \perp XF_{\rm e}$, so we conclude that $F_{\rm e}$, A_c, X, A_b are concyclic. Because AA_cXA_b is a parallellogram, we see that the reflection in the midpoint of A_b and A_c of the circle through A_b, A_c, X, $F_{\rm e}$ is in fact the circumcircle of triangle AA_bA_c. We deduce that the reflection of $F_{\rm e}$ in A_bA_c lies on the circumcircle of triangle AA_bA_c. Since $F_{\rm e} \neq H_a$ lies on the Euler line of triangle AA_bA_c and E_a lies on the circumcircle of triangle AA_bA_c, we have proven that the reflection of $F_{\rm e}$ in A_bA_c is the Euler reflection point of triangle AA_bA_c.

By theorem 7, it immediately follows that E_a lies on OI. This completes the proof. □

We know that we can see E_a as an intersection point of the perpendicular to A_bA_c through $F_{\rm e}$ with the circumcircle of triangle AA_bA_c. This line intersects the

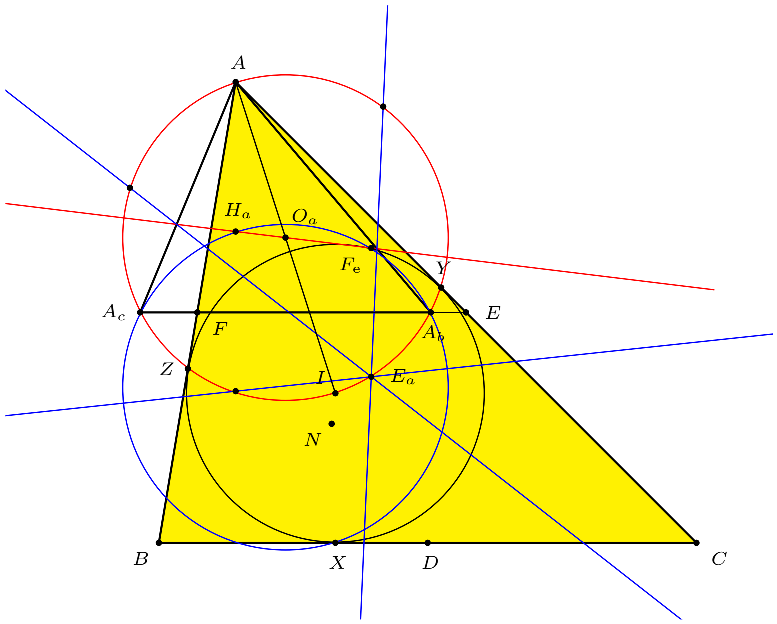

Figure 9.

circle in another point, which we will call U. Similarly define V and W on the circumcircles of triangles BB_aB_c and CC_aC_b.

Theorem 9. *The lines AU, BV, CW are concurrent at X_{80}, the reflection of I in* $F_{\rm e}$ (see Figure 10).

Proof. The previous theorem tells us that E_a lies on OI. It follows that $\angle E_aIO_a = \angle OIA$. In the proof of Theorem 5, we prove that $\angle AIO = \angle AH_aO_a$. Since F_eE_a and AH_a are parallel, we deduce that E_a, I, O_a and F_e are concyclic. If we call U' the intersection of $E_aF_{\rm e}$ and the line through A parallel to $O_aF_{\rm e}$, then we have that $\angle E_aIA = \angle E_aF_{\rm e}O_a = \angle E_aU'A$. It follows that A, U', E_a, I are concyclic, so $U \equiv U'$.

Now consider a homothety centered at I with factor 2. Clearly, $O_aF_{\rm e}$ is mapped to a parallel line through A, which is shown to pass through U. The image of $F_{\rm e}$ however is X_{80}, so AU passes through X_{80}. Similar arguments for BV, CW complete the proof. □

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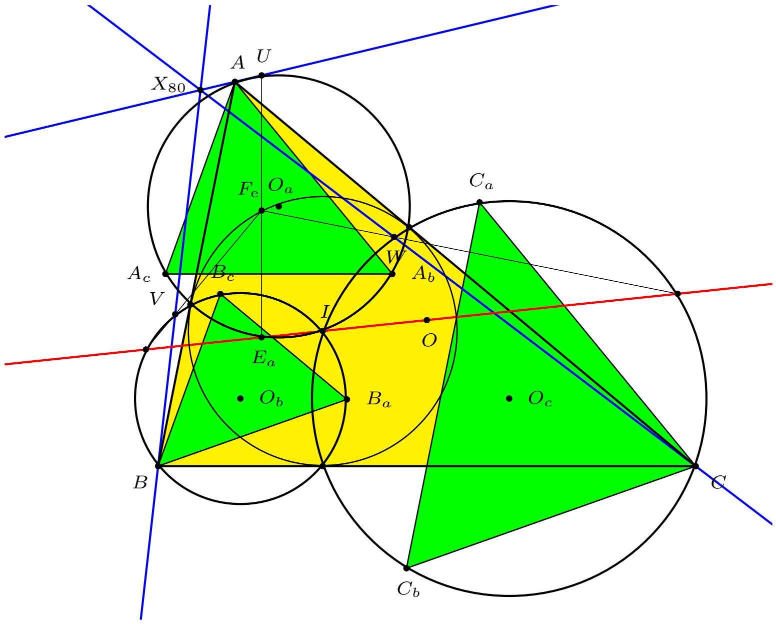

Figure 10.

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Jan Vonk: Groenstraat 70, 9340 Lede, Belgium
E-mail address: `jan.vonk.jv@gmail.com`

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Rings of Squares Around Orthologic Triangles

Zvonko Čerin

Abstract. We explore some properties of the geometric configuration when a ring of six squares with the same orientation are erected on the segments BD, DC, CE, EA, AF and FB connecting the vertices of two orthologic triangles ABC and DEF. The special case when DEF is the pedal triangle of a variable point P with respect to the triangle ABC was studied earlier by Bottema [1], Deaux [5], Erhmann and Lamoen [4], and Sashalmi and Hoffmann [8]. We extend their results and discover several new properties of this interesting configuration.

1. Introduction – Bottema's Theorem

The orthogonal projections P_a, P_b and P_c of a point P onto the sidelines BC, CA and AB of the triangle ABC are vertices of its pedal triangle.

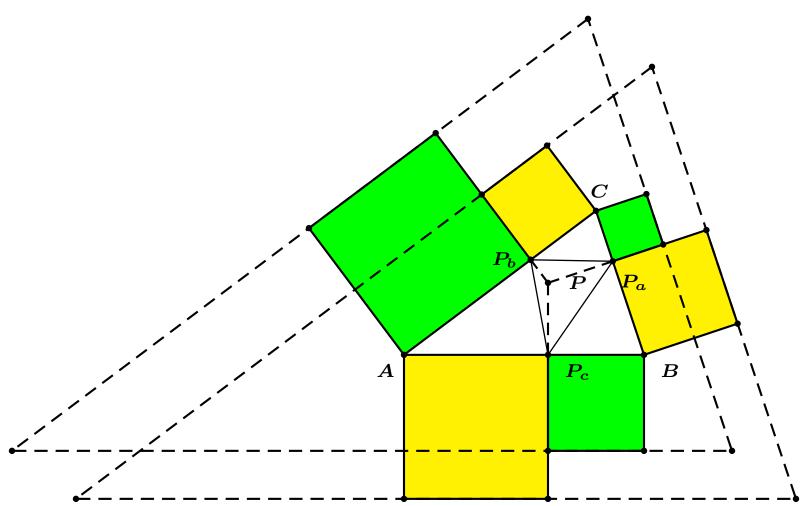

Figure 1. Bottema's Theorem on sums of areas of squares.

In [1], Bottema made the remarkable observation that

$$|BP_a|^2 + |CP_b|^2 + |AP_c|^2 = |P_aC|^2 + |P_bA|^2 + |P_cB|^2.$$

This equation has an interpretation in terms of area which is illustrated in Figure 1. Rather than using geometric squares, other similar figures may be used as in [8].

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Figure 1 also shows two congruent triangles homothetic with the triangle ABC that are studied in [4] and [8].

The primary purpose of this paper is to extend Bottema's Theorem (see Figure 2). The longer version of this paper is available at the author's Web home page `http://math.hr/~cerin/`. We thank the referee for many useful suggestions that improved greatly our exposition.

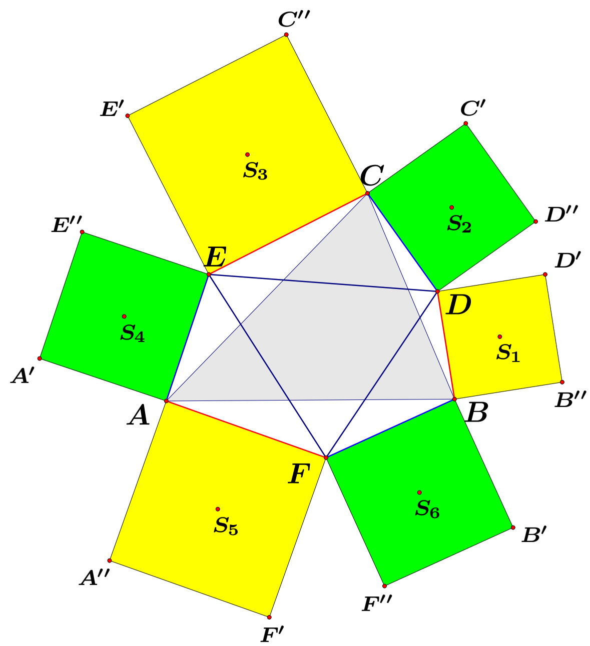

Figure 2. Notation for a ring of six squares around two triangles.

2. Connection with orthology

The origin of our generalization comes from asking if it is possible to replace the pedal triangle $P_aP_bP_c$ in Bottema's Theorem with some other triangles. In other words, if ABC and DEF are triangles in the plane, when will the following equality hold?

$$|BD|^2 + |CE|^2 + |AF|^2 = |DC|^2 + |EA|^2 + |FB|^2 \tag{1}$$

The straightforward analytic attempt to answer this question gives the following simple characterization of the equality (1).

Throughout, triangles will be non-degenerate.

Theorem 1. *The relation (1) holds for triangles ABC and DEF if and only if they are orthologic.*

Recall that triangles ABC and DEF are *orthologic* provided the perpendiculars at vertices of ABC onto sides EF, FD and DE of DEF are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, DEF]$. It is

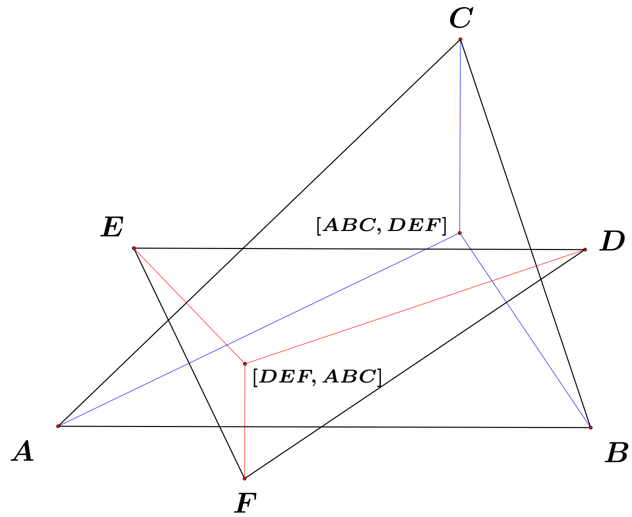

Figure 3. The triangles ABC and DEF are orthologic.

well-known that this relation is reflexive and symmetric. Hence, the perpendiculars from vertices of DEF onto the sides BC, CA, and AB are concurrent at the point $[DEF,\, ABC\,]$. These points are called the *first* and *second orthology centers* of the (orthologic) triangles ABC and DEF.

It is obvious that a triangle and the pedal triangle of any point are orthologic so that Theorem 1 extends Bottema's Theorem and the results in [8] (Theorem 3 and the first part of Theorem 5).

Proof. The proofs in this paper will all be analytic.

In the rectangular co-ordinate system in the plane, we shall assume throughout that $A(0,\,0)$, $B(1,\,0)$, $C(u,\,v)$, $D(d,\,\delta)$, $E(e,\,\varepsilon)$ and $F(f,\,\varphi)$ for real numbers $u,\ v,\ d,\ \delta,\ e,\ \varepsilon,\ f$ and φ. The lines will be treated as ordered triples of coefficients $(a,\,b,\,c)$ of their (linear) equations $a\,x + b\,y + c = 0$. Hence, the perpendiculars from the vertices of DEF onto the corresponding sidelines of ABC are $(u-1,\,v,\,d(1-u) - v\,\delta)$, $(u,\,v,\,-(u\,e + v\,\varepsilon))$ and $(1,\,0,\,-f)$. They will be concurrent provided the determinant $v\,\Delta = v((u-1)\,d - ue + f + v\,(\delta - \varepsilon))$ of the matrix from them as rows is equal to zero. In other words, $\Delta = 0$ is a necessary and sufficient condition for ABC and DEF to be orthologic.

On the other hand, the difference of the right and the left side of (1) is $2\,\Delta$ which clearly implies that (1) holds if and only if ABC and DEF are orthologic triangles. □

3. The triangles $S_1S_3S_5$ and $S_2S_4S_6$

We continue our study of the ring of six squares with the Theorem 2 about two triangles associated with the configuration. Like Theorem 1, this theorem detects when two triangles are orthologic. Recall that $S_1, \ldots,\ S_6$ are the centers of the squares in Figure 2. Note that a similar result holds when the squares are folded inwards, and the proof is omitted.

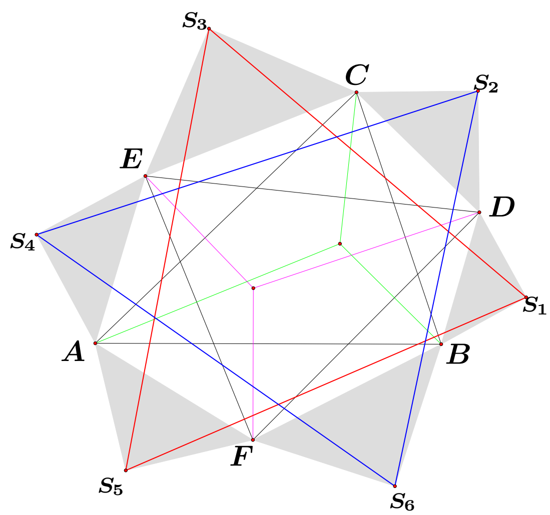

Figure 4. $|S_1S_3S_5| = |S_2S_4S_6|$ iff ABC and DEF are orthologic.

Theorem 2. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. The vertices V and U of the square $DEVU$ have co-ordinates $(e+\varepsilon-\delta,\, \varepsilon+d-e)$ and $(d+\varepsilon-\delta,\, \delta+d-e)$. From this we infer easily co-ordinates of all points in Figure 2. With the notation $u_+ = u+v$, $u_- = u-v$, $d_+ = d+\delta$, $d_- = d-\delta$, $e_+ = e+\varepsilon$, $e_- = e-\varepsilon$, $f_+ = f+\varphi$ and $f_- = f-\varphi$ they are the following.

$$A'(-\varepsilon,\, e), \qquad A''(\varphi,\, -f), \qquad B'(1-\varphi,\, f-1), \qquad B''(1+\delta,\, 1-d),$$
$$C'(u_+-\delta,\, u_-+d), \quad C''(u_-+\varepsilon,\, u_+-e), \quad D'(d_+,\, 1-d_-),$$
$$D''(d_-+v,\, d_+-u), \qquad E'(e_+-v,\, u-e_-), \qquad E''(e_-,\, e_+),$$
$$F'(f_+,\, -f_-), \quad F''(f_-,\, f_+-1), \quad S_1\left(\tfrac{1+d_+}{2},\, \tfrac{1-d_-}{2}\right), \quad S_2\left(\tfrac{d_-+u_+}{2},\, \tfrac{d_+-u_-}{2}\right),$$
$$S_3\left(\tfrac{u_-+e_+}{2},\, \tfrac{u_+-e_-}{2}\right), \quad S_4\left(\tfrac{e_-}{2},\, \tfrac{e_+}{2}\right), \quad S_5\left(\tfrac{f_+}{2},\, -\tfrac{f_-}{2}\right), \quad S_6\left(\tfrac{f_-+1}{2},\, \tfrac{f_+-1}{2}\right).$$

Let P^x and P^y be the x– and y– co-ordinates of the point P. Since the area $|DEF|$ is a half of the determinant of the matrix with the rows $(D^x,\, D^y,\, 1)$, $(E^x,\, E^y,\, 1)$ and $(F^x,\, F^y,\, 1)$, the difference $|S_2S_4S_6| - |S_1S_3S_5|$ is $\frac{\Delta}{4}$. We conclude that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

4. The first family of pairs of triangles

The triangles $S_1S_3S_5$ and $S_2S_4S_6$ are just one pair from a whole family of triangle pairs which all have the same property with a single notable exception.

For any real number t different from -1 and 0, let $S_1^t, \ldots, S_6^t$ denote points that divide the segments AS_1, AS_2, BS_3, BS_4, CS_5 and CS_6 in the ratio $t:1$. Let $\rho(P, \theta)$ denote the rotation about the point P through an angle θ. Let G_σ and G_τ be the centroids of ABC and DEF.

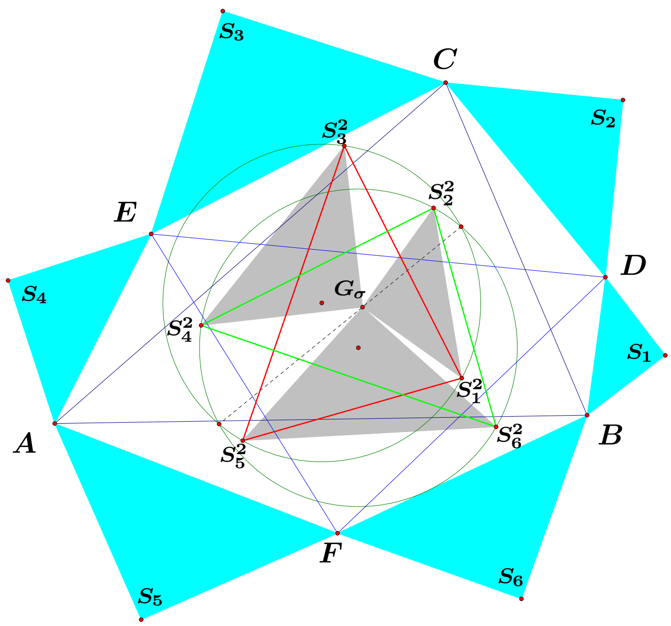

Figure 5. The triangles $S_1^2S_3^2S_5^2$ and $S_2^2S_4^2S_6^2$ are congruent.

The following result is curios (See Figure 5) because the particular value $t = 2$ gives a pair of congruent triangles regardless of the position of the triangles ABC and DEF.

Theorem 3. *The triangle $S_2^2S_4^2S_6^2$ is the image of the triangle $S_1^2S_3^2S_5^2$ under the rotation $\rho\left(G_\sigma, \frac{\pi}{2}\right)$. The radical axis of their circumcircles goes through the centroid G_σ.*

Proof. Since the point that divides the segment DE in the ratio $2:1$ has coordinates $\left(\frac{d+2\,e}{3}, \frac{\delta+2\,\varepsilon}{3}\right)$, it follows that

$$S_1^2\left(\frac{1+d_+}{3}, \frac{1-d_-}{3}\right) \quad \text{and} \quad S_2^2\left(\frac{d_-+u_+}{3}, \frac{d_+-u_-}{3}\right).$$

Since $G_\sigma\left(\frac{1+u}{3}, \frac{v}{3}\right)$, it is easy to check that S_2^2 is the vertex of a (negatively oriented) square on $G_\sigma S_1^2$. The arguments for the pairs $\left(S_3^2, S_4^2,\right)$ and $\left(S_5^2, S_6^2,\right)$ are analogous.

Finally, the proof of the claim about the radical axis starts with the observation that since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent it suffices to show that $|G_\sigma O_{odd}|^2 = |G_\sigma O_{even}|^2$, where O_{odd} and O_{even} are their circumcenters. This routine task was accomplished with the assistance of a computer algebra system. □

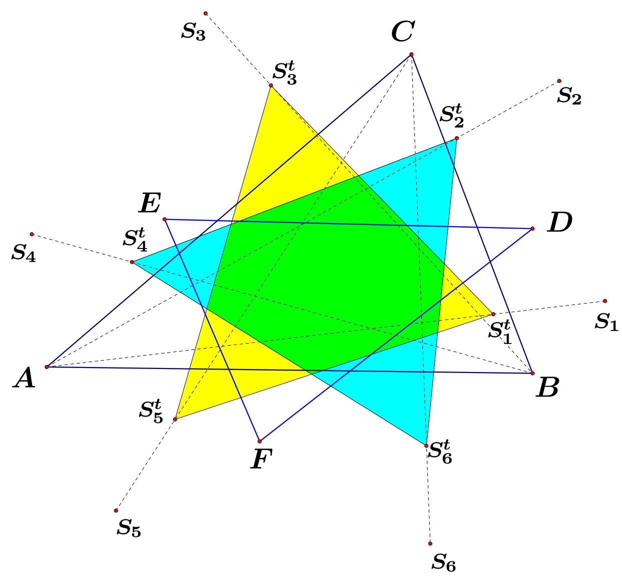

Figure 6. $|S_1^tS_3^tS_5^t| = |S_2^tS_4^tS_6^t|$ iff ABC and DEF are orthologic.

The following result resembles Theorem 2 (see Figure 6) and shows that each pair of triangles from the first family could be used to detect if the triangles ABC and DEF are orthologic.

Theorem 4. *For any real number t different from -1, 0 and 2, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. Since the point that divides the segment DE in the ratio $t : 1$ has co-ordinates $\left(\frac{d+te}{t+1}, \frac{\delta+t\varepsilon}{t+1}\right)$, it follows that the points S_i^t have the co-ordinates

$$S_1^t\left(\tfrac{t(1+d_+)}{2(t+1)}, \tfrac{t(1-d_-)}{2(t+1)}\right),\ S_2^t\left(\tfrac{t(d_-+u_+)}{2(t+1)}, \tfrac{t(d_+-u_-)}{2(t+1)}\right),\ S_3^t\left(\tfrac{2+t(u_-+e_+)}{2(t+1)}, \tfrac{t(u_+-e_-)}{2(t+1)}\right),$$

$$S_4^t\left(\tfrac{2+t\,e_-}{2(t+1)}, \tfrac{t\,e_+}{2(t+1)}\right),\quad S_5^t\left(\tfrac{2\,u+t\,f_+}{2(t+1)}, \tfrac{2\,v-t\,f_-}{2(t+1)}\right),\quad S_6^t\left(\tfrac{2\,u+t(1+f_-)}{2(t+1)}, \tfrac{2\,v-t(1-f_+)}{2(t+1)}\right).$$

As in the proof of Theorem 2, we find that the difference of areas of the triangles $S_2^tS_4^tS_6^t$ and $S_1^tS_3^tS_5^t$ is $\frac{t(2-t)\Delta}{4(t+1)^2}$. Hence, for $t \neq -1,\ 0,\ 2$, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. □

5. The second family of pairs of triangles

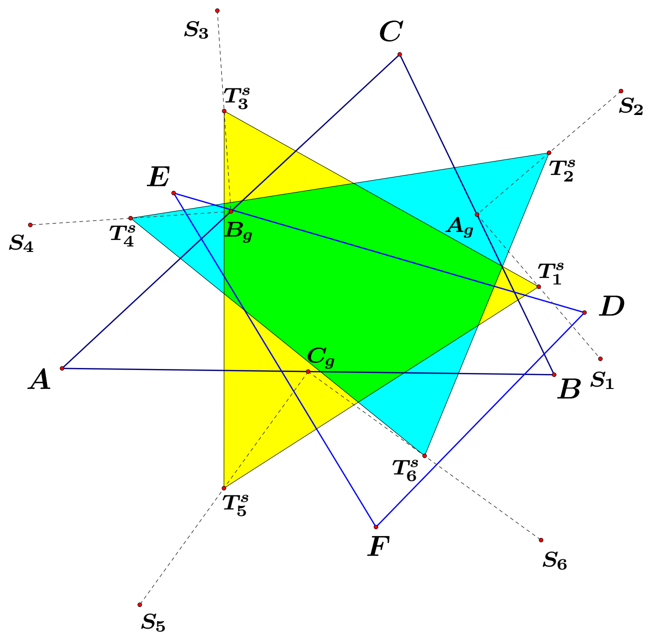

Figure 7. $|T_1^sT_3^sT_5^s| = |T_2^sT_4^sT_6^s|$ iff ABC and DEF are orthologic.

The first family of pairs of triangles was constructed on lines joining the centers of the squares with the vertices A, B and C. In order to get the second analogous family we shall use instead lines joining midpoints of sides with the centers of the squares (see Figure 7). A slight advantage of the second family is that it has no exceptional cases.

Let A_g, B_g and C_g denote the midpoints of the segments BC, CA and AB. For any real number s different from -1, let T_1^s, …, T_6^s denote points that divide the segments A_gS_1, A_gS_2, B_gS_3, B_gS_4, C_gS_5 and C_gS_6 in the ratio $s : 1$. Notice that $T_1^sT_2^sA_g$, $T_3^sT_4^sB_g$ and $T_5^sT_6^sC_g$ are isosceles triangles with the right angles at the vertices A_g, B_g and C_g.

Theorem 5. *For any real number s different from -1 and 0, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. As in the proof of Theorem 4, we find that the difference of areas of the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ is $\frac{s\,\Delta}{4(s+1)}$. Hence, for $s \neq -1,\ 0$, the triangles $T_1^tT_3^tT_5^t$ and $T_2^tT_4^tT_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. □

6. The third family of pairs of triangles

When we look for reasons why the previous two families served our purpose of detecting orthology it is clear that the vertices of a triangle homothetic with ABC should be used. This leads us to consider a family of pairs of triangles that depend on two real parameters and a point (the center of homothety).

For any real numbers s and t different from -1 and any point P the points X, Y and Z divide the segments PA, PB and PC in the ratio $s : 1$ while the points $U_i^{(s,t)}$ for $i = 1, \dots, 6$ divide the segments XS_1, XS_2, YS_3, YS_4, ZS_5 and ZS_6 in the ratio $t : 1$.

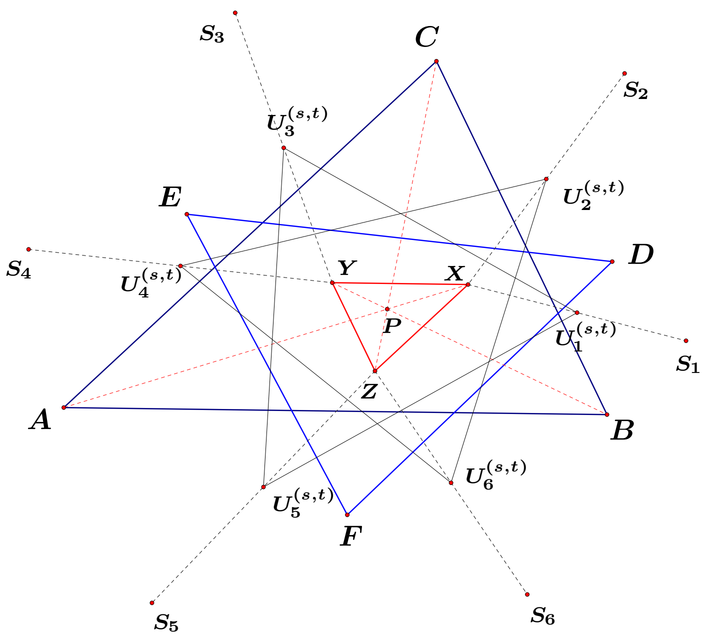

Figure 8. $|U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}| = |U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}|$ iff ABC and DEF are orthologic.

The above results (Theorems 4 and 5) are special cases of the following theorem (see Figure 8).

Theorem 6. *For any point P and any real numbers $s \neq -1$ and $t \neq -1, \frac{2\,s}{s+1}$, the triangles $U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}$ and $U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}$ have equal areas if and only if the triangles ABC and DEF are orthologic.*

The proof is routine. See that of Theorem 4.

7. The triangles $A_0B_0C_0$ and $D_0E_0F_0$

In this section we shall see that the midpoints of the sides of the hexagon $S_1S_2S_3S_4S_5S_6$ also have some interesting properties.

Let A_0, B_0, C_0, D_0, E_0 and F_0 be the midpoints of the segments S_1S_2, S_3S_4, S_5S_6, S_4S_5, S_6S_1 and S_2S_3. Notice that the triangles $A_0B_0C_0$ and $D_0E_0F_0$ have as centroid the midpoint of the segment $G_\sigma G_\tau$.

Recall that triangles ABC and XYZ are *homologic* provided the lines AX, BY, and CZ are concurrent. In stead of homologic many authors use *perspective*.

Theorem 7. (a) *The triangles ABC and $A_0B_0C_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.*

(b) *The triangles DEF and $D_0E_0F_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.*

(c) *If the triangles ABC and DEF are orthologic, then the triangles $A_0B_0C_0$ and $D_0E_0F_0$ are homologic.*

Proof. Let $D_1(d_1,\,\delta_1)$, $E_1(e_1,\,\varepsilon_1)$ and $F_1(f_1,\,\varphi_1)$. Recall from [2] that the triangles DEF and $D_1E_1F_1$ are orthologic if and only if $\Delta_0 = 0$, where

$$\Delta_0 = \Delta_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} d & d_1 & 1 \\ e & e_1 & 1 \\ f & f_1 & 1 \end{vmatrix} + \begin{vmatrix} \delta & \delta_1 & 1 \\ \varepsilon & \varepsilon_1 & 1 \\ \varphi & \varphi_1 & 1 \end{vmatrix}.$$

Then (a) and (b) follow from the relations

$$\Delta_0(ABC,\, A_0B_0C_0) = -\frac{\Delta}{2} \quad \text{and} \quad \Delta_0(DEF,\, D_0E_0F_0) = \frac{\Delta}{2}.$$

The line DD_1 is $(\delta-\delta_1, d_1-d, \delta_1\,d - d_1\,\delta)$, so that the triangles DEF and $D_1E_1F_1$ are homologic if and only if $\Gamma_0 = 0$, where

$$\Gamma_0 = \Gamma_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} \delta-\delta_1 & d_1-d & \delta_1\,d - d_1\,\delta \\ \varepsilon-\varepsilon_1 & e_1-e & \varepsilon_1\,e - e_1\,\varepsilon \\ \varphi-\varphi_1 & f_1-f & \varphi_1\,f - f_1\,\varphi \end{vmatrix}.$$

Part (c) follows from the observation that $\Gamma_0(A_0B_0C_0,\, D_0E_0F_0)$ contains Δ as a factor. □

8. Triangles from centroids

Let G_1, G_2, G_3 and G_4 denote the centroids of the triangles $G_{12A}G_{34B}G_{56C}$, $G_{12D}G_{34E}G_{56F}$, $G_{45A}G_{61B}G_{23C}$ and $G_{45D}G_{61E}G_{23F}$ where G_{12A}, G_{12D}, G_{34B}, G_{34E}, G_{56C}, G_{56F}, G_{45A}, G_{45D}, G_{61B}, G_{61E}, G_{23C} and G_{23F} are centroids of the triangles S_1S_2A, S_1S_2D, S_3S_4B, S_3S_4E, S_5S_6C, S_5S_6F, S_4S_5A, S_4S_5D, S_6S_1B, S_6S_1E, S_2S_3C and S_2S_3F.

Theorem 8. *The points G_1 and G_2 are the points G_3 and G_4 respectively. The points G_1 and G_2 divide the segments $G_\sigma G_\tau$ and $G_\tau G_\sigma$ in the ratio $1:2$.*

Proof. The centroids G_{12A}, G_{34B} and G_{56C} have the co-ordinates $\left(\frac{2\,d+1+v+u}{6},\, \frac{2\,\delta+1+v-u}{6}\right)$, $\left(\frac{2(e+1)+u-v}{6},\, \frac{2\,\varepsilon+u+v}{6}\right)$ and $\left(\frac{2(f+u)+1}{6},\, \frac{2(\varphi+v)-1}{6}\right)$. It follows that G_1 and G_2 have coordinates $\left(\frac{d+e+f+2(u+1)}{9},\, \frac{\delta+\varepsilon+\varphi+2\,v}{9}\right)$ and $\left(\frac{2(d+e+f)+u+1}{9},\, \frac{2(\delta+\varepsilon+\varphi)+v}{9}\right)$ respectively. It is now easy to check that $G_3 = G_1$ and $G_4 = G_2$.

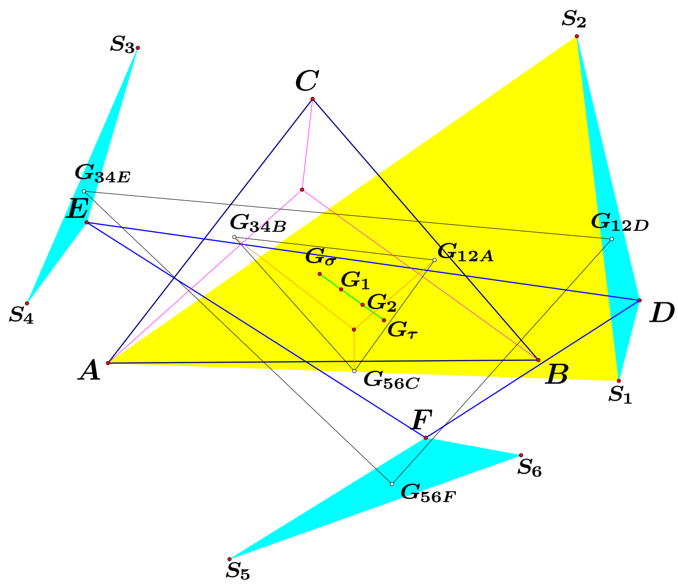

Figure 9. G_1 and G_2 divide $G_\sigma G_\tau$ in four equal parts and ABC is orthologic with $G_{12A}G_{34B}G_{56C}$ iff it is orthologic with DEF.

Let G_1' divide the segment $G_\sigma G_\tau$ in the ratio $1:2$. Since $G_\tau\left(\frac{d+e+f}{3}, \frac{\delta+\varepsilon+\varphi}{3}\right)$ and $G_\sigma\left(\frac{1+u}{3}, \frac{v}{3}\right)$, we have $(G_1')^x = \frac{2\,(G_\sigma)^x+(G_\tau)^x}{3} = \frac{(2+2\,u)+(d+e+f)}{9} = (G_1)^x$. Of course, in the same way we see that $(G_1')^y = (G_1)^y$ and that G_2 divides $G_\tau G_\sigma$ in the same ratio $1:2$. □

Theorem 9. *The following statements are equivalent:*

(a) *The triangles ABC and $G_{12A}G_{34B}G_{56C}$ are orthologic.*
(b) *The triangles ABC and $G_{12D}G_{34E}G_{56F}$ are orthologic.*
(c) *The triangles DEF and $G_{45A}G_{61B}G_{23C}$ are orthologic.*
(d) *The triangles DEF and $G_{45D}G_{61E}G_{23F}$ are orthologic.*
(e) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*
(f) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*
(g) *The triangles ABC and DEF are orthologic.*

Proof. The equivalence of (a) and (g) follows from the relation

$$\Delta_0(ABC,\, G_{12A}G_{34B}G_{56C}) = \frac{\Delta}{3}.$$

The equivalence of (g) with (b), (c), (d), (e) and (f) one can prove in the same way. □

9. Four triangles on vertices of squares

In this section we consider four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$, $D''E''F''$ which have twelve outer vertices of the squares as vertices. The sum of areas of the first two is equal to the sum of areas of the last two. The same relation holds if we replace the word ”area” by the phrase ”sum of the squares of the sides”.

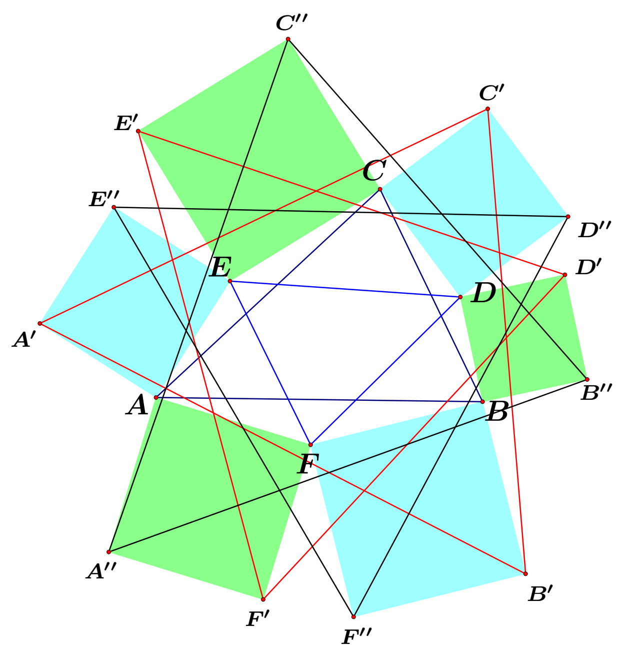

Figure 10. Four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$.

For a triangle XYZ let $|XYZ|$ and $s_2(XYZ)$ denote its (oriented) area and the sum $|YZ|^2 + |ZX|^2 + |XY|^2$ of squares of lengths of its sides.

Theorem 10. (a) *The following equality for areas of triangles holds:*

$$|A'B'C'| + |D'E'F'| = |A''B''C''| + |D''E''F''|.$$

(b) *The following equality also holds:*

$$s_2(A'B'C') + s_2(D'E'F') = s_2(A''B''C'') + s_2(D''E''F'').$$

The proofs of both parts can be accomplished by a routine calculation.

Let A'_1, B'_1 and C'_1 denote centers of squares of the same orientation built on the segments $B'C'$, $C'A'$ and $A'B'$. The points D'_1, E'_1, F'_1, A''_1, B''_1, C''_1, D''_1, E''_1 and F''_1 are defined analogously. Notice that $(A'B'C', A'_1B'_1C'_1)$, $(A''B''C'', A''_1B''_1C''_1)$, $(D'E'F', D'_1E'_1F'_1)$ and $(D''E''F'', D''_1E''_1F''_1)$ are four pairs of both orthologic and homologic triangles.

The following theorem claims that the four triangles from these centers of squares retain the same property regarding sums of areas and sums of squares of lengths of sides.

Theorem 11. (a) *The following equality for areas of triangles holds:*

$$|A'_1B'_1C'_1| + |D'_1E'_1F'_1| = |A''_1B''_1C''_1| + |D''_1E''_1F''_1|.$$

(b) *The following equality also holds:*

$$s_2(A_1'B_1'C_1') + s_2(D_1'E_1'F_1') = s_2(A_1''B_1''C_1'') + s_2(D_1''E_1''F_1'').$$

The proofs of both parts can be accomplished by a routine calculation.

Notice that in the above theorem we can take instead of the centers any points that have the same position with respect to the squares erected on the sides of the triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$. Also, there are obvious extensions of the previous two theorems from two triangles to the statements about two n-gons for any integer $n > 3$.

Of course, it is possible to continue the above sequences of triangles and define for every integer $k \geq 0$ the triangles $A_k'B_k'C_k'$, $A_k''B_k''C_k''$, $D_k'E_k'F_k'$ and $D_k''E_k''F_k''$. The sequences start with $A'B'C'$, $A''B''C''$, $D'E'F'$ and $D''E''F''$. Each member is homologic, orthologic, and shares the centroid with all previous members and for each k an analogue of Theorem 11 is true.

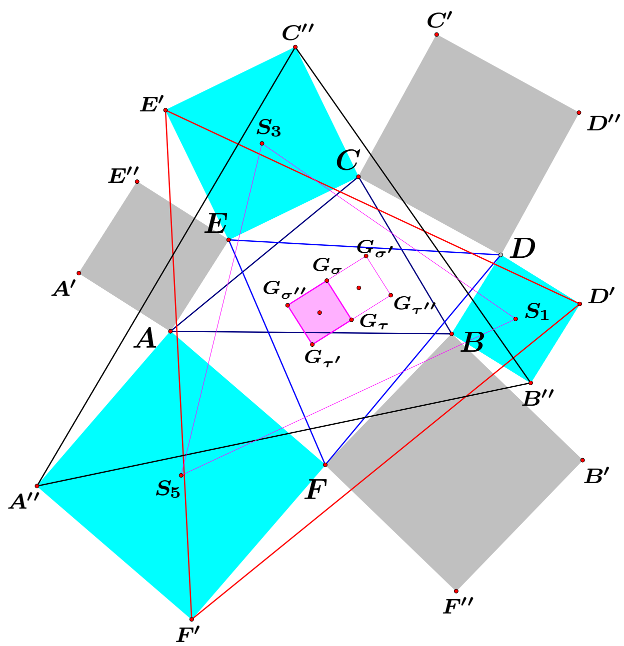

Figure 11. $G_\sigma G_\tau G_{\tau'} G_{\sigma''}$ and $G_\sigma G_\tau G_{\tau''} G_{\sigma'}$ are squares.

10. The centroids of the four triangles

Let $G_{\sigma'}$, $G_{\tau'}$, $G_{\sigma''}$, $G_{\tau''}$, G_o and G_e be shorter notation for the centroids $G_{A'B'C'}$, $G_{D'E'F'}$, $G_{A''B''C''}$, $G_{D''E''F''}$, $G_{S_1S_3S_5}$ and $G_{S_2S_4S_6}$. The following theorem shows that these centroids are the vertices of three squares associated with the ring of six squares.

Theorem 12. (a) *The centroids $G_{\sigma''}$, $G_{\tau'}$, G_τ and G_σ are vertices of a square.*

(b) *The centroids* $G_{\sigma'}$ *and* $G_{\tau''}$ *are reflections of the centroids* $G_{\sigma''}$ *and* $G_{\tau'}$ *in the line* $G_\sigma G_\tau$. *Hence, the centroids* $G_{\tau''}$, $G_{\sigma'}$, G_σ *and* G_τ *are also vertices of a square.*

(c) *The centroids* G_e *and* G_o *are the centers of the squares in* (a) *and (b), respectively. Hence, the centroids* G_σ, G_e, G_τ *and* G_o *are also vertices of a square.*

The proofs are routine.

11. Extension of Ehrmann–Lamoen results

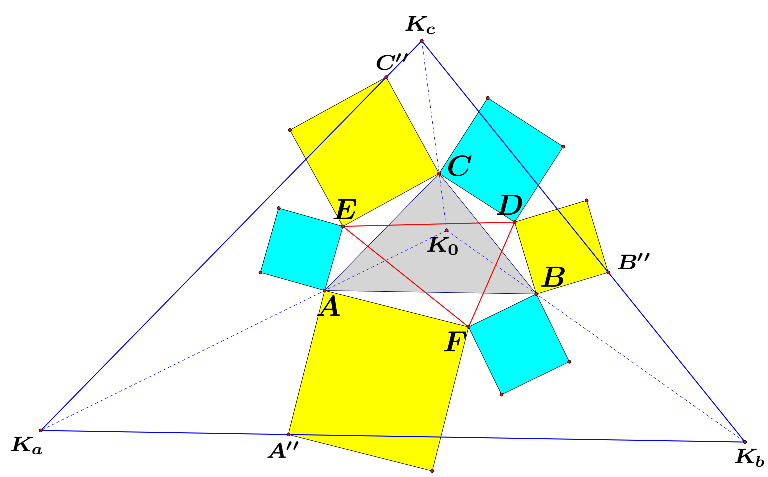

Figure 12. The triangle $K_aK_bK_c$ from parallels to BC, CA, AB through B'', C'', A'' is homothetic to ABC from the center K_0.

Let $K_aK_bK_c$ be a triangle from the intersections of parallels to the lines BC, CA and AB through the points B'', C'' and A''. Similarly, Let $K_aK_bK_c$ be a triangle from intersections of parallels to the lines BC, CA and AB through the points B'', C'' and A''. Similarly, the triangles $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are constructed in the same way through the triples of points (C', A', B'), (D'', E'', F''), (D', E', F'), (S_1, S_3, S_5) and (S_2, S_4, S_6), respectively. Some of these triangles have been considered in the case when the triangle DEF is the pedal triangle $P_aP_bP_c$ of the point P. Work has been done by Ehrmann and Lamoen in [4] and also by Hoffmann and Sashalmi in [8]. In this section we shall see that natural analogues of their results hold in more general situations.

Theorem 13. (a) *The triangles* $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ *and* $Q_aQ_bQ_c$ *are each homothetic with the triangle* ABC.

(b) *The quadrangles* $K_aL_aM_aN_a$, $K_bL_bM_bN_b$ *and* $K_cL_cM_cN_c$ *are parallelograms.*

(c) *The centers* J_a, J_b *and* J_c *of these parallelograms are the vertices of a triangle that is also homothetic with the triangle* ABC.

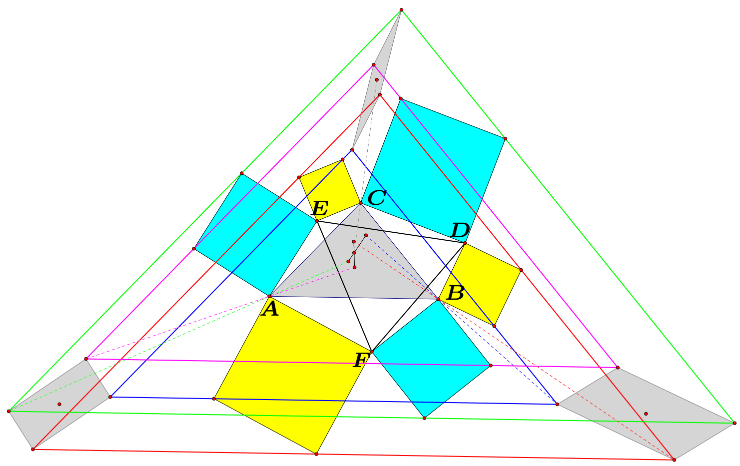

Figure 13. The triangles $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$ and $N_aN_bN_c$ together with three parallelograms.

Proof of parts (a) and (c) are routine while the simplest method to prove the part (b) is to show that the midpoints of the segments K_xM_x and L_xN_x coincide for $x = a,\, b,\, c$.

Let J_0, K_0, L_0, M_0, N_0, P_0 and Q_0 be centers of the above homotheties. Notice that J_0 is the intersection of the lines K_0M_0 and L_0N_0.

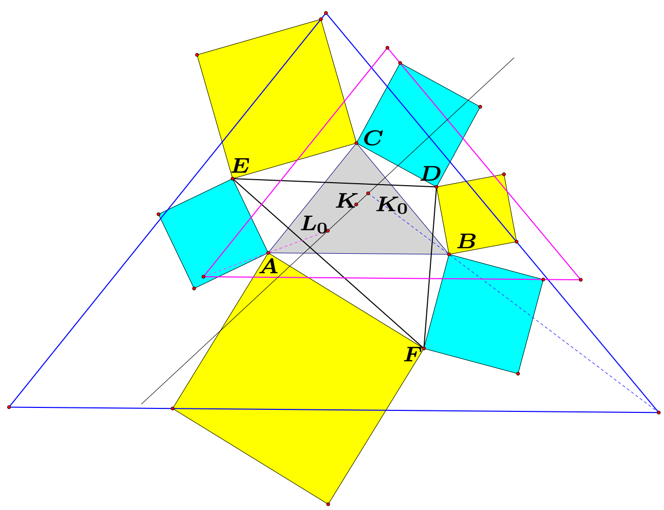

Figure 14. The line K_0L_0 goes through the symmedian point K of the triangle ABC.

Theorem 14. (a) *The symmedian point K of the triangle ABC lies on the line K_0L_0.*

(b) *The points P_0 and Q_0 coincide with the points N_0 and M_0.*

(c) *The equalities* $2 \cdot \overrightarrow{P_v Q_v} = \overrightarrow{K_v L_v}$ *hold for* $v = a,\, b,\, c$.

Proof. (a) It is straightforward to verify that the symmedian point with co-ordinates $\left(\frac{u^2+u+v^2}{2(u^2-u+v^2+1)}, \frac{v}{2(u^2-u+v^2+1)}\right)$ lies on the line K_0L_0.

(b) That the center P_0 coincides with the center N_0 follows easily from the fact that $(A,\, N_a,\, P_a)$ and $(B,\, N_b,\, P_b)$ are triples of collinear points.

(c) Since $Q_a^y = \frac{f+\varphi-1}{2}$, $P_a^y = \frac{\varphi-f}{2}$, $L_a^y = f-1$ and $K_a^y = -f$, we see that

$$2 \cdot (Q_a^y - P_a^y) = L_a^y - K_a^y.$$

Similarly, $2 \cdot (Q_a^x - P_a^x) = L_a^x - K_a^x$. This proves the equality $2 \cdot \overrightarrow{P_a Q_a} = \overrightarrow{K_a L_a}$. □

Theorem 15. *The triangles $K_aK_bK_c$ and $L_aL_bL_c$ are congruent if and only if the triangles ABC and DEF are orthologic.*

Proof. Since the triangles $K_aK_bK_c$ and $L_aL_bL_c$ are both homothetic to the triangle ABC, we conclude that they will be congruent if and only if $|K_aK_b| = |L_aL_b|$. Hence, the theorem follows from the equality

$$|K_aK_b|^2 - |L_aL_b|^2 = \frac{\left[(2u-1)^2 + (2v+1)^2 + 2\right]\,\Delta}{v^2}.$$

□

Let O and ω denote the circumcenter and the Brocard angle of the triangle ABC.

Theorem 16. *If the triangles ABC and DEF are orthologic then the following statements are true.*

(a) *The symmedian point K of the triangle ABC is the midpoint of the segment K_0L_0.*

(b) *The triangles $M_aM_bM_c$ and $N_aN_bN_c$ are congruent.*

(c) *The triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ are congruent.*

(d) *The common ratio of the homotheties of the triangles $K_aK_bK_c$ and $L_aL_bL_c$ with the triangle ABC is $(1+\cot\,\omega) : 1$.*

(e) *The translations $K_aK_bK_c \mapsto L_aL_bL_c$ and $N_aN_bN_c \mapsto M_aM_bM_c$ are for the image of the vector $2 \cdot \overrightarrow{O[DEF,\, ABC]}$ under the rotation $\rho\left(O,\, \frac{\pi}{2}\right)$.*

(f) *The vector of the translation $P_aP_bP_c \mapsto Q_aQ_bQ_c$ is the image of the vector $\overrightarrow{O[DEF,\, ABC]}$ under the rotation $\rho\left(O,\, \frac{\pi}{2}\right)$.*

Proof. (a) Let $\xi = u^2 - u + v^2$. Let the triangles ABC and DEF be such that the centers K_0 and L_0 are well-defined. In other words, let $M,\, N \neq 0$, where $M,\, N = (u-1)d + v\delta - ue - v\varepsilon + f \pm (\xi+1)$. Let Z_0 be the midpoint of the segment K_0L_0. Then $|Z_0K|^2 = \frac{\Delta^2\, P}{4(\xi+1)^2\, M^2\, N^2}$, where

$$P = \frac{Q\, S^2}{(\xi+u)^2(\xi+3u+1)^2} + \frac{4v^2(\xi+1)^2\, T^2}{(\xi+u)(\xi+3u+1)},$$

$$S = (ue+v\varepsilon)(\xi^2+\xi-3u(u-1)) + (\xi+u)$$
$$[(\xi+3u+1)((u-1)d+v\delta) + ((1-2\xi)u-\xi-1)f - (\xi+1)(\xi+u-1)],$$

$Q = \xi^2 + (4u+1)\xi + u(3u+1)$ and $T = ue + v\varepsilon + (\xi + u)(f-1)$. Hence, if the triangles ABC and DEF are orthologic (i. e., $\Delta = 0$), then $K = Z_0$. The converse is not true because the factors S and T can be simultaneously equal to zero. For example, this happens for the points $A(0,0)$, $B(1,0)$, $C\left(\frac{1}{3},\, 1\right)$, $D(2,5)$, $E\left(4,\, -\frac{32}{9}\right)$ and $F(3,\, -1)$. An interesting problem is to give geometric description for the conditions $S = 0$ and $T = 0$.

(b) This follows from the equality

$$|N_aN_b|^2 - |M_aM_b|^2 = \frac{4(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + 1)\,\Delta}{v^2}.$$

(c) This follows similarly from the equality

$$|P_aP_b|^2 - |Q_aQ_b|^2 = \frac{(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + v + 1)\,\Delta}{v^2}.$$

(d) The ratio $\frac{|K_aK_b|}{|AB|}$ is $\frac{|\Delta + u^2 - u + v^2 + v + 1|}{v}$. Hence, when the triangles ABC and DEF are orthologic, then $\Delta = 0$ and this ratio is

$$\frac{u^2 - u + v^2 + v + 1}{v} = 1 + \frac{|BC|^2 + |CA|^2 + |AB|^2}{4 \cdot |ABC|} = 1 + \cot\omega.$$

(e) The tip of the vector $\overrightarrow{K_aL_a}$ (translated to the origin) is at the point

$$V(\xi - 2(ue + v\varepsilon - uf),\, 2f - 1).$$

The intersection of the perpendiculars through the points D and E onto the sidelines BC and CA is the point

$$U\left((1-u)d - v\delta + ue + v\varepsilon,\, \frac{uv\delta + (u-1)(du - ue - v\varepsilon)}{v}\right).$$

When the triangles ABC and DEF are orthologic this point will be the second orthology center $[DEF,\, ABC]$. Since the circumcenter O has the co-ordinates $\left(\frac{1}{2},\, \frac{\xi}{2v}\right)$, the tip of the vector $2 \cdot \overrightarrow{OU}$ is at the point

$$W^*\left(2((1-u)d - v\delta + ue + v\varepsilon) - 1,\, \frac{2((u-1)(ud - ue - v\varepsilon) + uv\delta) - \xi}{v}\right).$$

Its rotation about the circumcenter by $\frac{\pi}{2}$ has the tip at $W(-(W^*)^y,\, (W^*)^x)$. The relations $U^x - W^x = \frac{2u\Delta}{v}$ and $U^y - W^y = 2\Delta$ now confirm that the claim (e) holds.

(f) The proof for this part is similar to the proof of the part (e). □

12. New results for the pedal triangle

Let a, b, c and S denote the lengths of sides and the area of the triangle ABC. In this section we shall assume that DEF is the pedal triangle of the point P with respect to ABC. Our goal is to present several new properties of Bottema's original configuration. It is particularly useful for the characterizations of the Brocard axis.

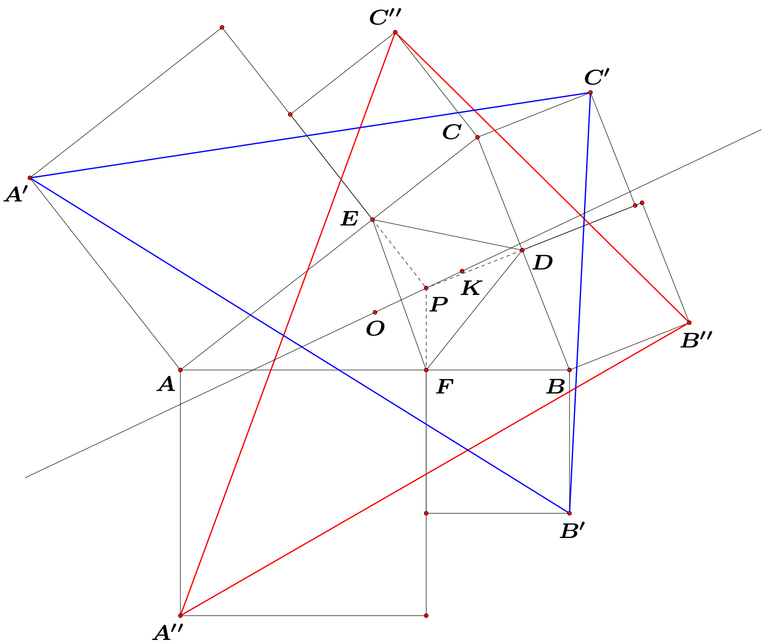

Figure 15. $s_2(A'B'C') = s_2(A''B''C'')$ iff P is on the Brocard axis.

Theorem 17. *There is a unique central point P with the property that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent. The first trilinear co-ordinate of this point P is $a((b^2+c^2+2\,S)a^2-b^4-c^4-2\,S(b^2+c^2))$. It lies on the Brocard axis and divides the segment OK in the ratio $(-\cot\omega):(1+\cot\omega)$ and is also the image of K under the homothety $h(O,\,-\cot\omega)$.*

Proof. Let $P(p,\,q)$. The orthogonal projections P_a, P_b and P_c of the point P onto the sidelines BC, CA and AB have the co-ordinates

$$\left(\frac{(u-1)^2p+v(u-1)q+v^2}{\xi-u+1},\ \frac{v((u-1)p+vq-u+1)}{\xi-u+1}\right),$$

$\left(\frac{u(up+vq)}{\xi+u},\ \frac{v(up+vq)}{\xi+u}\right)$ and $(p,\,0)$.

Since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area, it is easy to prove using the Heron formula that they will be congruent if and only if two of their corresponding sides have equal length. In other words, we must find the solution of the equations

$$|S_3S_5|^2-|S_4S_6|^2=\frac{v\xi p}{\xi+u}-\frac{v^2q}{\xi+u}+\frac{\xi+u-1}{2}=0,$$

$$|S_5S_1|^2-|S_6S_2|^2=\frac{v(\xi\,p+v\,q)}{\xi-u+1}-\frac{\xi^2-(2(u-v)-1)\,\xi+u\,(u-1)}{2(\xi-u+1)}=0.$$

As this is a linear system it is clear that there is only one solution. The required point is $P\left(\frac{1-2\,u+v}{2v},\ \frac{\xi^2+(v+1)\xi-v^2}{2v^2}\right)$. Let $s=-\frac{1}{1+\frac{v}{\xi+1}}=\frac{-\cot\omega}{1+\cot\omega}$. The point P divides the segment OK in the ratio $s:1$, where $O\left(\frac{1}{2},\ \frac{\xi}{2v}\right)$ and $K\left(\frac{\xi+2u}{2(\xi+1)},\ \frac{v}{2(\xi+1)}\right)$. □

Theorem 18. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have the same centroid if and only if the point P is the circumcenter of the triangle ABC.*

Proof. We get $|G_oG_e|^2=\frac{M^2+N^2}{9(\xi-u+1)(\xi+u)(1+4\xi)}$, with

$$M=3\xi(2u-1)p+v(1+4\xi)q-\xi(2\xi+3u-1)$$

and $N=v(1+\xi)(2p-1)$. Hence, $G_o=G_e$ if and only if $N=0$ and $M=0$. In other words, the centroids of the triangles $S_1S_3S_5$ and $S_2S_4S_6$ coincide if and only if $p=\frac{1}{2}$ and $q=\frac{\xi}{2v}$ (i. e., if and only if the point P is the circumcenter O of the triangle ABC). □

Recall that the Brocard axis of the triangle ABC is the line joining its circumcenter with the symmedian point.

Let s be a real number different from 0 and -1. Let the points A_s, B_s and C_s divide the segments BD, CE and AF in the ratio $s:1$ and let the points D_s, E_s and F_s divide the segments DC, EA and FB in the ratio $1:s$.

Theorem 19. *For the pedal triangle DEF of a point P with respect to the triangle ABC the following statements are equivalent:*

(a) *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.*

(b) *The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(c) *The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(d) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(e) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(f) *The triangles $A'B'C'$ and $A''B''C''$ have the same area.*

(g) *The triangles $A'B'C'$ and $A''B''C''$ have the same sums of squares of lengths of sides.*

(h) *The triangles $D'E'F'$ and $D''E''F''$ have the same area.*

(i) *The triangles $D'E'F'$ and $D''E''F''$ have the same sums of squares of lengths of sides.*

(j) *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal sums of squares of lengths of sides.*

(k) *For any real number $t\neq-1,\ 0,\ 2$, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal sums of squares of lengths of sides.*

(l) *For any real number $s\neq-1,\ 0$, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal sums of squares of lengths of sides.*

(m) *The triangles $A_sB_sC_s$ and $D_sE_sF_s$ have the same area.*

(n) *The point P lies on the Brocard axis of the triangle ABC.*

Proof. (a) The orthology criterion $\Delta_0(A_0B_0C_0,\ D_0E_0F_0)$ is equal to the quotient $\frac{-v\,M}{8(\xi+u)(\xi-u+1)}$, with M the following linear polynomial in p and q.

$$M = 2\,\left(\xi^2+\xi-v^2\right)p + 2\,v\,(2\,u-1)\,q - (\xi+u)\,(\xi+u-1)\,.$$

In fact, $M=0$ is the equation of the Brocard axis because the co-ordinates $\left(\frac{1}{2},\ \frac{\xi}{2v}\right)$ and $\left(\frac{\xi+2u}{2(\xi+1)},\ \frac{v}{2(\xi+1)}\right)$ of the circumcenter O and the symmedian point K satisfy this equation. Hence, the statements (a) and (n) are equivalent.

(f) It follows from the equality $|A''B''C''| - |A'B'C'| = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (f) and (n) are equivalent.

(i) It follows from the equality $s_2(D'E'F') - s_2(D''E''F'') = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (i) and (n) are equivalent. □

It is well-known that $\cot\omega = \frac{a^2+b^2+c^2}{4S}$ so that we shall assume that the degenerate triangles do not have well-defined Brocard angle. It follows that the statement ”The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles” could be added to the list of the previous theorem provided we exclude the points for which the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are degenerate. The following result explains when this happens. Let $K_{-\omega}$ denote the point described in Theorem 17.

Theorem 20. *The following statements are equivalent:*

(a) *The points S_1, S_3 and S_5 are collinear.*

(b) *The points S_2, S_4 and S_6 are collinear.*

(c) *The point P is on the circle with the center $K_{-\omega}$ and the radius equal to the circumradius R of the triangle ABC times the number $\sqrt{(1+\cot\omega)^2+1}$.*

Proof. Let M be the following quadratic polynomial in p and q:

$$v^2(p^2+q^2) + v\,(2\,u-w)\,p - \left(\xi^2+w\,\xi-v^2\right)q - (\xi+u)\,(\xi-u+w)\,,$$

where $w=v+1$. The points S_1, S_3 and S_5 are collinear if and only if

$$0 = \begin{vmatrix} S_1^x & S_1^y & 1 \\ S_3^x & S_3^y & 1 \\ S_5^x & S_5^y & 1 \end{vmatrix} = \frac{v\,M}{2(u-1-\xi)(u+\xi)}.$$

The equivalence of (a) and (c) follows from the fact that $M=0$ is the equation of the circle described in (c). Indeed, we see directly that the co-ordinates of its center are $\left(\frac{w-2u}{v},\ \frac{\xi^2+w\,\xi-v^2}{2v^2}\right)$ so that this center is the point $K_{-\omega}$ while the square of its radius is $\frac{(\xi-u+1)(\xi+u)\left((\xi+w)^2+v^2\right)}{4v^4} = \frac{(\xi-u+1)(\xi+u)}{4v^2}\cdot\frac{(\xi+w)^2+v^2}{v^2} = R^2\cdot\beta^2$, where β is equal to the number $\sqrt{(1+\cot\omega)^2+1}$ because $\cot\omega = \frac{\xi+1}{v}$.

The equivalence of (b) and (c) is proved in the same way. □

Theorem 21. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ always have different sums of squares of lengths of sides.*

Proof. The difference $s_2(A_0B_0C_0) - s_2(D_0E_0F_0)$ is equal to $\frac{3v^3\,N}{4(\xi-u+1)(u+\xi)}$, where N denotes the following quadratic polynomial in variables p and q:

$$\left(p-\frac{1}{2}\right)^2+\left(q-\frac{\xi}{2v}\right)^2+\frac{3(\xi-u+1)(\xi+u)}{4v^2}.$$

However, this polynomial has no real roots. □

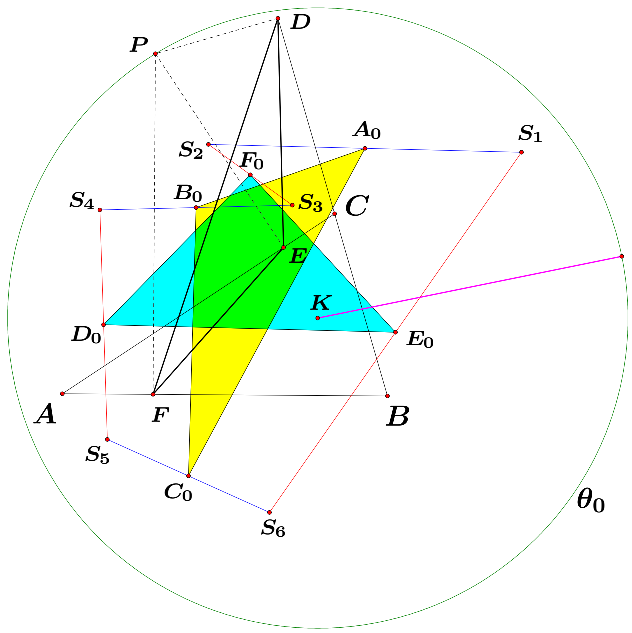

Figure 16. $|A_0B_0C_0| = |D_0E_0F_0|$ iff P is on the circle θ_0.

Theorem 22. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ have the same areas if and only if the point P lies on the circle θ_0 with the center at the symmedian point K of the triangle ABC and the radius $R\sqrt{4-3\tan^2\omega}$, where R and ω have their usual meanings associated with triangle ABC.*

Proof. The difference $|D_0E_0F_0| - |A_0B_0C_0|$ is equal to the quotient $\frac{v^2\,\zeta^2\,M}{16\,\mu\,(\zeta-u)}$, where $\zeta = \xi+1$, $\mu = \xi+u$ and M denotes the following quadratic polynomial in variables p and q:

$$\left(p-\frac{\mu+u}{2\,\zeta}\right)^2+\left(q-\frac{v}{2\,\zeta}\right)^2-\frac{\mu\,(\zeta-u)\,\left(4\,\zeta^2-3\,v^2\right)}{4\,\zeta^2\,v^2}.$$

The third term is clearly equal to $-R^2(4-3\tan^2\omega)$. Hence, $M=0$ is the equation of the circle whose center is the symmedian point of the triangle ABC with the co-ordinates $\left(\frac{\mu+u}{2\,\zeta},\ \frac{v}{2\,\zeta}\right)$ and the radius $R\sqrt{4-3\tan^2\omega}$. □

Let A^*, B^*, C^*, D^*, E^* and F^* denote the midpoints of the segments $A'A''$, $B'B''$, $C'C''$, $D'D''$, $E'E''$ and $F'F''$. Notice that the points A^*, B^*, C^*, D^*, E^* and F^* are the centers of squares built on the segments S_4S_5, S_6S_1, S_2S_3, S_1S_2, S_3S_4 and S_5S_6, respectively. Also, the triangles $A^*B^*C^*$ and $D^*E^*F^*$ share the centroids with the triangles ABC and DEF.

Notice that the lines AA^*, BB^* and CC^* intersect in the isogonal conjugate of the point P with respect to the triangle ABC.

Theorem 23. *The triangles $A^*B^*C^*$ and $D^*E^*F^*$ have the same sums of squares of lengths of sides if and only if the point P lies on the circle θ_0.*

Proof. The proof is almost identical to the proof of the previous theorem since the difference $s_2(D^*E^*F^*) - s_2(A^*B^*C^*)$ is equal to $\frac{v^2(\xi+1)^2\,M}{2(\xi-u+1)(\xi+u)}$. □

Theorem 24. *For any point P the triangles $A^*B^*C^*$ and $D^*E^*F^*$ always have different areas.*

Proof. The proof is similar to the proof of Theorem 21 since the difference $|D^*E^*F^*| - |A^*B^*C^*|$ is equal to $\frac{v^3\,N}{8(\xi-u+1)(\xi+u)}$. □

13. New results for the antipedal triangle

Recall that the antipedal triangle $P_a^*P_b^*P_c^*$ of a point P not on the side lines of the triangle ABC has as vertices the intersections of the perpendiculars erected at A, B and C to PA, PB and PC respectively. Note that the triangle $P_a^*P_b^*P_c^*$ is orthologic with the triangle ABC so that Bottema's Theorem also holds for antipedal triangles.

Our final result is an analogue of Theorem 19 for the antipedal triangle of a point. It gives a nice connection of a Bottema configuration with the Kiepert hyperbola (i. e., the rectangular hyperbola which passes through the vertices, the centroid and the orthocenter [3]).

In the next theorem we shall assume that DEF is the antipedal triangle of the point P with respect to ABC. Of course, the point P must not be on the side lines BC, CA and AB.

Theorem 25. *The following statements are equivalent:*

(a) *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.*

(b) *The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(c) *The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(d) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(e) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(f) *The triangles $A'B'C'$ and $A''B''C''$ have the same area.*

(g) *The triangles $A'B'C'$ and $A''B''C''$ have the same sums of squares of lengths of sides.*

(h) *The triangles $D'E'F'$ and $D''E''F''$ have the same area.*

(i) *The triangles $D'E'F'$ and $D''E''F''$ have the same sums of squares of lengths of sides.*

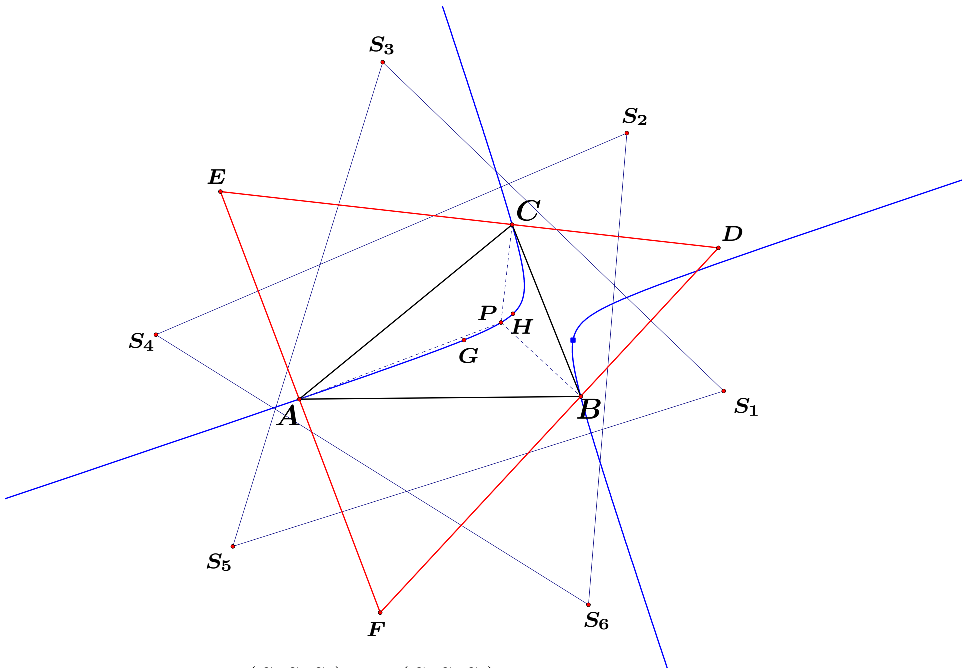

Figure 17. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ when P is on the Kiepert hyperbola.

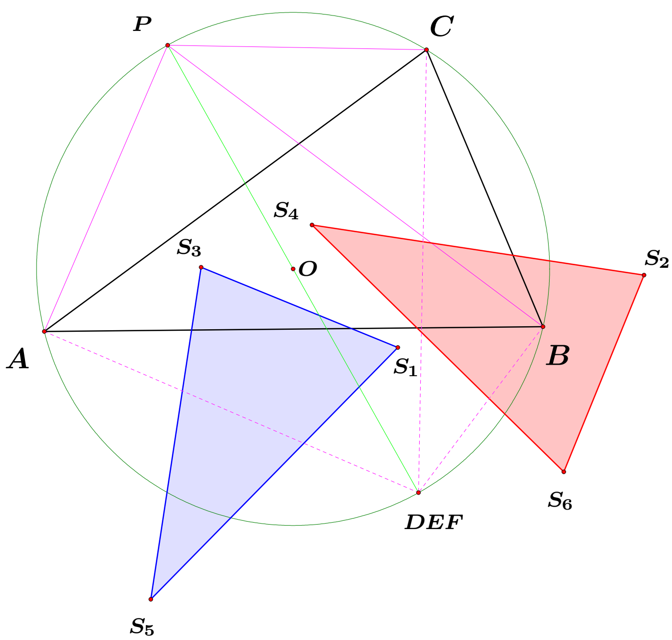

Figure 18. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ also when P is on the circumcircle.

(j) *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal sums of squares of lengths of sides.*

(k) *For any real number* $t \neq -1,\, 0,\, 2$, *the triangles* $S_1^t S_3^t S_5^t$ *and* $S_2^t S_4^t S_6^t$ *have equal sums of squares of lengths of sides.*

(l) *For any real number* $s \neq -1,\, 0$, *the triangles* $T_1^s T_3^s T_5^s$ *and* $T_2^s T_4^s T_6^s$ *have equal sums of squares of lengths of sides.*

(m) *The triangles* $A_s B_s C_s$ *and* $D_s E_s F_s$ *have the same area.*

(n) *The point* P *lies either on the Kiepert hyperbola of the triangle* ABC *or on its circumcircle.*

Proof. (g) $s_2(A''B''C'') - s_2(A'B'C') = \frac{2v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$, with

$$M = \left(p - \frac{1}{2}\right)^2 + \left(q - \frac{\xi}{2v}\right)^2 - \frac{\xi^2 + v^2}{4v^2},$$

$$N = v\,(2u-1)\left(p^2 - q^2 - p\right) - 2\,\left(u^2 - u - v^2 + 1\right) pq + \left(u^2 + u - v^2\right) q.$$

In fact, $M = 0$ is the equation of the circumcircle of the triangle ABC while $N = 0$ is the equation of its Kiepert hyperbola because the co-ordinates of the vertices A, B and C and the co-ordinates $\left(u,\, \frac{u(1-u)}{v}\right)$ and $\left(\frac{u+1}{3},\, \frac{v}{3}\right)$ of the orthocenter H and the centroid G satisfy this equation. Hence, the statements (g) and (n) are equivalent.

(j) It follows from the equality

$$s_2(S_2S_4S_6) - s_2(S_1S_3S_5) = \frac{v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$$

that the statements (j) and (n) are equivalent.

(m) It follows from the equality

$$|D_sE_sF_s| - |A_sB_sC_s| = \frac{s\,v\,M\,N}{2(s+1)^2\,q(vp-uq)(v(p-1)-(u-1)q)}$$

that the statements (m) and (n) are equivalent. □

Of course, as in the case of the pedal triangles, we can add the statement "The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles." to the list in Theorem 25 but the points on the circle described in Theorem 20 must be excluded from consideration.

Notice that when the point P is on the circumcircle of ABC then much more could be said about the properties of the six squares built on segments BD, DC, CE, EA, AF and FB. A considerable simplification arises from the fact that the antipedal triangle DEF reduces to the antipodal point Q of the point P. For example, the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are the images under the rotations $\rho(U,\, \frac{\pi}{4})$ and $\rho(V,\, -\frac{\pi}{4})$ of the triangle $A_\diamond B_\diamond C_\diamond = h(O,\, \frac{\sqrt{2}}{2})(ABC)$ (the image of ABC under the homothety with the circumcenter O as the center and the factor $\frac{\sqrt{2}}{2}$). The points U and V are constructed as follows.

Let the circumcircle $\sigma_\diamond$ of the triangle $A_\diamond B_\diamond C_\diamond$ intersect the segment OQ in the point R, let ℓ be the perpendicular bisector of the segment QR and let T be the midpoint of the segment OQ. Then the point U is the intersection of the line ℓ with $\rho(T, \frac{\pi}{4})(PQ)$ while the point V is the intersection of the line ℓ with $\rho(T, -\frac{\pi}{4})(PQ)$.

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Zvonko Čerin:Kopernikova 7, 10010 Zagreb, CROATIA
E-mail address: `cerin@math.hr`

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On the Newton Line of a Quadrilateral

Paris Pamfilos

Abstract. We introduce the idea of the conjugate polygon of a point relative to another polygon and examine the closing property of polygons inscribed in others and having sides parallel to a conjugate polygon. Specializing for quadrangles we prove a characterization of their Newton line related to the possibility to inscribe a quadrangle having its sides parallel to the sides of a conjugate one.

1. Introduction

Given two quadrangles $a = A_1A_2A_3A_4$ and $b = B_1B_2B_3B_4$ one can ask whether it is possible to inscribe in the first a quadrangle $c = C_1C_2C_3C_4$ having its sides parallel to corresponding sides of the second. It is also of importance to know how many solutions to the problem exist and which is their structure. The corresponding problem for triangles is easy to solve, well known and has relations to pivoting around a pivot-point of which there are twelve in the generic case ([9, p. 297], [8, p. 109]). Here I discuss the case of quadrangles and in some extend the case of arbitrary polygons. While in the triangle case the inscribed one is *similar* to a given triangle, for quadrangles and more general polygons this is no more possible. I start the discussion by examining properties of polygons inscribed in others to reveal some general facts. In this frame it is natural to introduce the class of *conjugate polygons* with respect to a point, which generalize the idea of the *precevian* triangle, having for vertices the *harmonic associates* of a point [12, p.100]. Then I discuss some properties of them, which in the case of quadrangles relate the inscription-problem to the Newton line of their associated *complete* quadrilateral (in this sense I speak of the *Newton line of the quadrangle* [13, p.169], [6, p.76], [3, p.69], [4], [7]). After this preparatory discussion I turn to the examination of the case of quadrangles and prove a characteristic property of their Newton line (§5, Theorems 11, 14).

2. Periodic polygon with respect to another

Consider two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ and pick a point C_1 on side A_1A_2 of the first. From this draw a parallel to side B_1B_2 of the second polygon until it hits side A_2A_3 to a point C_2 (see Figure 1). Continue in this way picking points C_i on the sides of the first polygon so that C_iC_{i+1} is parallel to side B_iB_{i+1} of the second polygon (indices $i > n$ are reduced modulo n if corresponding points X_i are not defined). In the last step draw a parallel to B_nB_1 from C_n until it hits the initial side A_1A_2 at a point C_{n+1}. I call polygon $c = C_1 \cdots C_{n+1}$ *parallel to* b *inscribed in* a *and starting at* C_1. In general polygon c is not closed. It can even have self-intersections and/or some side(s) degenerate to

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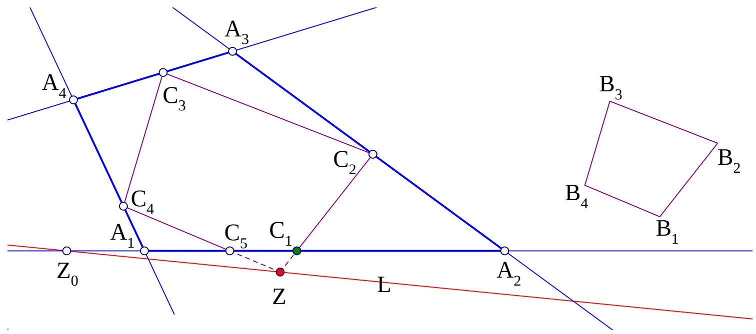

Figure 1. Inscribing a polygon

points (identical with vertices of a). One can though create a corresponding closed polygon by extending segment C_nC_{n+1} until to hit C_1C_2 at a point Z. Polygon $ZC_2\cdots C_n$ has sides parallel to corresponding sides of $B_1\cdots B_n$. Obviously triangle $C_{n+1}ZC_1$ has fixed angles and remains similar to itself if the place of the starting point C_1 changes on A_1A_2. Besides one can easily see that the function expressing the coordinate y of C_{n+1} in terms of the coordinate x of C_1 is a linear one $y = ax + b$. This implies that point Z moves on a fixed line L ([10, Tome 2, p. 10]) as point C_1 changes its position on line A_1A_2 (see Figure 1). This in turn shows that there is, in general, a unique place for C_1 on side A_1A_2 such that points C_{n+1}, C_1 coincide and thus define a *closed* polygon $C_1\cdots C_n$ inscribed in the first polygon and having its sides parallel to corresponding sides of the second. This place for C_1 is of course the intersection point Z_0 of line L with side A_1A_2. In the exceptional case in which L is parallel to A_1A_2 there is no such polygon. By the way notice that, for obvious reasons, in the case of triangles line L passes through the vertex opposite to side A_1A_2.

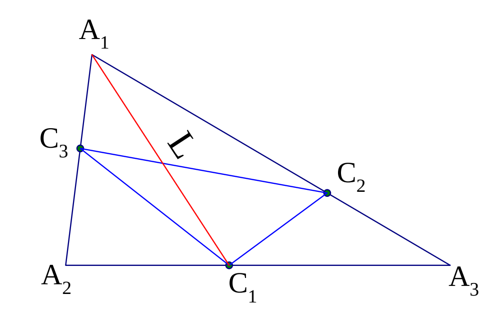

Figure 2. The triangle case

This example shows that the answer to next question is not in general in the affirmative. The question is: Under which conditions for the two polygons is line L identical with side A_1A_2, so that the above procedure produces always closed polygons $C_1\cdots C_n$? If this is the case then I say that polygon $B_1\cdots B_n$ is *periodic* with respect to $A_1 \cdots A_n$. Below it will be shown that this condition is independent of the side A_1A_2 selected. If it is satisfied by starting points C_1 on this side and drawing a parallel to B_1B_2 then it is satisfied also by picking the starting point C_i on side A_iA_{i+1}, drawing a parallel to B_iB_{i+1} and continuing in this way.

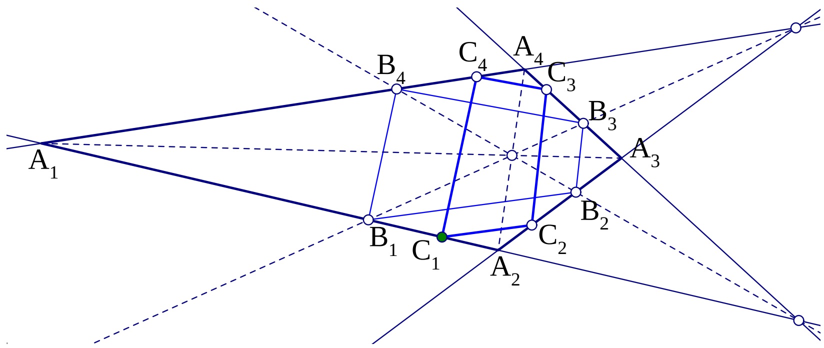

Figure 3. $B_1B_2B_3B_4$ periodic with respect to $A_1A_2A_3A_4$

There are actually plenty of examples of pairs of polygons satisfying the periodicity condition. For instance take an arbitrary quadrangle $A_1A_2A_3A_4$ and consider its $dual$ quadrangle $B_1B_2B_3B_4$, created through the intersections of its sides with the lines joining the intersection of its diagonals with the two intersection points of its pairs of opposite sides (see Figure 3). For every point C_1 on A_1A_2 the procedure described above closes and defines a quadrangle $C_1C_2C_3C_4$ inscribed in $A_1A_2A_3A_4$ and having its sides parallel to $B_1B_2B_3B_4$. This will be shown to be a consequence of Theorem 11 in combination with Proposition 16. It should be noticed though that periodicity, as defined here, is a relation depending on the *ordered* sets of vertices of two polygons. $B_1 \cdots B_n$ can be periodic with respect to $A_1 \cdots A_n$ but $B_2 \cdots B_nB_1$ not. Figure 4 displays such an example.

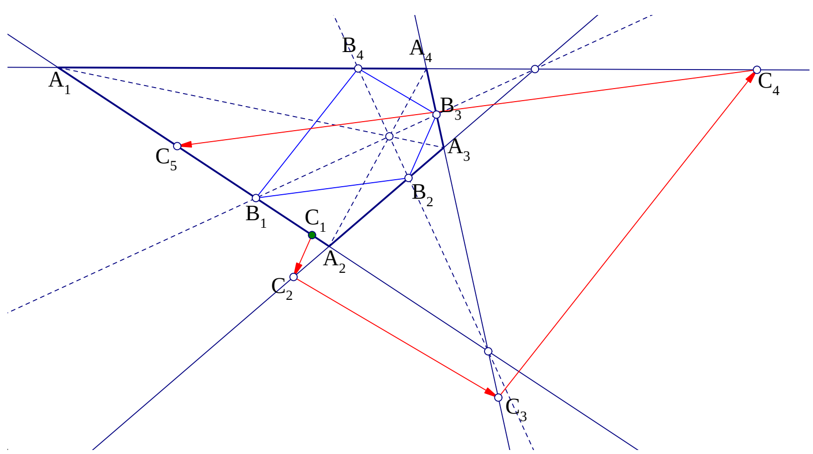

Figure 4. $B_2B_3B_4B_1$ not periodic with respect to $A_1A_2A_3A_4$

To handle the question in a systematic way I introduce some structure into the problem, which obviously is affinely invariant ([1], [2, vol.I, pp.32–66], [5]). I will consider the correspondence $C_1 \longmapsto C_{n+1}$ as the restriction on line A_1A_2 of a globally defined affine transformation G_1 and investigate the properties of this map. Figure 5 shows how transformation G_1 is constructed. It is the composition of *affine reflections* F_i ([5, p. 203]). The affine reflection F_i has its *axis* along A_iY_i which is the harmonic conjugate line of A_iX_i with respect to the two adjacent sides $A_{i-1}A_i$, A_iA_{i+1} at A_i. Its *conjugate direction* is A_iX_i which is parallel to side $B_{i-1}B_i$. By its definition map F_i corresponds to each point X the point Y such that the line-segment XY is parallel to the conjugate direction A_iX_i and has its middle on the axis A_iY_i of the map.

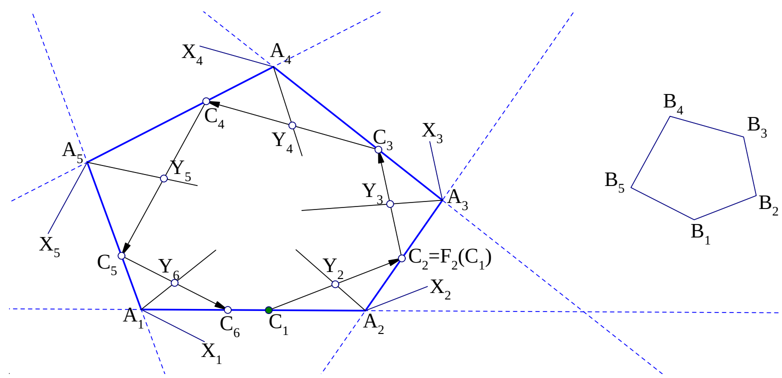

Figure 5. An affine transformation

The map $G_1 = F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_2$ is a globally defined affine transformation, which on line A_1A_2 coincides with correspondence $C_1 \longmapsto C_{n+1}$. I call it *the first recycler of* b *in* a. Line A_1A_2 remains invariant by G_1 as a whole and each solution to our problem having $C_1 = C_{n+1}$ represents a fixed point of G_1. Thus, if there are more than one solutions, then line A_1A_2 will remain pointwise fixed under G_1. Assume now that G_1 leaves line A_1A_2 pointwise fixed. Then it is either an affine reflection or a *shear* ([5, p.203]) or it is the identity map, since these are the only affine transformations fixing a whole line and having determinant ± 1. Since G_1 is a product of affine reflections, its kind depends only on the number n of sides of the polygon. Thus for n even it is a shear or the identity map and for n odd it is an affine reflection. For n even it is shown by examples that both cases can happen: map G_1 can be a shear as well as the identity. In the second case I call $B_1 \cdots B_n$ *strongly periodic* with respect to $A_1 \cdots A_n$. The strongly periodic case delivers closed polygons $D_1 \cdots D_n$ with sides parallel to those of $B_1 \cdots B_n$ and the position of D_1 can be arbitrary. To construct such polygons start with an arbitrary point D_1 of the plane and define $D_2 = F_2(D_1), D_3 = F_3(D_2), \cdots, D_n = F_n(D_{n-1})$. The previous example of the dual of a quadrangle is a strongly periodic one (see Figure 6).

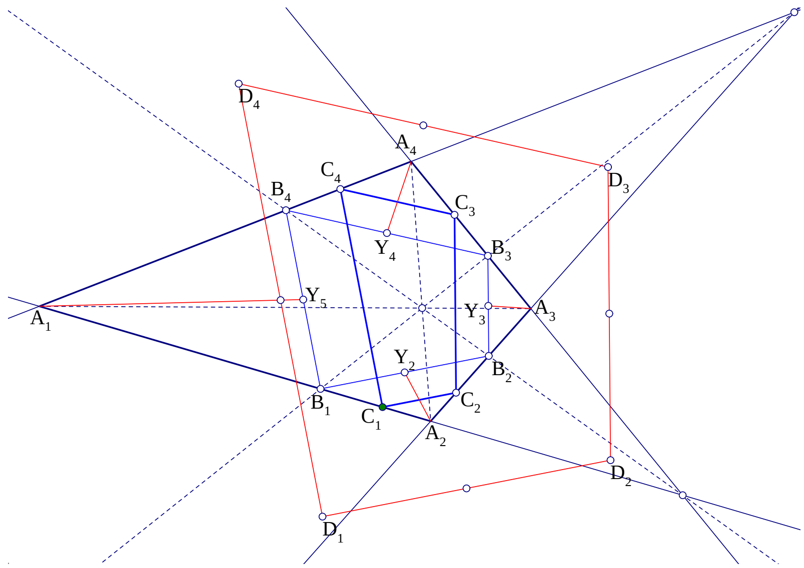

Figure 6. Strongly periodic case

Another case delivering many strongly periodic examples is that of a square $A_1A_2A_3A_4$ and the inscribed in it quadrangle $B_1B_2B_3B_4$, resulting by projecting an arbitrary point X on the sides of the square (see Figure 7).

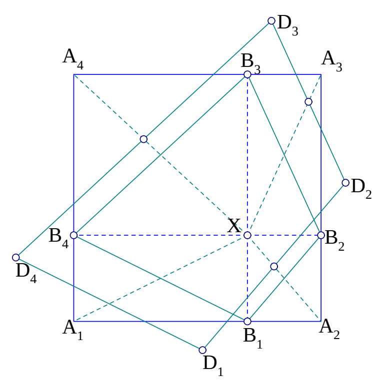

Figure 7. Strongly periodic case II

Analogously to G_1 one can define the affine map $G_2 = F_2 \circ F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_3$, which I call *second recycler of* b *in* a. This does the same work in constructing a polygon $D_2 \cdots D_n D_1$ inscribed in $A_1 \cdots A_n$ and with sides parallel to those of $B_2 \cdots B_n B_1$ but now the starting point D_2 is to be taken on side A_2A_3, whereas the sides will be parallel successively to $B_2B_3, B_3B_4, \cdots$. Analogously are defined the affine maps $G_i, i = 3, \cdots, n$ (*i-th recycler of* b *in* a). It follows immediately from their definition that G_i are conjugate to each other. Obviously, since the F_i are involutive, we have $G_2 = F_2 \circ G_1 \circ F_2$ and more general $G_k = F_k \circ G_{k-1} \circ F_k$.

Thus, if there is a fixed point X_1 of G_1 on side A_1A_2, then $X_2 = F_2(X_1)$ will be a fixed point of G_2 on A_2A_3 and more general $X_k = F_k \circ \cdots \circ F_2(X_1)$ will be a fixed point of G_k on side A_kA_{k+1}. Corresponding property will be also valid in the case A_1A_2 remains pointwise fixed under G_1. Then every side A_kA_{k+1} will remain fixed under the corresponding G_k. The discussion so far is summarized in the following proposition.

Proposition 1. (1) *Given two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ there is in the generic case only one closed polygon $c = C_1 \cdots C_n$ having its vertex C_i on side A_iA_{i+1} and its sides C_iC_{i+1} parallel to B_iB_{i+1} for $i = 1, \cdots, n$. If there are two such polygons then there are infinite many and their corresponding point C_1 can be an arbitrary point of A_1A_2. In this case b is called periodic with respect to a.*

(2) *Using the sides of polygons a and b one can construct an affine transformation G_1 leaving invariant the side A_1A_2 and having the property: b is periodic with respect to a precisely when G_1 leaves side A_1A_2 pointwise fixed.*

(3) *In the periodic case, if n is odd then G_1 is an affine reflection with axis (mirror) line A_1A_2 and if n is even then it is a shear with axis A_1A_2 or the identity map. In the last case b is called strongly periodic with respect to a.*

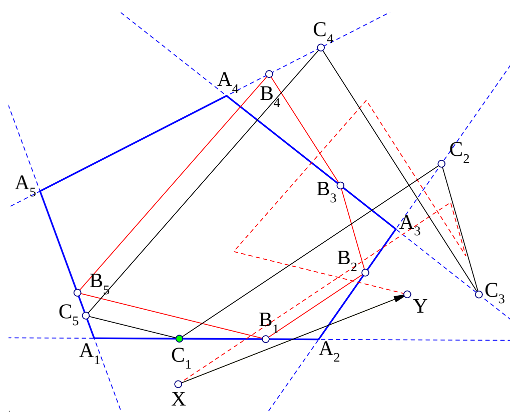

Figure 8. Periodic pentagons

Figure 8 shows a periodic case for $n = 5$. The figure shows also a typical pair $Y = G_1(X)$ of points related by the affine reflection G_1 resulting in this case.

3. Conjugate polygon

Given a closed polygon $a = A_1 \cdots A_n$ and a point P not lying on the side-lines of a, consider for each $i = 1, \dots, n$ the harmonic conjugate line A_iX_i of line A_iP with respect to the two adjacent sides of a at A_i. The polygon $b = B_1 \cdots B_n$ having sides these lines is called *conjugate of a with respect to P*. The definition generalizes the idea of the *precevian triangle* of a triangle $a = A_1A_2A_3$ with respect to a point P, which is the triangle $B_1B_2B_3$ having vertices the *harmonic associates* B_i of P with respect to a ([12, p.100]).

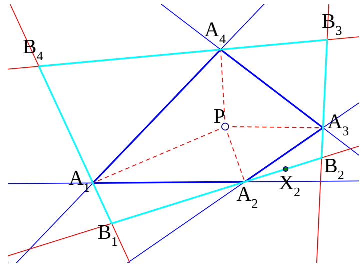

Figure 9. Conjugate quadrangle with respect to P

Proposition 2. *Given a closed polygon $a = A_1 \cdots A_n$ with n odd and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 is an affine reflection the axis of which passes through P and its conjugate direction is that of line A_1A_2.*

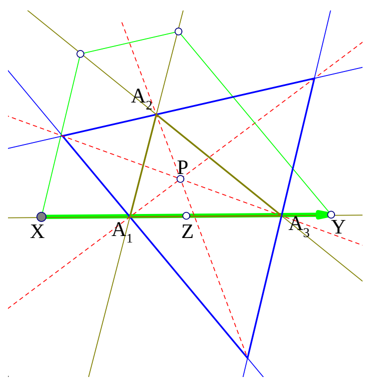

Figure 10. G_1 is an affine reflection

That point P remains fixed under G_1 is obvious, since G_1 is a composition of affine reflections all of whose axes pass through P. From this, using the preservation of proportions by affinities and the invariance of A_1A_2 follows also the that the parallels to A_1A_2 remain also invariant under G_1. Let us introduce coordinates (x, y) with origin at P and x-axis parallel to A_1A_2. Then G_1 has a representation of the form $\{x' = ax + by, y' = y\}$. Since its determinant is -1 it follows that $a = -1$. Thus, on every line $y = y_0$ parallel to A_1A_2 the transformation acts through $x' = -x + by_0 \Leftrightarrow x' + x = by_0$, showing that the action on line $y = y_0$ is a point symmetry at point Z with coordinates $(by_0/2, y_0)$, which remains also fixed by G_1 (see Figure 10). Then the whole line PZ remains fixed by G_1, thus showing it to be an affine reflection as claimed. The previous proposition completely solves the initial problem of inscription for conjugate polygons with n sides and n odd. In fact, as noticed at the beginning, such an inscription possibility corresponds to a fixed point of the map G_1 and this has a unique such point on A_1A_2. Thus we have next corollary.

Corollary 3. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is odd, then there is exactly one closed polygon $C_1 \cdots C_n$ with $C_i \in A_iA_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b. In particular, for n odd there are no periodic conjugate polygons.*

The analogous property for conjugate polygons and n even is expressed by the following proposition.

Proposition 4. *Given a closed polygon $a = A_1 \cdots A_n$ with n even and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 either is a shear the axis of which is the parallel to side A_1A_2 through P, or it is the identity map.*

The proof, up to minor changes, is the same with the previous one, so I omit it. The analogous corollary distinguishes now two cases, the second corresponding to G_1 being the identity. Periodicity and strong periodicity coincide when n is even and when b is the conjugate of a with respect to some point.

Corollary 5. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is even, then there is either no closed polygon $C_1 \cdots C_n$ with $C_i \in A_iA_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b, or b is strongly periodic with respect to a.*

Remark. Notice that the existence of even one fixed point not lying on the parallel to A_1A_2 through P (the axis of the shear) imply that G_1 is the identity or equivalently, the corresponding conjugate polygon is strongly periodic.

The next propositions deal with some properties of conjugate polygons needed, in the case of quadrangles, in relating the periodicity to the Newton's line.

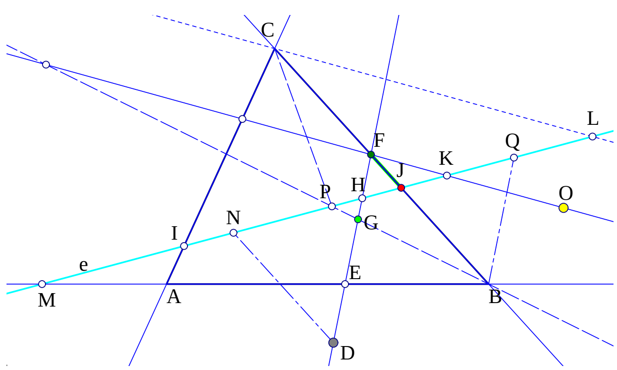

Figure 11. Fixed point O

Lemma 6. *Let $\{ABC, D, e\}$ be correspondingly a triangle, a point and a line. Consider a variable line through D intersecting sides AB, BC correspondingly at points E, F. Let G be the middle of EF and P the intersection point of lines*

e and BG. Let further CL be the harmonic conjugate of line CP with respect to CA, CB. Then the parallel to CL from F passes through a fixed point O.

To prove the lemma introduce affine coordinates with axes along lines $\{BC, e\}$ and origin at J, where $I = e \cap CA, J = e \cap CB$ (see Figure 11). The points on line e are: $M = e \cap AB, N = e \cap (||BC, D), H = e \cap DE, Q = e \cap (||DE, B)$, where the symbol $(||XY, Z)$ means: *the parallel to XY from Z*. Denote abscissas/ordinates by the small letters corresponding to labels of points, with the exceptions of $a = DN$, the abscissa x of F and the ordinate y of K. The following relations are easily deduced.

$$h = \frac{hx}{x+a}, \quad q = b\frac{h}{x}, \quad p = \frac{mq}{2q-m}, \quad l = \frac{pi}{2p-i}, \quad y = \frac{lx}{c}.$$

Successive substitutions produce a homographic relation between variables x, y:

$$p_1x + p_2y + p_3xy = 0,$$

with constants (p_1, p_2, p_3), which is equivalent to the fact that line FK passes through point O with coordinates $(-\frac{p_2}{p_3}, -\frac{p_1}{p_3})$.

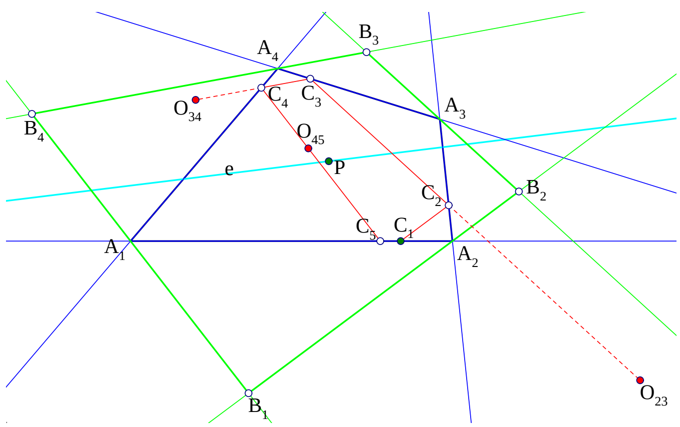

Figure 12. Sides through fixed points

Lemma 7. *Let $\{A_1 \cdots A_n, C_1, e\}$ be correspondingly a closed polygon, a point on side A_1A_2 and a line. Consider a point P varying on line e and the corresponding conjugate polygon $b = B_1 \cdots B_n$. Construct the parallel to b polygon $c = C_1 \cdots C_{n+1}$ starting at C_1. As P varies on e, every side of polygon c passes through a corresponding fixed point.*

The proof results by inductively applying the previous lemma to each side of c, starting with side C_1C_2, which by assumption passes through C_1 (see Figure 12). Next prove that side C_2C_3 passes through a point O_{23} by applying previous lemma to the triangle with sides A_1A_2, A_2A_3, A_3A_4 and by taking C_1 to play the role of D in the lemma. Then apply the lemma to the triangle with sides A_2A_3, A_3A_4, A_4A_5 taking for D the fixed point O_{23} of the previous step. There results a fixed point

O_{34} through which passes side C_3C_4. The induction continues in the obvious way, using in each step the fixed point obtained in the previous step, thereby completing the proof.

Lemma 8. *Let* $\{A_1 \cdots A_n, C_1, e\}$ *be correspondingly a closed polygon, a point on side* A_1A_2 *and a line. Consider a point* P *varying on line* e*, the corresponding conjugate polygon* $b = B_1 \cdots B_n$ *and the corresponding parallel to* b *polygon* $c = C_1 \cdots C_{n+1}$ *starting at* C_1*. Then the correspondence* $P \longmapsto C_{n+1}$ *is either constant or a projective one from line* e *onto line* A_1A_2*.*

Assume that the correspondence is not a constant one. Proceed then by applying the previous lemma and using the fixed points $O_{23}, O_{34}, \cdots$ through which pass the sides of the inscribed polygons c as P varies on line e. It is easily shown inductively that correspondences $f_1 : P \mapsto C_2,\ f_2 : P \mapsto C_3,\ \ldots, f_n : P \mapsto C_{n+1}$ are projective maps between lines. That f_1 is a projectivity is a trivial calculation. Map f_2 is the composition of f_1 and the perspectivity between lines A_3A_2, A_3A_4 from O_{23}, hence also projective. Map f_3 is the composition of f_2 and the perspectivity between lines A_4A_3, A_4A_5 from O_{34}, hence also projective. The proof is completed by the obvious induction.

4. The case of parallelograms

The only quadrangles not possessing a Newton line are the parallelograms. For these though the periodicity question is easy to answer. Next two propositions show that parallelograms are characterized by the strong periodicity of their conjugates with respect to *every* point not lying on their side-lines.

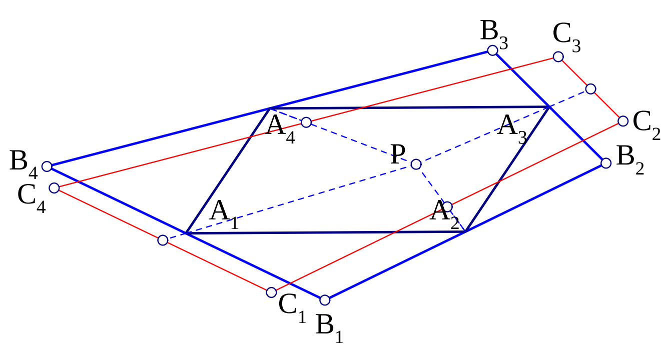

Figure 13. Parallelograms and periodicity

Proposition 9. *For every parallelogram* $a = A_1A_2A_3A_4$ *and every point* P *not lying on its side-lines the corresponding conjugate quadrangle* $b = B_1B_2B_3B_4$ *is strongly periodic.*

The proposition (see Figure 13) is equivalent to the property of the corresponding first recycler G_1 to be the identity. To prove this it suffices to show that G_1 fixes a point not lying on the parallel to A_1A_2 through P (see the remark after corollary 5 of previous paragraph). In the case of parallelograms however it is easily seen that a is the parallelogram of the middles of the sides of the conjugates b.

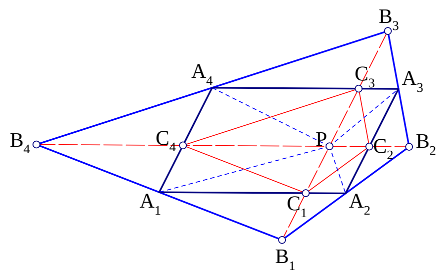

Figure 14. C_1 fixed by G_1

In fact, let $b = B_1B_2B_3B_4$ be the conjugate of a with respect to P and consider the intersection points $C_1, C_2, \cdots$ of the sides $A_1A_2, A_2A_3, \cdots$ of the parallelogram correspondingly with lines $PB_1, PB_2, \ldots$ (see Figure 14). The bundles of lines $A_1(B_1, P, C_1, A_4)$ at A_1 and $A_2(B_1, P, C_1, A_3)$ at A_2 are harmonic by the definition of b. Besides their three first rays intercept on line PB_1 correspondingly the same three points B_1, P, C_1 hence the fourth harmonic of these three points is the intersection point of their fourth rays A_1A_4, A_2A_3, which is the point at infinity. Consequently C_1 is the middle of PB_1. The analogous property for C_2, C_3, C_4 implies that quadrangle $c = C_1C_2C_3C_4$ has its sides parallel to those of b and consequently lines PA_i are the medians of triangles $PB_{i-1}B_i$. Thus point B_1 is a fixed point of G_1 not lying on its axis, consequently G_1 is the identity.

Proposition 10. *If for every point P not lying on the side-lines of the quadrangle $a = A_1A_2A_3A_4$ the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ is strongly periodic, then a is a parallelogram.*

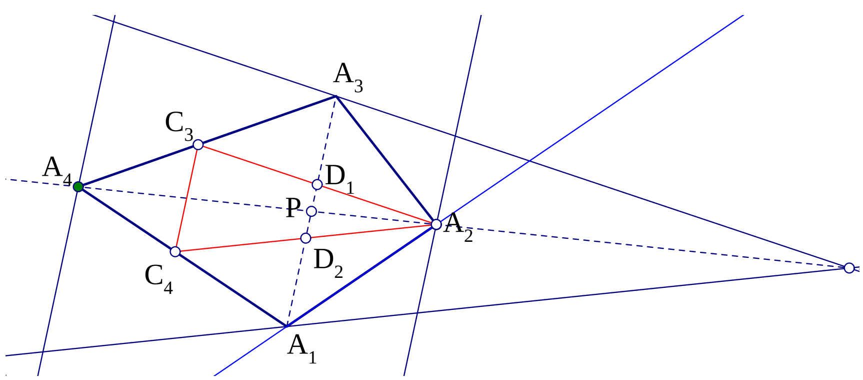

Figure 15. Parallelogram characterization

This is seen by taking for P the intersection point of the diagonals of the quadrangle. Consider then the parallel to b polygon starting at A_2. By assumption this must be closed, thus defining a triangle $A_2C_3C_4$ (see Figure 15). The middles D_1, D_2 of the sides of the triangle are by definition on the diagonal A_1A_3, which is parallel to C_3C_4. Thus the diagonal A_1A_3 is parallel to the conjugate direction of the other diagonal A_2A_4, consequently P is the middle of A_1A_3. Working in the same way with side A_2A_3 and the recycler G_2 it is seen that P is also the middle of A_2A_4, hence the quadrangle is a parallelogram.

5. A property of the Newton line

By the convention made above the *Newton line* of a quadrangle, which is not a parallelogram, is the line passing through the middles of the diagonals of the associated *complete* quadrilateral. In this paragraph I assume that the quadrangle of reference is not a parallelogram, thus has a Newton line. The points of this line are then characterized by having their corresponding conjugate quadrangle strongly periodic.

Theorem 11. *Given a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and a point P on its Newton-line, the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ with respect to P is strongly periodic.*

Before starting the proof I supply two lemmata which reduce the periodicity condition to a simpler geometric condition that can be easily expressed in projective coordinates.

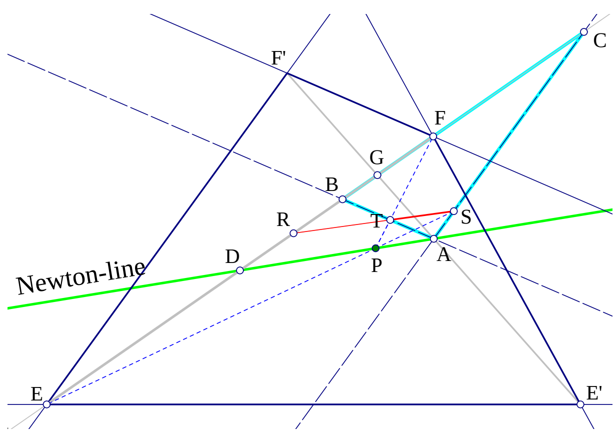

Figure 16. A fixed point

Lemma 12. *Let $a = EE'FF'$ be a quadrangle with diagonals $EF, E'F'$ and corresponding middles on them D, A. Draw from A parallels AB, AC correspondingly to sides $FF', F'E$ intersecting the diagonal EF correspondingly at points B, C. For every point P on the Newton-line AD of the quadrangle lines PE, PF*

intersect correspondingly lines AC, AB at points S, T. Line ST intersects the diagonal EF always at the same point R, which is the harmonic conjugate of the intersection point G of the diagonals with respect to B, C.

The proof is carried out using barycentric coordinates with respect to triangle ABC. Then points $D, E, F, \ldots$ on line BC are represented using the corresponding small letters for parameters $D = B + dC$, $E = B + eC$, $F = B + fC, \ldots$ (see Figure 16). In addition P is represented through a parameter p in $P = D + pA$. First we calculate E', F' in terms of these parameters:

$$E' = (f + g + 2fg)A - fB - (fg)C,$$
$$F' = (g - f)A + fB + fgC.$$

Then the coordinates of S, T are easily shown to be:

$$S = pA + (d - e)C,$$
$$T = (pf)A + (f - d)B.$$

From these the intersection point R of line ST with BC is seen to be:

$$R = (d - f)B + (f(d - e))C.$$

This shows that R is independent of the value of parameter p hence the same for all points P on the Newton-line. Some more work is needed to verify the claim about its precise location on line AB. For this the parallelism EF' to AC and the fact that D is the middle of EF are proved to be correspondingly equivalent to the two conditions:

$$g = \frac{f(1+e)}{1+f}, \quad d = \frac{f(e+1) + e(f+1)}{(e+1) + (f+1)}.$$

These imply in turn the equation

$$g = \frac{f(e-d)}{d-f},$$

which is easily shown to translate to the fact that R is the harmonic conjugate of G with respect to B, C.

Lemma 13. *Let $a = ABCD$ be a quadrangle with diagonals AC, BD and corresponding middles on them M, N. Draw from M parallels ME, MF correspondingly to sides AB, AD intersecting the diagonal BD at points E, F. Let P be a point of the Newton-line MN and S, T correspondingly the intersections of line-pairs $(PB, ME), (PD, MF)$. The conjugate quadrangle of P is periodic precisely when the harmonic conjugate of AP with respect to AB, AD is parallel to ST.*

In fact, consider the transformation $G_1 = F_4 \circ F_3 \circ F_2 \circ F_1$ composed by the affine reflections with corresponding axes PC, PD, PA, PB. By the discussion in the previous paragraph, the periodicity of the conjugate quadrilateral to P is equivalent to G_1 being the identity. Since G_1 is a shear and acts on BC in general as a translation by a vector $\mathbf{v}$ to show that $\mathbf{v} = \mathbf{0}$ it suffices to show that it fixes

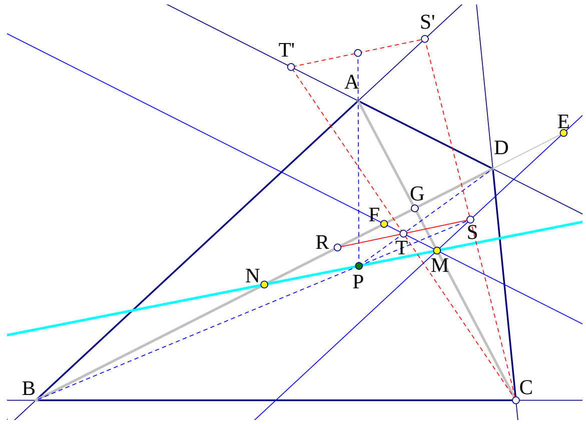

Figure 17. Equivalent problem

an arbitrary point on BC. This criterion applied to point C means that for $T' = F_2(C), S' = F_3(T')$ point $C' = F_4(S')$ is identical with C (see Figure 17). Since T, S are the middles of CT', CS', this implies the lemma.

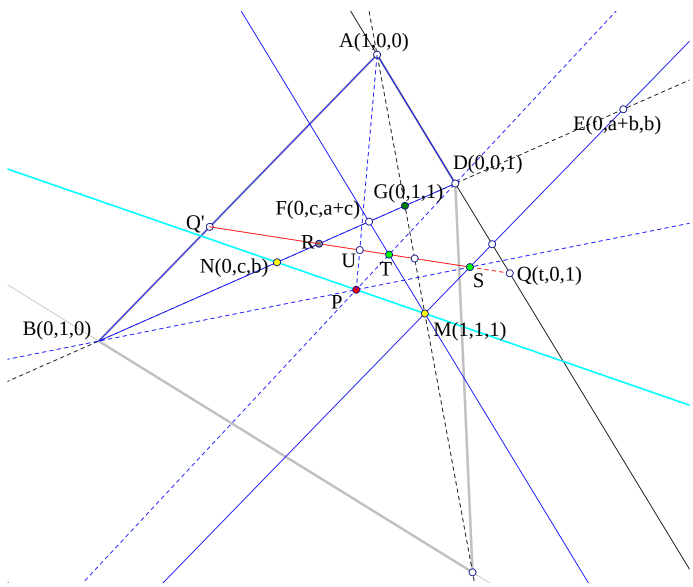

Figure 18. Representation in coordinates

Proof of the theorem: Because of the lemmata 12 and 13 one can consider the variable point P not as an independent point varying on the Newton-line MN but as a construct resulting by varying a line through R which is the harmonic conjugate of the intersection point G of the diagonals with respect to E, F. Such a line intersects the parallels ME, MF to sides AB, AD at S, T and determines P as intersection of lines BS, DT. Consider the coordinates defined by the projective basis (see Figure 18) $\{A(1,0,0), B(0,1,0), D(0,0,1), M(1,1,1)\}$. Assume further that the line at infinity is represented by an equation in the form

$$ax + by + cz = 0.$$

Then all relevant points and lines of the figure can be expressed in terms of the constants (a, b, c). In particular

$$ax - (a + c)y + cz = 0, \quad ax + by - (a + b)z = 0, \quad (c - b)x + by - cz = 0,$$

are the equations of lines MF, ME and the Newton-line MN. Point R has coordinates $(0, a', b')$, where $a' = (c - a - b)$, $b' = (a + c - b)$. Assume further that the parametrization of a line through R is done by a point $Q(t, 0, 1)$ on line AD. This gives for line RQ the equation $RQ : a'x + (tb')y - (ta')z = 0$. Point S has coordinates (a'', b'', c'') where $a'' = t(a'b - b'(a+b)),\ b'' = a'(a+b) - taa',\ c'' = a'b - tab'$. This gives for P the coordinates $(ba'', cc'' - a''(c - b), bc'')$ and the coordinates of the intersection point U of PA with RQ can be shown to be $U = c''Q - (c + at)Q'$, where $Q'(tb', -a', 0)$ is the intersection point of AB and RQ. From these follows easily that U is the middle of QQ' showing the claim according to Lemma 13.

Theorem 14. *For a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ only the points P on its Newton-line have the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ strongly periodic.*

The previous theorem guarantees that all points of the Newton line have a strongly periodic corresponding conjugate polygon b. Assume now that there is an additional point P_0, not on the Newton line, which has also a strongly periodic corresponding conjugate polygon. In addition fix a point C_1 on A_1A_2. Take then a point P_1 on the Newton line and consider line $e = P_0P_1$. By Lemma 8 the correspondence $f : e \longrightarrow A_1A_2$ sending to each point $P \in e$ the end-point C_{n+1} of the polygon parallel to the conjugate b of a with respect to P starting at a fixed point C_1 is either a constant or a projective map. Since f takes for two points P_0, P_1 the same value (namely $f(P_0) = f(P_1) = C_1$) this map is constant. Hence the whole line e consists of points having corresponding conjugate polygon strongly periodic. This implies that any point of the plane has the same property. In fact, for an arbitrary point Q consider a line e_Q passing through Q and intersecting e and the Newton line at two points Q_0 and Q_1. By the same reasoning as before we conclude that all points of line e_Q have corresponding conjugate polygons strongly periodic, hence Q has the same property. By Proposition 10 of the preceding paragraph it follows that the quadrangle must be a parallelogram, hence a contradiction to the hypothesis for the quadrangle.

6. The dual quadrangle

In this paragraph I consider a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and its *dual* quadrangle $b = B_1B_2B_3B_4$, whose vertices are the intersections of the sides of the quadrangle with the lines joining the intersection point of its diagonals with the intersection points of its two pairs of opposite sides. After a preparatory lemma, Proposition 16 shows that b is the conjugate polygon with respect to an appropriate point on the Newton line, hence b is strongly periodic.

Lemma 15. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its associated complete quadrilateral. Let also $b = B_1B_2B_3B_4$ be the dual quadrangle of a.*
(1) *Line EG intersects the parallel A_4N to the side B_1B_4 of b at its middle M.*
(2) *Side B_1B_2 of b intersects the segment MN at its middle O.*

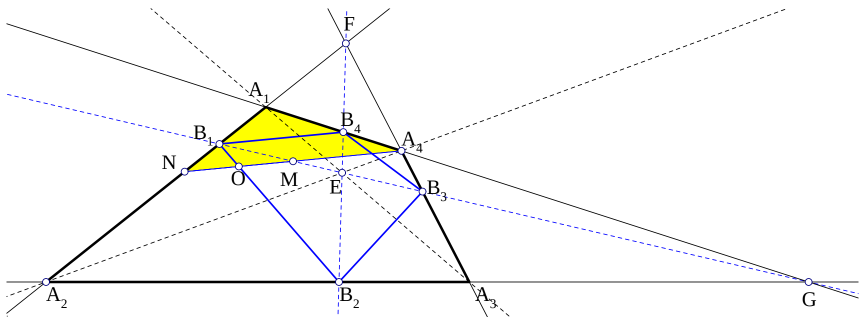

Figure 19. Dual property

$MN/MA_4 = 1$, since Menelaus theorem applied to triangle A_1NA_4 with secant line B_1B_3G gives $(B_1N/B_1A_1)(MA_4/MN)(GA_1/GA_4) = 1$. But $B_1N/B_1A_1 = B_4A_4/B_4A_1 = GA_4/GA_1$. Later equality because (B_4, G) are harmonic conjugate to (A_1, A_4). Also $ON/OM = 1$, since the bundle $B_1(B_2, B_4, E, F)$ is harmonic. Thus the parallel NM to line B_1B_4 of the bundle is divided in two equal parts by the other three rays of the bundle.

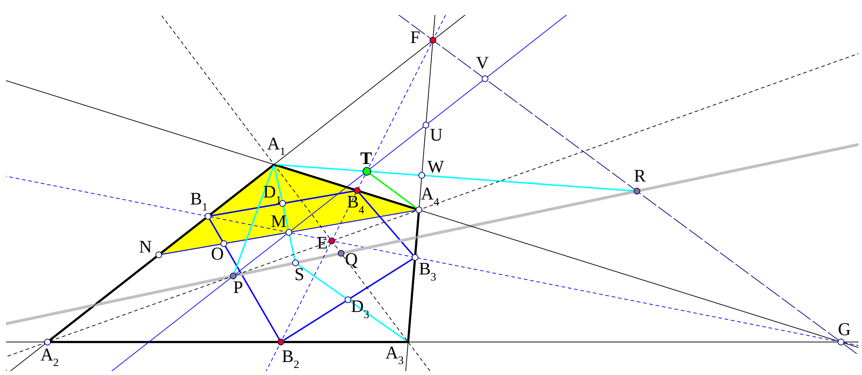

Figure 20. Dual is strongly periodic

Proposition 16. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its corresponding complete quadrilateral and $\{P, Q, R\}$ the middles of the diagonals $\{A_2A_4, A_1A_3, FG\}$ contained in the Newton line of the quadrilateral. Let $b =$*

$B_1B_2B_3B_4$ be the dual quadrangle of a
(1) *The four medians $\{A_1D_1, A_2D_2, A_3D_3, A_4D_4\}$ of triangles $\{A_1B_1B_4$, $A_2B_2B_1$, $A_3B_3B_2$, $A_4B_4B_3\}$ respectively meet at a point S on the Newton line.*
(2) *S is the harmonic conjugate of the diagonal middle R with respect to the two others (P, Q).*

Start with the intersection point T of diagonal B_2B_4 with line A_1R (see Figure 20). Draw from T line TV parallel to side A_1A_2 intersecting side A_3A_4 at U. Since the bundle $F(V, T, U, A_1)$ is harmonic and TV is parallel to ray FA_1 of it point U is the middle of TV. Since $A_4(A_1, W, T, R)$ is a harmonic bundle and R is the middle of FG, its ray A_4T is parallel to FG. It follows that A_4TFV is a parallelogram. Thus U is the middle of A_4F, hence the initial parallel TV to line A_1A_2 passes through the middles of segments having one end-point at A_4 and the other on line A_1A_2. Among them it passes through the middles of $\{A_1A_4, A_4N, A_4A_2\}$ the last being P the middle of the diagonal A_2A_4. Extend the median A_1D_1 of triangle $A_1B_1B_4$ to intersect the Newton line at S. Bundle $A_1(P, Q, S, R)$ is harmonic. In fact, using Lemma 15 it is seen that it has the same traces on line TV with those of the harmonic bundle $E(P, A_1, M, T)$. Thus S is the harmonic conjugate of R with respect to (P, Q).

Remarks. (1) Poncelet in a preliminary chapter [10, Tome I, p. 308] to his celebrated *porism* (see [2, Vol. II, pp. 203–209] for a modern exposition) examined the idea of *variable* polygons $b = B_1 \cdots B_n$ having *all but one* of their vertices on fixed lines (sides of another polygon) and restricted by having their sides to pass through corresponding fixed points $E_1, \cdots, E_n$. Maclaurin had previously shown that in the case of triangles ($n = 3$) the free vertex describes a conic([11, p. 248]). This generalizes to polygons with arbitrary many sides. If the fixed points through which pass the variable sides are *collinear* then the free vertex describes a line ([10, Tome 2, p. 10]). This is the case here, since the fixed points are the points on the line at infinity determining the directions of the sides of the inscribed polygons.

(2) In fact one could formulate the problem handled here in a somewhat more general frame. Namely consider polygons inscribed in a fixed polygon $a = A_1 \cdots A_n$ and having their sides passing through corresponding fixed *collinear* points. This case though can be reduced to the one studied here by a projectivity f sending the line carrying the fixed points to the line at infinity. The more general problem lives of course in the projective plane. In this frame the affine reflections F_i, considered above, are replaced by *harmonic homologies* ([5, p. 248]). The center of each F_i is the corresponding fixed point E_i through which passes a side B_iB_{i+1} of the variable polygon. The axis of the homology is the polar of this fixed point with respect to the side-pair $(A_{i_1}A_i, A_iA_{i+1})$ of the fixed polygon. The definitions of periodicity and the related results proved here transfer to this more general frame without difficulty.

(3) Though I am speaking all the time about a quadrangle, the property proved in §5 essentially characterizes the associated *complete quadrilateral*. If a point P has a periodic conjugate with respect to one, out of the three, quadrangles embedded

in the complete quadrilateral then it has the same property also with respect to the other two quadrangles embedded in the quadrilateral.

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Paris Pamfilos: Department of Mathematics, University of Crete, 71409 Herakleion, Greece
E-mail address: `pamfilos@math.uoc.gr`

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Folding a Square to Identify Two Adjacent Sides

Cristinel Mortici

Abstract. The purpose of this paper is to establish some properties that appear in a square cut by two rays at 45 degrees passing through a vertex of the square. Elementary proofs and other interesting comments are provided.

1. A simple problem and a reformulation

The starting point of this work is the following problem from [3], partially discussed in [4]. [1]

Proposition 1. *Two points M and N on the hypotenuse BD of the isosceles, right-angled triangle ABD, with M between B and N, define an angle $\measuredangle MAN = 45^\circ$ if and only if $BM^2 + ND^2 = MN^2$* (see Figure 1).

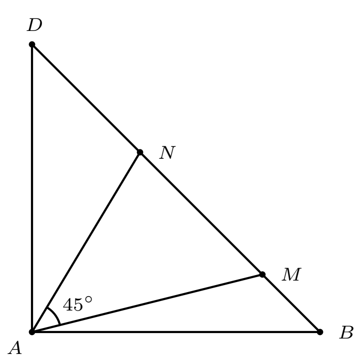

Figure 1

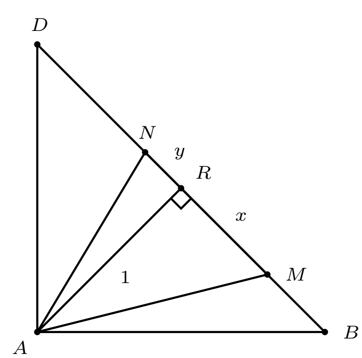

Figure 3

Proof. Let R be the midpoint of BD so that $AR = BR = BR$, and AR is an altitude of triangle ABD. We assume $AR = 1$ and denote $RM = x$, $RN = y$ (see Figure 3). Note that

$$\tan(\measuredangle MAN) = \tan(\measuredangle MAR + \measuredangle NAR) = \frac{x+y}{1-xy} = 1.$$

It follows that $\measuredangle MAN = 45^\circ$ if and only if $x + y = 1 - xy$. On the other hand, $BM^2 + ND^2 = MN^2$ if and only if $(1-x)^2 + (1-y)^2 = (x+y)^2$. Equivalently, $x + y = 1 - xy$, the same condition for $\measuredangle MAN = 45^\circ$. $\square$

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[1]This problem (erroneously attributed to another author in [1]) was considered by Boskoff and Suceavă as an example of an elliptic projectivity characterized by the Pythagorean relation.

This necessary and sufficient condition assumes new, interesting forms if we consider the isosceles right triangle as a half-square, and fold the adjacent sides AB and AC along the lines AM and AN. Without loss of generality we assume $AB = AC = 1$.

Theorem 2. *Let $ABCD$ be a unit square. Two half-lines through A meet the diagonal BD at M and N, and the sides BC, CD at M and P and Q respectively* (see Figure 2). *Assume $AP \not\equiv AQ$.*

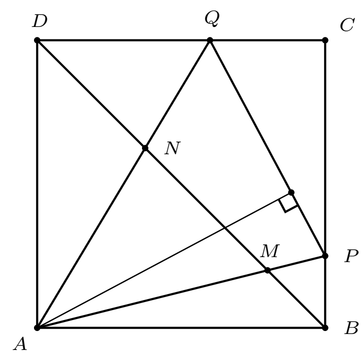

Figure 2

The following statements are equivalent:
(i) $\measuredangle PAQ = 45^\circ$.
(ii) $MN^2 = BM^2 + ND^2$.
(iii) *The perimeter of triangle CPQ is equal to* 2.
(iv) $PQ = BP + QD$.
(v) *The distance from A to line PQ is equal to* 1.
(vi) *The area of triangle AMN is half of the area of triangle APQ.*
(vii) $PQ = \sqrt{2} \cdot MN$.
(viii) $PQ^2 = 2(BM^2 + ND^2)$.
(ix) *The line passing through A and $MQ \cap NP$ is perpendicular on PQ.*
(x) $AN = NP$.
(xi) $AM = MQ$.

Remark. In the excluded case $AP = AQ$, statement (ix) does not imply the other statements.

Proof of Theorem 2. With Cartesian coordinates $A(0,\,0)$, $B(1,\,0)$, $C(1,\,1)$, $D(0,\,1)$ and $P(1,\ a)$, $Q(b,\ 1)$ for some distinct $a, b \in (0,\ 1)$, we have $M(\frac{1}{1+a},\ \frac{a}{1+a})$ and $N(\frac{b}{1+b},\ \frac{1}{1+b})$. Then (i)-(xi) are each equivalent to

$$a + b + ab = 1. \tag{1}$$

This is clear from the following, which are obtained from routine calculations.
(i): $\tan \measuredangle PAQ = 1 - \frac{a+b+ab-1}{a+b}$.
(ii): $MN^2 - BM^2 - ND^2 = -\frac{2(a+b+ab-1)}{(b+1)(a+1)}$.

(iii, iv): $(PQ - BP - QD) + 2 = CP + PQ + QC = 2 - \frac{2(a+b+ab-1)}{a+b+\sqrt{(1-a)^2+(1-b)^2}}$.
(v): $\operatorname{dist}(A, PQ) = 1 + \frac{(1-a)(1-b)(a+b+ab-1)}{(1-ab+\sqrt{(1-a)^2+(1-a)^2})\sqrt{(1-a)^2+(1-b)^2}}$.
(vi): $\frac{\text{area}[AMN]}{\text{area}[APQ]} = \frac{1}{2} + \frac{a+b+ab-1}{2(1+a)(1+b)}$.
(vii): $PQ^2 - 2MN^2 = (ab + a + b - 1) \cdot \frac{(a+b)(a-b)^2+(ab^3+a^3b+a^2+b^2+2-6ab)}{(1+a)^2(1+b)^2}$.
(viii): $PQ^2 - 2(BM^2 + ND^2) = (a + b + ab - 1) \cdot f(a, b)$, where

$$f(a, b) := \frac{-4a - 4b - ab^2 - a^2b + ab^3 + a^3b - 10ab + a^2 + a^3 + b^2 + b^3 - 2}{(a+1)^2(b+1)^2}.$$

(ix): If O is the intersection of PN and QN, then

$$m_{AO}m_{PQ} = -1 + \frac{(a-b)(a+b+ab-1)}{b(1-b)(a+1)}.$$

(x): $AN^2 - NP^2 = (a + b + ab - 1) \cdot \frac{1-a}{1+b}$.
(xi) $AM^2 - MQ^2 = (a + b + ab - 1) \cdot \frac{1-b}{1+a}$.

The expression (vii) is indeed equivalent with (1), if we take into account that

$$\frac{a^2 + b^2 + a^3b + ab^3 + 1 + 1}{6} > \sqrt[6]{a^2 \cdot b^2 \cdot a^3b \cdot ab^3 \cdot 1 \cdot 1} = ab.$$

For (viii), we prove that the $f(a, b) < 0$ for $a, b \in [0, 1]$. This is because, regarded as a function of $a \in [0, 1]$, $f''(a) = 6a + 6ab + 2(1 - b) > 0$. Since $f(0) < 0$ and $f(1) < 0$, we conclude that $f(a) < 0$ for $a \in [0, 1]$. □

2. A simple geometric proof of (i) ⇔ (ii)

Statement (ii) clearly suggests a right triangle with sides congruent to BM, ND and MN. One way to do this is indicated in Figure 5, where M' is chosen such that the segment DM' is perpendicular to BD and is congruent to BM. Under the hypothesis (ii), we have $M'N = MN$. Moreover, $\Delta AMB \equiv \Delta AM'D$, and $\measuredangle MAM' = 90°$. It also follows that the triangles AMN and $AM'N$ have three pairs of equal corresponding sides, and are congruent. From this, $\measuredangle MAN = \measuredangle NAM' = 45°$. This shows that (ii) $\implies$ (i).

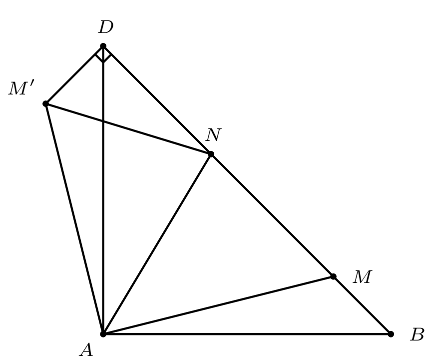

Figure 5

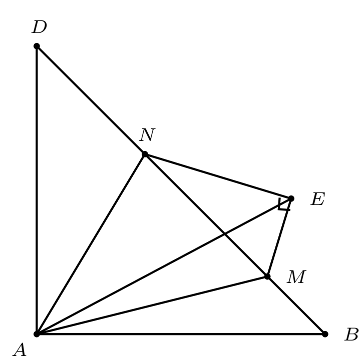

Figure 6

Another idea is to build an auxiliary right triangle with the hypotenuse MN, whose legs have lengths equal to BM and ND. This is based on the simple idea of folding the half-square ABD along AM and AN to identify the adjacent sides AB and AC. Let E be the reflection of B in the line AM (see Figure 6). Note that $BM = ME$. Assuming $\measuredangle MAN = 45°$, we see that E is also the reflection of D in the line AN. Now the triangles AMB and AME are congruent, so are the triangles ANE and AND. Thus, $\measuredangle MEN = \measuredangle MEA + \measuredangle NEA = \measuredangle MBA + \measuredangle NDA = 45° + 45° = 90°$. By the Pythagorean theorem, $MN^2 = ME^2 + EN^2 = BM^2 + ND^2$. This shows that (i) $\implies$ (ii).

3. A generalization

V. Proizolov has given in [6] the following nice result illustrating the beauty of the configuration of Theorem 2.

Proposition 3. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45°$, then $MN^2 = BM^2 + ND^2$* (see Figure 8).

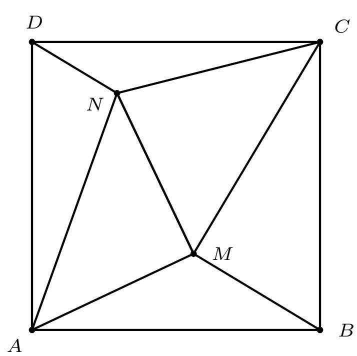

Figure 8

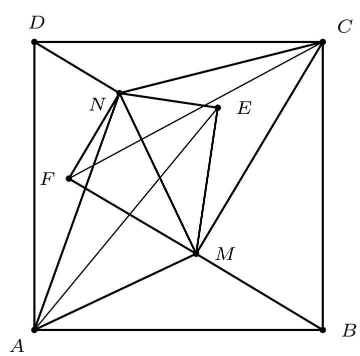

Figure 8A

This situation can be viewed as a surprising extension from the case of triangle ABD in Figure 6 is distorted into the polygon $ABMND$. In fact, by considering the symmetric of triangle ABD with respect to hypotenuse BD in Figure 1, a particular case of Proposition 3 is obtained. This analogy carries over to the general case. More precisely, we try to use the auxiliary construction from Figure 6, namely to consider the point E such that the triangles ANE and AND are symmetric and also the triangles AME and AMB are symmetric.

Let F be analogue defined, starting from the vertex C (see Figure 8A).

It follows that $\measuredangle MEN + \measuredangle MFN = 180°$, as the sum of the angles $\measuredangle B$ and $\measuredangle D$ of the square. But the triangles MEN and MFN are congruent, so $\measuredangle MEN = \measuredangle MFN = 90°$. The conclusion follows now from Pythagorean theorem applied in triangle MEN.

4. Rotation of the square

We show how to use the above auxiliary constructions to establish further interesting results. Complete the right triangle ABD from Figure 5 to an entire square $ABCD$. Triangle ADM' is obtained by rotating triangle ABM about A, through $90°$. This fact suggests us to make a clockwise rotation with center A of the entire figure to obtain the square $ADST$ (see Figure 9).

Denote the points corresponding to M, N, P, Q by M', N', P', Q' respectively. Assume that $\measuredangle PAQ = 45°$, or equivalently, $MN^2 = MB^2 + ND^2$.

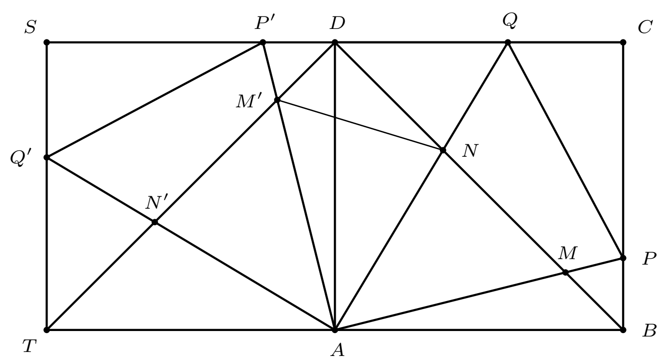

Figure 9

From $\Delta APQ \equiv \Delta AP'Q$ it follows that $PQ = P'Q$. If $AB = 1$, then

$$2 = SC = SP' + P'Q + QC = CP + PQ + QC$$

and we obtained the implication (i) $\Longrightarrow$ (iii).

The converse (iii) $\Longrightarrow$ (i) was first stated by A. B. Hodulev in [2].

5. Secants, tangents and lines external to a circle

We begin this section with an interesting question. Assuming $ABCD$ a unit square, how can we construct points P, Q such that the perimeter of triangle PQC is equal to 2? As we have already seen, one method is to make $\measuredangle PAQ = 45°$. Alternatively, note that the perimeter of triangle PQC is equal to 2 if and only if $PQ = BP + DQ$. This characterization allows us to construct points P, Q on the sides with the required property.

If we draw the arc with center A, passing through B and D, then every tangent line meeting the circle at T and the sides at P and Q determines the triangle ΔPQC of perimeter 2, because $PT = PB$ and $QT = QD$ (see Figure 10).

Moreover, if PQ does not meet the arc, then the length of PQ is less than the parallel tangent $P'Q'$ to the circle (see Figure 11). Consequently, if a segment PQ does not meet the circle, then $\measuredangle PAQ < 45°$. On the other hand, if PQ meets the circle twice, then $\measuredangle PAQ > 45°$.

We summarize these in the following theorem.

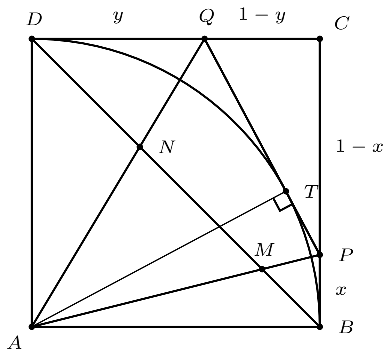

Figure 10

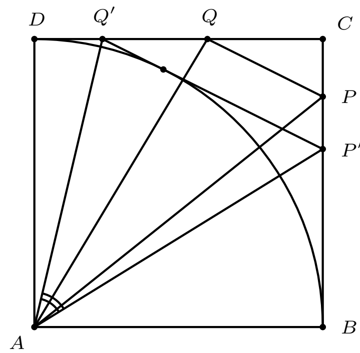

Figure 11

Theorem 4. *Let $ABCD$ be a unit square, and P, Q be points on the sides BC and CD respectively. Consider the quadrant ω of the circle with center A, passing through B and D.*
(a) $\measuredangle PAQ = 45^\circ$ *if and only if PQ is tangent to ω. Equivalently, the perimeter of triangle PQC is equal to 2.*
(b) $\measuredangle PAQ > 45^\circ$ *if and only if PQ intersects ω at two points. Equivalently, the perimeter of triangle PQC is greater than 2.*
(c) $\measuredangle PAQ < 45^\circ$ *if and only if PQ is exterior to ω. Equivalently, the perimeter of triangle PQC is less than 2.*

6. Comparison of areas

The implication (i) $\Longrightarrow$ (vi) was first discovered by Z. G. Gotman in [1].

In Figure 12 below, observe that the quadrilaterals $ABPN$ and $ADQM$ are cyclic, respectively because $\measuredangle NAP = \measuredangle NBP$ and $\measuredangle MAQ = \measuredangle MDQ$.

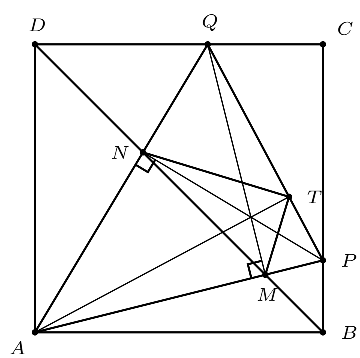

Figure 12

Consequently, AMQ and ANP are isosceles right-angled triangles. Hence,

$$\frac{S_{AMN}}{S_{APQ}} = \frac{AM \cdot AN}{AP \cdot AQ} = \frac{AM}{AQ} \cdot \frac{AN}{AP} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}.$$

Now we establish the implication (i) $\implies$ (ix).

In triangle ΔAPQ, QM and PN are altitudes, so the radius AT from Figure 10 is in fact the third altitude of the triangle ΔAPQ.

We can continue with the identifications, making use of the congruences $\Delta APB \equiv \Delta APT$ and $\Delta AQD \equiv \Delta AQT$. We deduce that $TM = MB$ and $TN = ND$. It follows that

$$MN^2 = MT^2 + TN^2 = BM^2 + ND^2.$$

Remark. The point E from Figure 6, coinciding with the point T from Figure 12, is more interesting than we have initially thought. It lies on the circumcircle of the given triangle ABD.

7. Two pairs of congruent segments

The implications (i) $\implies$ (x) and (xi) follow from the fact that ANP and AMQ are isosceles right-angled triangles.

For the converses, let us assume by way of contradiction that $\measuredangle MAN_1 = 45°$, with N_1 in BD, distinct from N. Then $AN_1 = N_1P$. As we have also $AN = NP$, it follows that NN_1 and consequently BD is the perpendicular bisector of AP, which is absurd.

8. Concluding remarks

Now let us return for a short time to the opposite angles drawn in Figure 8. It is the moment to celebrate the contribution of V. Proizvolov which proves in [5] the following nice result.

Proposition 5. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45°$, then*

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MAN} + S_{MBC} + S_{NCD}.$$

Having at hand the previous construction from Figure 8A (where F is defined by the conditions $\Delta CND \equiv \Delta CNF$ and $\Delta CMB \equiv \Delta CMF$), we have

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MCN} + S_{AMEN} = S_{AMCN} + S_{MEN}.$$

Similarly, $S_{MAN}+S_{MBC}+S_{NCD} = S_{AMCN}+S_{MFN}$ and the conclusion follows from the congruence of the triangles MEN and MFN.

We mention for example that the idea of folding a square as in Figure 6 leads to new results under weaker hypotheses. Indeed, if we consider that piece of paper as an isosceles triangle, not necessarily right-angled, then similar results hold. Thus, if triangle ABD is isosceles, then in triangle MEN, the angle $\measuredangle MEN$ is the sum of angles $\measuredangle ABD$ and $\measuredangle ADB$. Consequently, by applying the law of cosines to triangle MEN, we obtain the following extension of Proposition 1.

Proposition 6. *Let M and N be two points on side BD of the isosceles triangle ABD such that the angle $\measuredangle MAN = \frac{1}{2}\measuredangle BAD$. Then*

$$MB^2 - MN^2 + DN^2 = -2MB \cdot DN \cos A.$$

Another interesting extension is the following problem proposed by the author at the 5th Selection Test of the Romanian Team participating at 44th IMO Japan 2003.

Problem. Find the angles of a rhombus $ABCD$ with $AB = 1$ given that on sides CD (CB) *there exist points* P, *respective* Q *such that the angle* $\measuredangle PAQ = \frac{1}{2}\measuredangle BAD$ and the perimeter of triangle CPQ is equal to 2.

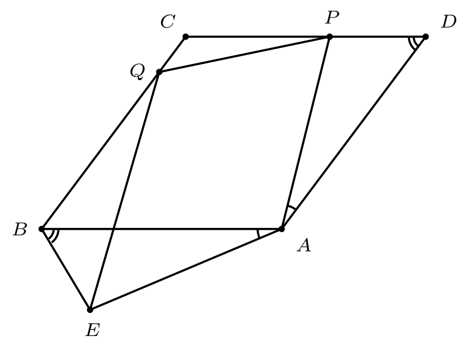

Figure 13

Let E be as in Figure 13 such that $\Delta APD \equiv \Delta AEB$. In fact we rotate triangle APD about A and what it is interesting for us is that $PQ = QE$ and $PD = BE$. Now, the equality $PQ = PD + QB$ can be written as $QE = BE + QB$, so the points Q, B, E are collinear.

This is possible only when $ABCD$ is square.

Finally, we consider replacing the square in Theorem 2 by a rhombus. Proposition 7 below was proposed by the author as a problem for the 12th Edition of the Clock-Tower School Competition, Râmnicu Vâlcea, Romania, 2009, then given at the first selection test for the Romanian team participating at the Junior Balkan Mathematical Olympiad, Neptun-Constanta, April, 15-th, 2009.

Proposition 7. *Let $ABCD$ be a rhombus. Two rays through A meet the diagonal BD at M, N, and the sides BC and CD at P, Q respectively* (see Figure 14). *Then $AN = NP$ if and only if $AM = MQ$.*

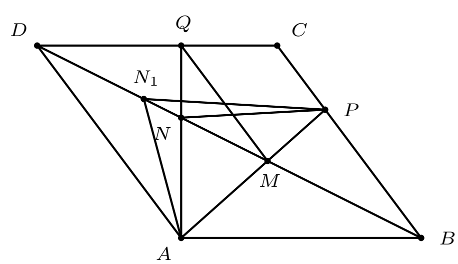

Figure 14

Proof. The key idea is that the statements $AN = NP$ and $AM = MQ$ are equivalent to $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$.

First, if $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$, then $\measuredangle NAP = \measuredangle NBP$, and the quadrilateral $ABPN$ is cyclic. As $\measuredangle ABN = \measuredangle PBN$, we have $AN = NP$.

For the converse, we consider N_1 on BD such that $\measuredangle PAN_1 = \frac{1}{2}\measuredangle BAD$. As above, we get $AN_1 = N_1P$. But $AN = NP$ so that BD must be the perpendicular bisector of the segment AP. This is absurd. $\square$

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Cristinel Mortici: Valahia University of Târgovişte, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte, Romania

E-mail address: cmortici@valahia.ro

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An Extension of Triangle Constructions from Located Points

Harold Connelly

Abstract. W. Wernick has tabulated 139 triangle construction problems using a list of sixteen points associated with the triangle. We add four points to his list and find an additional 140 construction problems.

William Wernick [3] and Leroy Meyers [2] discussed the problem of constructing a triangle with ruler and compass given the location of three points associated with the triangle. Wernick tabulated all the significantly distinct problems that could be formed from the following list of sixteen points:

$A,\ B,\ C$	Three vertices
$M_a,\ M_b,\ M_c$	Three midpoints of the sides
$H_a,\ H_b,\ H_c$	Three feet of the altitudes
$T_a,\ T_b,\ T_c$	Three feet of the internal angle bisectors
$G,\ H,\ I,\ O$	The centroid, orthocenter, incenter and circumcenter

Wernick found 139 triples that could be made from these points. They can be divided into the following four distinct types:

R – Redundant. Given the location of two of the points of the triple, the location of the third point is determined. An example would be: A, B, M_c.

L – Locus Restricted. Given the location of two points, the third must lie on a certain locus. Example: A, B, O.

S – Solvable. Known ruler and compass solutions exist for these triples.

U – Unsolvable. By using algebraic means, it is possible to prove that no ruler and compass solution exists for these triples. Example: O, H, I; see [1, 4].

To extend the work of Wernick and Meyers, we add the following four points to their list:

$E_a,\ E_b,\ E_c$	Three Euler points, which are the midpoints between the vertices and the orthocenter
N	The center of the nine-point circle.

Tabulated below, along with their types, are all of the 140 significantly distinct triples that can be formed by adding our new points to the original sixteen. Problems that remain unresolved as to type are left blank. In keeping with the spirit

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of Wernick's article, we have listed all of the possible combinations of points that are significantly distinct, even though many of them are easily converted, using redundancies, to problems in Wernick's list. We point out that although many of the problems are quite simple, a few provide a fine challenge. Our favorites include $A,\ E_b,\ G$ (Problem 17) and $E_a,\ E_b,\ O$ (Problem 50).

1. A, B, E_a	**S**	36. A, M_a, N	**S**	71. E_a, H, T_b	**U**	106. E_a, M_b, T_c	
2. A, B, E_c	**S**	37. A, M_b, N	**S**	72. E_a, H_a, H_b	**S**	107. E_a, N, O	**S**
3. A, B, N	**S**	38. A, N, O	**S**	73. E_a, H_a, I	**S**	108. E_a, N, T_a	
4. A, E_a, E_b	**S**	39. A, N, T_a		74. E_a, H_a, M_a	**L**	109. E_a, N, T_b	
5. A, E_a, G	**S**	40. A, N, T_b		75. E_a, H_a, M_b	**S**	110. E_a, O, T_a	
6. A, E_a, H	**R**	41. E_a, E_b, E_c	**S**	76. E_a, H_a, N	**L**	111. E_a, O, T_b	
7. A, E_a, H_a	**L**	42. E_a, E_b, G	**S**	77. E_a, H_a, O	**S**	112. E_a, T_a, T_b	
8. A, E_a, H_b	**L**	43. E_a, E_b, H	**S**	78. E_a, H_a, T_a	**L**	113. E_a, T_b, T_c	
9. A, E_a, I	**S**	44. E_a, E_b, H_a	**S**	79. E_a, H_a, T_b		114. G, H, N	**R**
10. A, E_a, M_a	**S**	45. E_a, E_b, H_c	**S**	80. E_a, H_b, H_c	**L**	115. G, H_a, N	**S**
11. A, E_a, M_b	**S**	46. E_a, E_b, I	**U**	81. E_a, H_b, I		116. G, I, N	**U**
12. A, E_a, N	**S**	47. E_a, E_b, M_a	**L**	82. E_a, H_b, M_a	**L**	117. G, M_a, N	**S**
13. A, E_a, O	**S**	48. E_a, E_b, M_c	**S**	83. E_a, H_b, M_b	**S**	118. G, N, O	**R**
14. A, E_a, T_a	**S**	49. E_a, E_b, N	**L**	84. E_a, H_b, M_c	**S**	119. G, N, T_a	**U**
15. A, E_a, T_b	**U**	50. E_a, E_b, O	**S**	85. E_a, H_b, N	**L**	120. H, H_a, N	**S**
16. A, E_b, E_c	**S**	51. E_a, E_b, T_a		86. E_a, H_b, O	**S**	121. H, I, N	**U**
17. A, E_b, G	**S**	52. E_a, E_b, T_c	**U**	87. E_a, H_b, T_a		122. H, M_a, N	**S**
18. A, E_b, H	**S**	53. E_a, G, H	**S**	88. E_a, H_b, T_b	**U**	123. H, N, O	**R**
19. A, E_b, H_a	**S**	54. E_a, G, H_a	**S**	89. E_a, H_b, T_c		124. H, N, T_a	**U**
20. A, E_b, H_b	**L**	55. E_a, G, H_b	**S**	90. E_a, I, M_a	**S**	125. H_a, H_b, N	**L**
21. A, E_b, H_c	**S**	56. E_a, G, I		91. E_a, I, M_b		126. H_a, I, N	**S**
22. A, E_b, I		57. E_a, G, M_a	**S**	92. E_a, I, N	**S**	127. H_a, M_a, N	**L**
23. A, E_b, M_a	**S**	58. E_a, G, M_b	**S**	93. E_a, I, O		128. H_a, M_b, N	**L**
24. A, E_b, M_b	**S**	59. E_a, G, N	**S**	94. E_a, I, T_a		129. H_a, N, O	**S**
25. A, E_b, M_c	**S**	60. E_a, G, O	**S**	95. E_a, I, T_b		130. H_a, N, T_a	
26. A, E_b, N	**S**	61. E_a, G, T_a		96. E_a, M_a, M_b	**L**	131. H_a, N, T_b	
27. A, E_b, O	**S**	62. E_a, G, T_b		97. E_a, M_a, N	**R**	132. I, M_a, N	**S**
28. A, E_b, T_a		63. E_a, H, H_a	**L**	98. E_a, M_a, O	**S**	133. I, N, O	**U**
29. A, E_b, T_b		64. E_a, H, H_b	**L**	99. E_a, M_a, T_a		134. I, N, T_a	
30. A, E_b, T_c		65. E_a, H, I	**S**	100. E_a, M_a, T_b		135. M_a, M_b, N	**L**
31. A, G, N	**S**	66. E_a, H, M_a	**S**	101. E_a, M_b, M_c	**S**	136. M_a, N, O	**S**
32. A, H, N	**S**	67. E_a, H, M_b	**S**	102. E_a, M_b, N	**L**	137. M_a, N, T_a	
33. A, H_a, N	**S**	68. E_a, H, N	**S**	103. E_a, M_b, O	**S**	138. M_a, N, T_b	
34. A, H_b, N	**S**	69. E_a, H, O	**S**	104. E_a, M_b, T_a		139. N, O, T_a	**U**
35. A, I, N		70. E_a, H, T_a	**S**	105. E_a, M_b, T_b		140. N, T_a, T_b	

Many of the problems in our list can readily be converted to one in Wernick's list. Here are those by the application of a redundancy.

Problem	5	7	8	9	10	11	13	14	15	31	32	38
Wernick	40	45	50	57	24	33	16	55	56	16	16	16
Problem	53	63	64	65	66	67	69	70	71			
Wernick	40	45	50	57	24	33	16	55	56			
Problem	115	116	117	119	120	121	122	124	129	133	136	139
Wernick	75	80	66	79	75	80	66	79	75	80	66	79

A few solutions follow.

Problem 41. Given points E_a, E_b, E_c.
Solution. The orthocenter of triangle $E_aE_bE_c$ is also the orthocenter, H, of triangle ABC. Since E_a is the midpoint of AH, A can be found. Similarly, B and C.

Problem 50. Given points E_a, E_b, O.
Solution. Let P and Q be the midpoints of E_aO and E_bO, respectively. Let R be the reflection of P through Q. The line through E_b, perpendicular to E_aE_b, intersects the circle with diameter OR at M_a. The circumcircle, with center O and radius E_aM_a, intersects M_aR at B and C. The line through E_a perpendicular to BC intersects the circumcircle at A. There are in general two solutions.

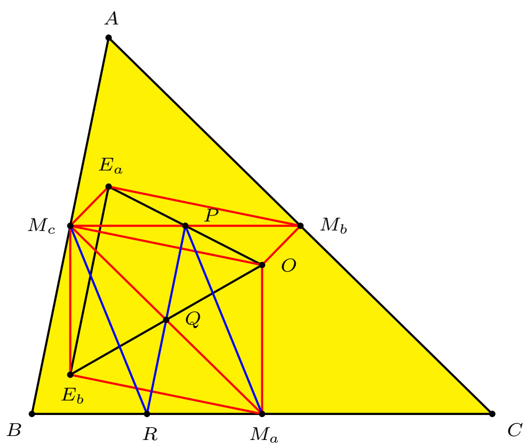

Figure 1.

Proof. In parallelogram $OM_aE_bM_c$, since diagonals bisect each other, Q is the midpoint of M_aM_c (see Figure 1). Similarly, P is the midpoint of M_bM_c. Since Q is also the midpoint of PR, PM_aRM_c is also a parallelogram and R must lie on BC. Therefore, the circle with diameter OR is a locus for M_a. Since M_aE_b is perpendicular to E_aE_b, the line through E_b perpendicular to PQ is a second locus for M_a. □

Problem 72. Given points E_a, H_a, H_b.
Solution. The line through Ha perpendicular to the line E_aH_a is the side BC. All three given points lie on the nine-point circle, so it can be found. The second intersection of the nine-point circle with BC gives M_a. The circle with M_a as

center and passing through H_b intersects the side BC at B and C. Finally, CH_b intersects E_aH_a at A.

Problem 103. Given points E_a, M_b, O.
Solution. The line through M_b, perpendicular to M_bO is AC. Reflecting AC through E_a, then dilating this line with O as center and ratio $\frac{1}{2}$ and finally intersecting this new line with the perpendicular bisector of E_aM_b gives N. Reflecting O through N gives H. E_aH intersects AC at A. The circumcircle, with center O passing through A, intersects AC again at C. The perpendicular from H to AC intersects the circumcircle at B.

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Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: cherylandharold@roadrunner.com

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Characterizations of a Tangential Quadrilateral

Nicuşor Minculete

Abstract. In this paper we will present several relations about the tangential quadrilaterals; among these, we have that the quadrilateral $ABCD$ is tangential if and only if the following equality

$$\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}$$

holds, where O is the point where the diagonals of convex quadrilateral $ABCD$ meet. This is equivalent to Wu's Theorem.

A tangential quadrilateral is a convex quadrilateral whose sides all tangent to a circle inscribed in the quadrilateral. [1] In a tangential quadrilateral, the four angle bisectors meet at the center of the inscribed circle. Conversely, a convex quadrilateral in which the four angle bisectors meet at a point must be tangential. A necessary and sufficient condition for a convex quadrilateral to be tangential is that its two pairs of opposite sides have equal sums (see [1, 2, 4]). In [5], Marius Iosifescu proved that a convex quadrilateral $ABCD$ is tangential if and only if

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2},$$

where x, y, z, w are the measures of the angles ABD, ADB, BDC, and DBC respectively (see Figure 1). In [3], Wu Wei Chao gave another characterization of tangential quadrilaterals. The two diagonals of any convex quadrilateral divide the quadrilateral into four triangles. Let r_1, r_2, r_3, r_4, in cyclic order, denote the radii of the circles inscribed in each of these triangles (see Figure 2). Wu found that the quadrilateral is tangential if and only if

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.$$

In this paper we find another characterization (Theorem 1 below) of tangential quadrilaterals. This new characterization is shown to be equivalent to Wu's condition and others (Proposition 2).

Consider a convex quadrilateral $ABCD$ with diagonals AC and BD intersecting at O. Denote the lengths of the sides AB, BC, CD, DA by a, b, c, d respectively.

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[1]Tangential quadrilateral are also known as circumscriptible quadrilaterals, see [2, p.135].

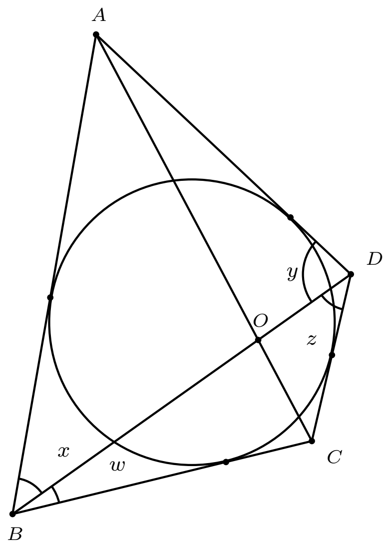

Figure 1

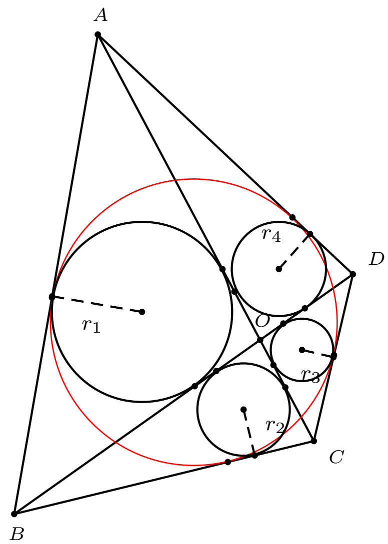

Figure 2

Theorem 1. *A convex quadrilateral $ABCD$ with diagonals intersecting at O is tangential if and only if*

$$\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}, \tag{1}$$

where $d(O,AB)$ is the distance from O to the line AB etc.

Proof. We first express (1) is an alternative form. Consider the projections M, N, P and Q of O on the sides AB, BC, CD, DA respectively.

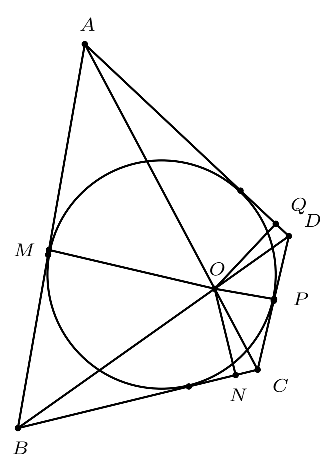

Figure 3

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. It is easy to see

$$\frac{OM}{d(C, AB)} = \frac{AO}{AC} = \frac{OQ}{d(C, AD)},$$
$$\frac{OM}{d(D, AB)} = \frac{BO}{BD} = \frac{ON}{d(D, BC)},$$
$$\frac{ON}{d(A, BC)} = \frac{OC}{AC} = \frac{OP}{d(A, DC)}.$$

This means

$$\frac{OM}{b \sin B} = \frac{OQ}{c \sin D}, \quad \frac{OM}{d \sin A} = \frac{ON}{c \sin C}, \quad \frac{ON}{a \sin B} = \frac{OP}{d \sin D}.$$

The relation (1) becomes

$$\frac{1}{OM} + \frac{1}{OP} = \frac{1}{ON} + \frac{1}{OQ},$$

which is equivalent to

$$1 + \frac{OM}{OP} = \frac{OM}{ON} + \frac{OM}{OQ},$$

or

$$1 + \frac{a \sin A \sin B}{c \sin C \sin D} = \frac{d \sin A}{c \sin C} + \frac{b \sin B}{c \sin D}.$$

Therefore (1) is equivalent to

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A. \tag{2}$$

Now we show that $ABCD$ is tangential if and only if (2) holds.

($\Rightarrow$) If the quadrilateral $ABCD$ is tangential, then there is a circle inscribed in the quadrilateral. Let r be the radius of this circle. Then

$$a = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right), \qquad b = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right),$$
$$c = r\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right), \qquad d = r\left(\cot\frac{D}{2} + \cot\frac{A}{2}\right).$$

Hence,

$$\begin{aligned} a\sin A\sin B &= r\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\cdot 4\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2} \\ &= 4r\left(\cos\frac{A}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2} \\ &= 4r\sin\frac{A+B}{2}\cos\frac{A}{2}\cos\frac{B}{2} \\ &= 4r\sin\frac{C+D}{2}\cos\frac{A}{2}\cos\frac{B}{2} \\ &= 4r\left(\cos\frac{D}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2} \\ &= 4r\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} b\sin B\sin C &= 4r\left(\tan\frac{D}{2}+\tan\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}, \\ c\sin C\sin D &= 4r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}, \\ d\sin D\sin A &= 4r\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}. \end{aligned}$$

From these relations it is clear that (2) holds.

($\Leftarrow$) We assume (2) and $ABCD$ not tangential. From these we shall deduce a contradiction.

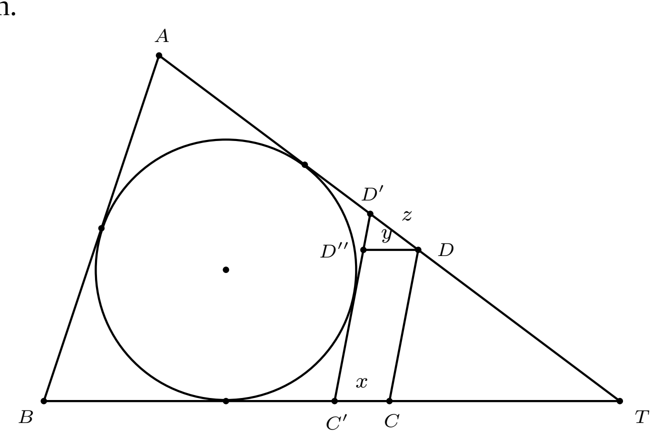

Figure 4.

Case 1. Suppose the opposite sides of $ABCD$ are not parallel.

Let T be the intersection of the lines AD and BC. Consider the incircle of triangle ABT (see Figure 4). Construct a parallel to the side DC which is tangent to the circle, meeting the sides BC and DA at C' and D' respectively. Let $BC' =$

b', $C'D' = c'$, $D'A = d'$, $C'C = x$, $D''D' = y$, and $D'D = z$, and where D'' is the point on $C'D'$ such that $C'CDD''$ is a parallelogram. Note that

$$b = b' + x, \quad c = c' - y, \quad d = d' + z.$$

Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c' \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \tag{3}$$

Comparison of (2) and (3) gives

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A,$$

we have

$$-y \sin C \sin D = x \sin B \sin C + z \sin D \sin A.$$

This is a contradiction since x, y, z all have the same sign, [2] and the trigonometric ratios are all positive.

Case 2. Now suppose $ABCD$ has a pair of parallel sides, say AD and BC. Consider the circle tangent to the sides AB, BC and DA (see Figure 5).

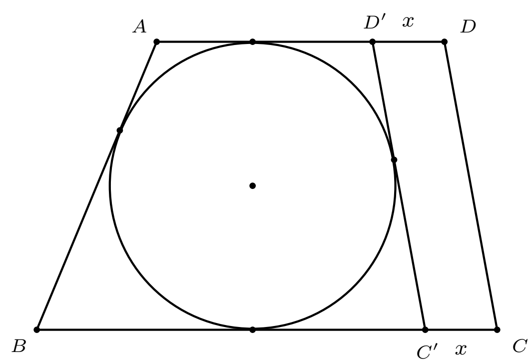

Figure 5.

Construct a parallel to DC, tangent to the circle, and intersecting BC, DA at C' and D' respectively. Let $C'C = D'D = x$, $BC' = b'$, and $D'A = d'$. [3] Clearly, $b' = b - x$, $d = d' + x$, and $C'D' = CD = c$. Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \tag{4}$$

Comparing this with (2), we have $x(\sin B \sin C + \sin D \sin A) = 0$. Since $x \neq 0$, $\sin A = \sin B$ and $\sin C = \sin D$, this reduces to $2 \sin A \sin C = 0$, a contradiction. □

Proposition 2. *Let O be the point where the diagonals of the convex quadrilateral $ABCD$ meet and r_1, r_2, r_3, and r_4 respectively the radii of the circles inscribed in the triangles AOB, BOC, COD and DOA respectively. The following statements are equivalent:*

[2] In Figure 4, the circle does not intersect the side CD. In case it does, we treat x, y, z as negative.

[3] Again, if the circle intersects CD, then x is regarded as negative.

(a) $\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}$.
(b) $\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}$.
(c) $\frac{a}{\Delta AOB}+\frac{c}{\Delta COD}=\frac{b}{\Delta BOC}+\frac{d}{\Delta DOA}$.
(d) $a\cdot\Delta COD+c\cdot\Delta AOB=b\cdot\Delta DOA+d\cdot\Delta BOC$.
(e) $a\cdot OC\cdot OD+c\cdot OA\cdot OB=b\cdot OA\cdot OD+d\cdot OB\cdot OC$.

Proof. (a) ⇔ (b). The inradius of a triangle is related to the altitudes by the simple relation
$$\frac{1}{r}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}.$$
Applying this to the four triangles AOB, BOC, COD, and DOA, we have
$$\begin{aligned}\frac{1}{r_1}&=\frac{1}{d(O,AB)}+\frac{1}{d(A,BD)}+\frac{1}{d(B,AC)},\\ \frac{1}{r_2}&=\frac{1}{d(O,BC)}+\frac{1}{d(C,BD)}+\frac{1}{d(B,AC)},\\ \frac{1}{r_3}&=\frac{1}{d(O,CD)}+\frac{1}{d(C,BD)}+\frac{1}{d(D,AC)},\\ \frac{1}{r_4}&=\frac{1}{d(O,DA)}+\frac{1}{d(A,BD)}+\frac{1}{d(D,AC)}.\end{aligned}$$
From these the equivalence of (a) and (b) is clear.

(b) ⇔ (c) is clear from the fact that $\frac{1}{d(O,AB)}=\frac{a}{a\cdot d(O,AB)}=\frac{a}{2\Delta AOB}$ etc.

The equivalence of (c), (d) and (e) follows from follows from
$$\Delta AOB=\frac{1}{2}\cdot OA\cdot OB\cdot\sin\varphi$$
etc., where φ is the angle between the diagonals. Note that
$$\Delta AOB\cdot\Delta COD=\Delta BOC\cdot\Delta DOA=\frac{1}{4}\cdot OA\cdot OB\cdot OC\cdot OD\cdot\sin^2\varphi.$$
□

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Nicuşor Minculete: "Dimitrie Cantemir" University of Braşov, Department of REI, Str. Bisericii Române nr. 107, Braşov, Romania
E-mail address: `minculeten@yahoo.com`

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A Note on the Anticomplements of the Fermat Points

Cosmin Pohoata

Abstract. We show that each of the anticomplements of the Fermat points is common to a triad of circles involving the triangle of reflection. We also generate two new triangle centers as common points to two other triads of circles. Finally, we present several circles passing through these new centers and the anticomplements of the Fermat points.

1. Introduction

The Fermat points $F_\pm$ are the common points of the lines joining the vertices of a triangle $\mathbf{T}$ to the apices of the equilateral triangles erected on the corresponding sides. They are also known as the isogonic centers (see [2, pp.107, 151]) and are among the basic triangle centers. In [4], they appear as the triangle centers X_{13} and X_{14}. Not much, however, is known about their anticomplements, which are the points $P_\pm$ which divide $F_\pm G$ in the ratio $F_\pm G : GP_\pm = 1 : 2$.

Given triangle $\mathbf{T}$ with vertices A, B, C,
(i) let A', B', C' be the reflections of the vertices A, B, C in the respective opposite sides, and
(ii) for $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1 (see Figures 1A and 1B).

Theorem 1. *For $\varepsilon = \pm 1$, the circumcircles of triangles $A'B_\varepsilon C_\varepsilon$, $B'C_\varepsilon A_\varepsilon$, $C'A_\varepsilon B_\varepsilon$ are concurrent at the anticomplement $P_{-\varepsilon}$ of the Fermat point $F_{-\varepsilon}$.*

2. Proof of Theorem 1

For $\varepsilon = \pm 1$, let $O_{a,\varepsilon}$ be the center of the equilateral triangle $A_\varepsilon BC$; similarly for $O_{b,\varepsilon}$ and $O_{c,\varepsilon}$.

(1) We first note that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε, and C_ε. Rotating triangle $O_{a,\varepsilon}AB$ through B by an angle $\varepsilon \cdot \frac{\pi}{3}$, we obtain triangle $O_{a,-\varepsilon}C_\varepsilon B$. Therefore, the triangles are congruent and $O_{a,-\varepsilon}C_\varepsilon = O_{a,\varepsilon}A$. Similarly, $O_{a,-\varepsilon}B_\varepsilon = O_{a,\varepsilon}A$. Clearly, $O_{a,\varepsilon}A = O_{a,-\varepsilon}A'$. It follows that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε and C_ε. Figures 1A and 1B illustrate the cases $\varepsilon = +1$ and $\varepsilon = -1$ respectively.

(2) Let $A_1B_1C_1$ be the anticomplementary triangle of ABC. Since AA_1 and A_+A_- have a common midpoint, $AA_+A_1A_-$ is a parallelogram. The lines $AA_{-\varepsilon}$ and A_1A_ε are parallel. Since A_1 is the anticomplement of A, it follows that the line A_1A_ε is the anticomplement of the line $AA_{-\varepsilon}$. Similarly, B_1B_ε and C_1C_ε are the anticomplements of the lines $BB_{-\varepsilon}$ and $CC_{-\varepsilon}$. Since $AA_{-\varepsilon}$, $BB_{-\varepsilon}$, $CC_{-\varepsilon}$

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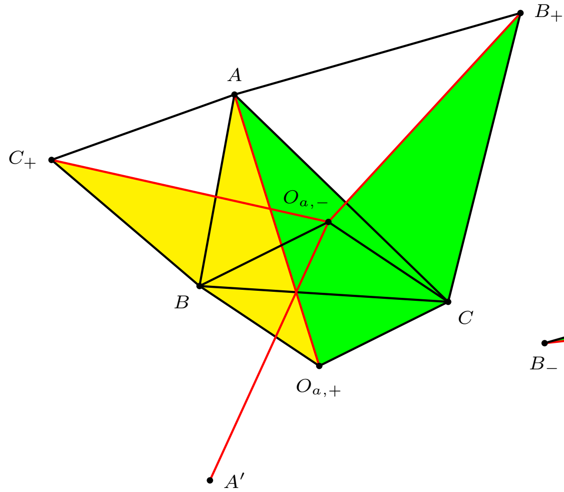

Figure 1A

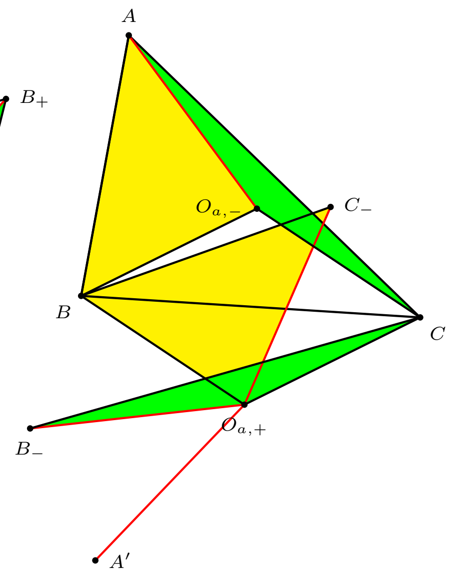

Figure 1B

concur at $F_{-\varepsilon}$, it follows that A_1A_ε, B_1B_ε, C_1C_ε concur at the anticomplement of $F_{-\varepsilon}$. This is the point $P_{-\varepsilon}$.

(3) $A_1B_1C_1$ is also the ε-Fermat triangle of $A_{-\varepsilon}B_{-\varepsilon}C_{-\varepsilon}$.

(i) Triangles $AB_\varepsilon C_\varepsilon$ and $CB_\varepsilon A_1$ are congruent, since $AB_\varepsilon = CB_\varepsilon$, $AC_\varepsilon = AB = CA_1$, and each of the angles $B_\varepsilon AC_\varepsilon$ and $B_\varepsilon CA_1$ is $\min\left(A + \frac{2\pi}{3},\ B + C + \frac{\pi}{3}\right)$. It follows that $B_\varepsilon C_\varepsilon = B_\varepsilon A_1$.

(ii) Triangles $AB_\varepsilon C_\varepsilon$ and BA_1C_ε are also congruent for the same reason, and we have $B_\varepsilon C_\varepsilon = A_1C_\varepsilon$.

It follows that triangle $A_1B_\varepsilon C_\varepsilon$ is equilateral, and $\angle C_\varepsilon A_1B_\varepsilon = \frac{\pi}{3}$.

(4) Because $P_{-\varepsilon}$ is the second Fermat point of $A_1B_1C_1$, we may assume $\angle C_\varepsilon P_{-\varepsilon}B_\varepsilon = \frac{\pi}{3}$. Therefore, $P_{-\varepsilon}$ lies on the circumcircle of $A_1B_\varepsilon C_\varepsilon$, which is the same as that of $A'B_\varepsilon C_\varepsilon$. On the other hand, since the quadrilateral $AA_+A_1A_-$ is a parallelogram (the diagonals AA_1 and A_-A_+ have a common midpoint D, the midpoint of segment BC), the anticomplement of the line AA_- coincides with A_1A_+. It now follows that the lines A_1A_+, B_1B_+, C_1C_+ are concurrent at the anticomplement P_- of the second Fermat points, and furthermore, $\angle C_+P_-B_+ = \frac{\pi}{3}$. Since

$$\begin{aligned}
\angle C_+AB_+ &= 2\pi - (\angle BAC_+ + \angle CAB + \angle B_+AC) \\
&= \frac{4\pi}{3} - \angle CAB \\
&= \frac{\pi}{3} + \angle ABC + \angle BCA \\
&= \angle ABC_+ + \angle ABC + \angle CBM \\
&= \angle C_+BM,
\end{aligned}$$

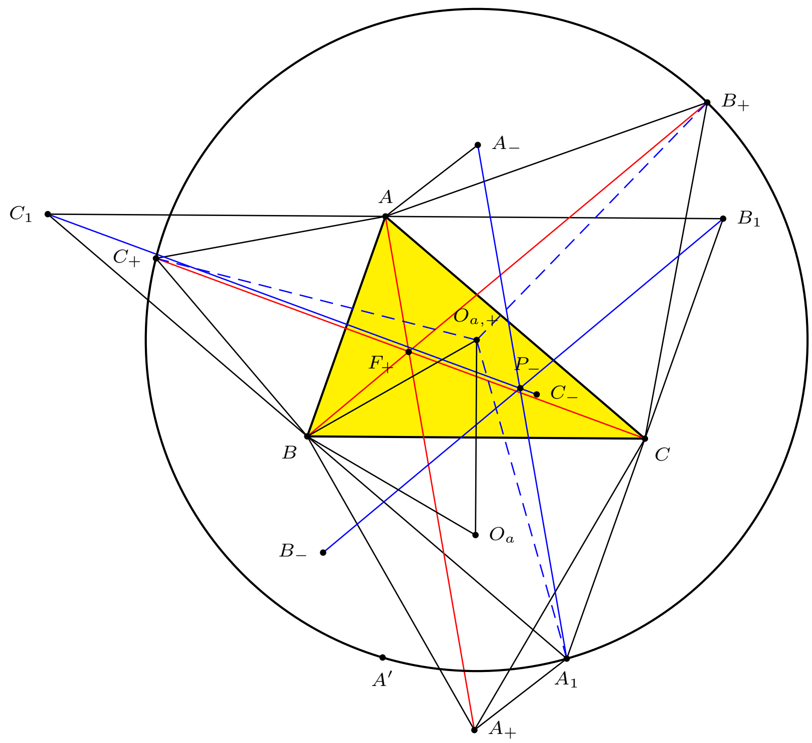

Figure 2

it follows that the triangles C_+AB_+ and C_+BM are congruent. Likewise, $\angle C_+AB_+ = \angle MCB_+$, and so the triangles C_+AB_+ and MCB_+ are congruent. Therefore, the triangle C_+MB_+ is equilateral, and thus $\angle C_+MB_+ = \frac{\pi}{3}$. Combining this with $\angle C_+P_-B_+ = 60°$, yields that the quadrilateral $MP_-B_+C_+$ is cyclic, and since $A'MB_+C_+$ is also cyclic, we conclude that the anticomplement P_- of the second Fermat point F_- lies on the circumcircle of triangle $A'B_+C_+$. Similarly, P_- lies on the circumcircles of triangles $B'C_+A_+$, and $C'A_+B_+$, respectively. This completes the proof of Theorem 1.

3. Two new triangle centers

By using the same method as in [6], we generate two other concurrent triads of circles.

Theorem 2. *For $\varepsilon = \pm 1$, the circumcircles of the triangles $A_\varepsilon B'C'$, $B_\varepsilon C'A'$, $C_\varepsilon A'B'$ are concurrent.*

Proof. Consider the inversion Ψ with respect to the anticomplement of the second Fermat point. According to Theorem 1, the images of the circumcircles of triangles $A'B_+C_+$, $B'C_+A_+$, $C'A_+B_+$ are three lines which bound a triangle $A'_+B'_+C'_+$, where A'_+, B'_+, C'_+ are the images of A_+, B_+, and C_+, respectively. Since the images A'', B'', C'' of A', B', C' under Ψ lie on the sidelines $B'_+C'_+$, $C'_+A'_+$, $A'_+B'_+$, respectively, by Miquel's theorem, we conclude that the circumcircles of triangles $A'_+B''C''$, $B'_+C''A''$, $C'_+A''B''$ are concurrent. Thus, the circumcircles of triangles $A_+B'C'$, $B_+C'A'$, $C_+A'B'$ are also concurrent (see Figure 3).

Similarly, inverting with respect to the anticomplement of the first Fermat point, by Miquel's theorem, one can deduce that the circumcircles of triangles $A_-B'C'$, $B_-C'A'$, $C_-A'B'$ are concurrent. □

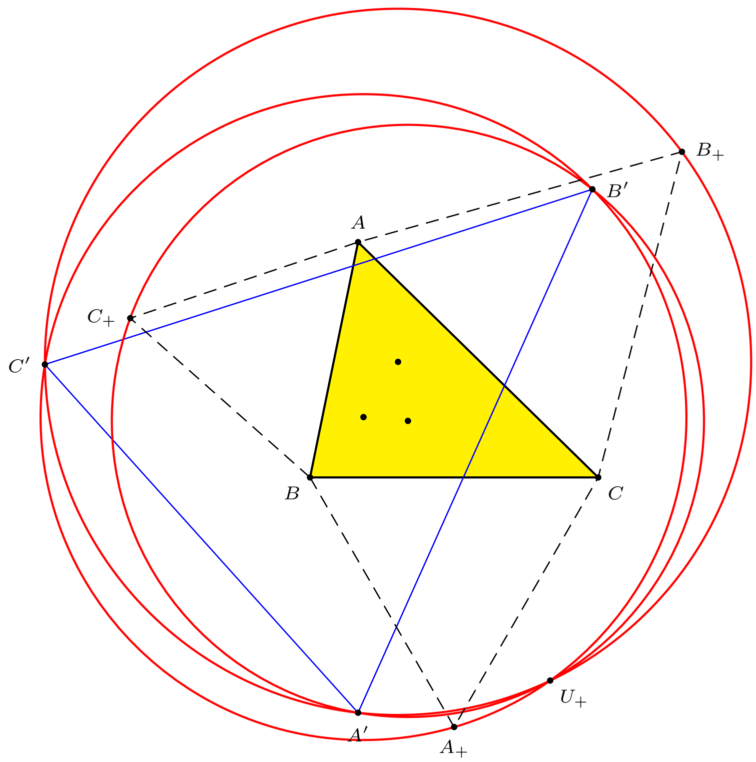

Figure 3.

Javier Francisco Garcia Capitan has kindly communicated that their points of concurrency do not appear in [4]. We will further denote these points by U_+, and U_-, respectively. We name these centers U_+, U_- the *inversive associates* of the anticomplements P_+, P_- of the Fermat points.

4. Circles around $P_\pm$ and their inversive associates

We denote by O, H the circumcenter, and orthocenter of triangle ABC. Let J_+, J_- be respectively the inner and outer isodynamic points of the triangle. Though the last two are known in literature as the common two points of the Apollonius circles, L. Evans [1] gives a direct relation between them and the Napoleonic configuration, defining them as the perspectors of the triangle of reflections $A'B'C'$

with each of the Fermat triangles $A_+B_+C_+$, and $A_-B_-C_-$. They appear as X_{15}, X_{16} in [4].

Furthermore, let W_+, W_- be the Wernau points of triangle ABC. These points are known as the common points of the folloing triads of circles: AB_+C_+, BC_+A_+, CA_+B_+, and respectively AB_-C_-, BC_-A_-, CA_-B_- [3]. According to the above terminology, W_+, W_- are the inversive associates of the Fermat points F_+, and F_-. They appear as X_{1337} and X_{1338} in [4].

We conclude with a list of concyclic quadruples involving these triangle centers. The first one is an immediate consequence of the famous Lester circle theorem [5]. The other results have been verified with the aid of Mathematica.

Theorem 3. *The following quadruples of points are concyclic:*
(i) P_+, P_-, O, H;
(ii) P_+, P_-, F_+, J_+;
(ii′) P_+, P_-, F_-, J_-;
(iii) P_+, U_+, F_+, O;
(iii′) P_-, U_-, F_-, O;
(iv) P_+, U_-, F_+, W_+;
(iv′) P_-, U_+, F_-, W_-;
(v) U_+, J_+, W_+, W_-;
(v′) U_-, J_-, W_+, W_-.

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Cosmin Pohoata: 13 Pridvorului Street, Bucharest, Romania 010014
E-mail address: `pohoata_cosmin2000@yahoo.com`

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Heptagonal Triangles and Their Companions

Paul Yiu

Abstract. A heptagonal triangle is a non-isosceles triangle formed by three vertices of a regular heptagon. Its angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. As such, there is a unique choice of a companion heptagonal triangle formed by three of the remaining four vertices. Given a heptagonal triangle, we display a number of interesting companion pairs of heptagonal triangles on its nine-point circle and Brocard circle. Among other results on the geometry of the heptagonal triangle, we prove that the circumcenter and the Fermat points of a heptagonal triangle form an equilateral triangle. The proof is an interesting application of Lester's theorem that the Fermat points, the circumcenter and the nine-point center of a triangle are concyclic.

1. The heptagonal triangle $\mathbf{T}$ and its companion

A heptagonal triangle $\mathbf{T}$ is one with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. Its vertices are three vertices of a regular heptagon inscribed in its circumcircle. Among the remaining four vertices of the heptagon, there is a unique choice of three which form another (congruent) heptagonal triangle $\mathbf{T}'$. We call this the companion of $\mathbf{T}$, and the seventh vertex of the regular heptagon the residual vertex of $\mathbf{T}$ and $\mathbf{T}'$ (see Figure 1). In this paper we work with complex number coordinates, and take the unit circle

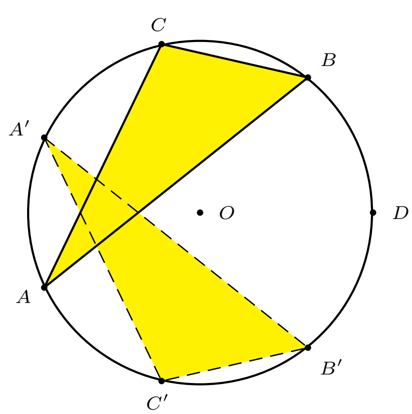

Figure 1. A heptagonal triangle and its companion

in the complex plane for the circumcircle of $\mathbf{T}$. By putting the residual vertex D at 1, we label the vertices of $\mathbf{T}$ by

$$A = \zeta^4, \qquad B = \zeta, \qquad C = \zeta^2,$$

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The author thanks the Editor for suggesting the configuration studied in §8, leading to, among other results, Theorem 20 on six circles concurring at the Feuerbach point of the heptagonal triangle.

and those of $\mathbf{T}'$ by

$$A' = \zeta^3, \qquad B' = \zeta^6, \qquad C' = \zeta^5,$$

where $\zeta := \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a primitive 7-th root of unity.

We study the triangle geometry of $\mathbf{T}$, some common triangle centers, lines, circles and conics associated with it. We show that the Simson lines of A', B', C' with respect to $\mathbf{T}$ are concurrent (Theorem 4). We find a number of interesting companion pairs of heptagonal triangles associated with $\mathbf{T}$. For example, the medial triangle and the orthic triangle of $\mathbf{T}$ form such a pair on the nine-point circle (Theorem 5), and the residual vertex is a point on the circumcircle of $\mathbf{T}$. It is indeed the Euler reflection point of $\mathbf{T}$. In the final section we prove that the circumcenter and the Fermat points form an equilateral triangle (Theorem 22). The present paper can be regarded as a continuation of Bankoff-Garfunkel [1].

2. Preliminaries

2.1. *Some simple coordinates.* Clearly, the circumcenter O of $\mathbf{T}$ has coordinate 0, and the centroid is the point $G = \frac{1}{3}(\zeta + \zeta^2 + \zeta^4)$. Since the orthocenter H and the nine-point center N are points (on the Euler line) satisfying

$$OG : GN : NH = 2 : 1 : 3,$$

we have

$$\begin{aligned} H &= \zeta + \zeta^2 + \zeta^4, \\ N &= \frac{1}{2}(\zeta + \zeta^2 + \zeta^4). \end{aligned} \tag{1}$$

This reasoning applies to any triangle with vertices on the unit circle. The bisectors of angles A, B, C of $\mathbf{T}$ intersect the circumcircle at $-C'$, A', B' respectively. These form a triangle whose orthocenter is the incenter I of $\mathbf{T}$ (see Figure 2). This latter is therefore the point

$$I = \zeta^3 - \zeta^5 + \zeta^6. \tag{2}$$

Similarly, the external bisectors of angles A, B, C intersect the circumcircle at C', $-A'$, $-B'$ respectively. Identifying the excenters of $\mathbf{T}$ as orthocenters of triangles with vertices on the unit circle, we have

$$\begin{aligned} I_a &= -(\zeta^3 + \zeta^5 + \zeta^6), \\ I_b &= \zeta^3 + \zeta^5 - \zeta^6, \\ I_c &= -\zeta^3 + \zeta^5 + \zeta^6. \end{aligned} \tag{3}$$

Figure 2 shows the tritangent circles of the heptagonal triangle $\mathbf{T}$.

2.2. *Representation of a companion pair*. Making use of the simple fact that the complex number coordinates of vertices of a regular heptagon can be obtained from any one of them by multiplications by $\zeta, \ldots, \zeta^6$, we shall display a companion pair of heptagonal triangle by listing coordinates of the center, the residual vertex and the vertices of the two heptagonal triangles, as follows.

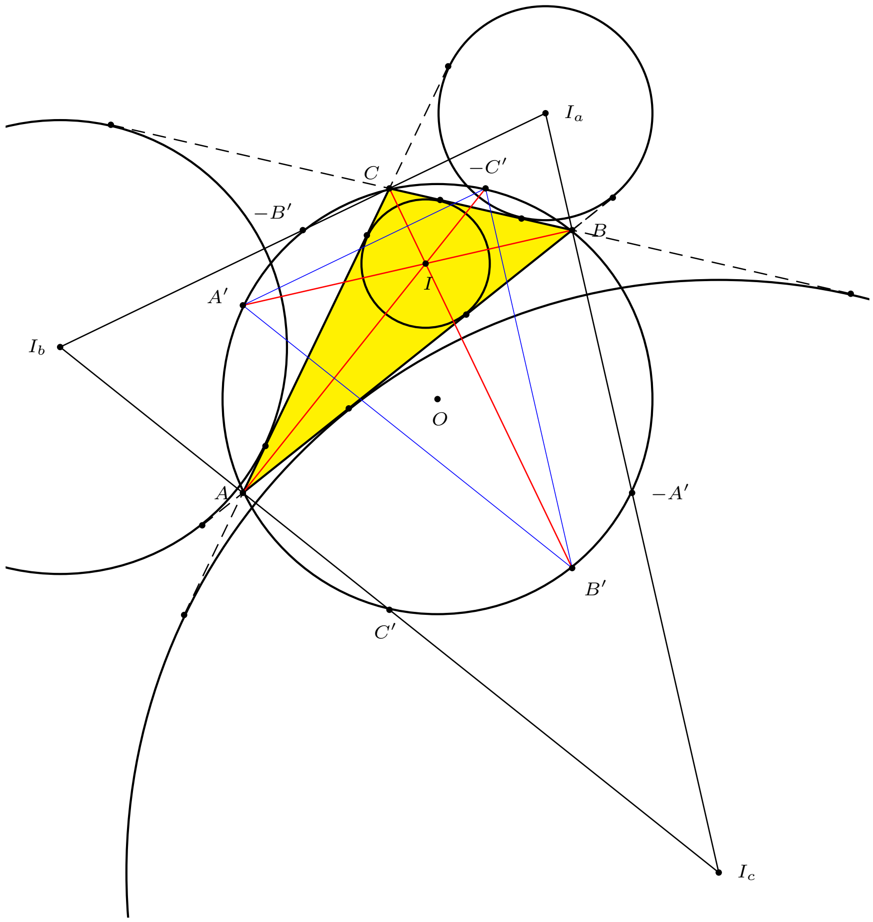

Figure 2. The tritangent centers

Center:	P
Residual vertex:	Q

Rotation	Vertices	Rotation	Vertices
ζ^4	$P + \zeta^4(Q - P)$	ζ^3	$P + \zeta^3(Q - P)$
ζ	$P + \zeta(Q - P)$	ζ^6	$P + \zeta^6(Q - P)$
ζ^2	$P + \zeta^2(Q - P)$	ζ^5	$P + \zeta^5(Q - P)$

2.3. While we shall mostly work in the cyclotomic field $\mathbb{Q}(\zeta)$, [1] the complex number coordinates of points we consider in this paper are *real* linear combinations of ζ^k for $0 \leq k \leq 6$, (the vertices of the regular heptagon on the circumcircle of

[1]See Corollary 23 for an exception.

$\mathbf{T}$). The real coefficients involved are rational combinations of

$$c_1 = \frac{\zeta+\zeta^6}{2} = \cos\frac{2\pi}{7}, \quad c_2 = \frac{\zeta^2+\zeta^5}{2} = \cos\frac{4\pi}{7}, \quad c_3 = \frac{\zeta^3+\zeta^4}{2} = \cos\frac{6\pi}{7}.$$

Note that $c_1 > 0$ and $c_2, c_3 < 0$. An expression of a complex number z as a real linear combination of ζ^4, ζ, ζ^2 (with sum of coefficients equal to 1) actually gives the absolute barycentric coordinate of the point z with reference to the heptagonal triangle $\mathbf{T}$. For example,

$$\begin{array}{rcrcrcr}
\zeta^3 & = & -2c_2\cdot\zeta^4 & + & 2c_2\cdot\zeta & + & 1\cdot\zeta^2, \\
\zeta^5 & = & 2c_1\cdot\zeta^4 & + & 1\cdot\zeta & - & 2c_1\cdot\zeta^2, \\
\zeta^6 & = & 1\cdot\zeta^4 & - & 2c_3\cdot\zeta & + & 2c_3\cdot\zeta^2, \\
1 & = & -2c_2\cdot\zeta^4 & - & 2c_3\cdot\zeta & - & 2c_1\cdot\zeta^2.
\end{array}$$

We shall make frequent uses of the important result.

Lemma 1 (Gauss). $1 + 2(\zeta + \zeta^2 + \zeta^4) = \sqrt{7}i$.

Proof. Although this can be directly verified, it is actually a special case of Gauss' famous theorem that if $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ for an odd integer n, then

$$\sum_{k=0}^{n-1}\zeta^{k^2} = \begin{cases}\sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ \sqrt{n}i & \text{if } n \equiv 3 \pmod 4.\end{cases}$$

For a proof, see [2, pp.75–76]. □

2.4. *Reflections and pedals.*

Lemma 2. *If α, β, γ are unit complex numbers, the reflection of γ in the line joining α and β is $\gamma' = \alpha + \beta - \alpha\beta\overline{\gamma}$.*

Proof. As points in the complex plane, γ' has equal distances from α and β as γ does. This is clear from

$$\begin{aligned}
\gamma' - \alpha &= \beta(1 - \alpha\overline{\gamma}) = \beta\overline{\gamma}(\gamma - \alpha), \\
\gamma' - \beta &= \alpha(1 - \beta\overline{\gamma}) = \alpha\overline{\gamma}(\gamma - \beta).
\end{aligned}$$

□

Corollary 3. (1) *The reflection of ζ^k in the line joining ζ^i and ζ^j is $\zeta^i + \zeta^j - \zeta^{i+j-k}$.*
(2) *The pedal* (orthogonal projection) *of ζ^k on the line joining ζ^i and ζ^j is*

$$\frac{1}{2}(\zeta^i + \zeta^j + \zeta^k - \zeta^{i+j-k}).$$

(3) *The reflections of A in BC, B in CA, and C in AB are the points*

$$\begin{aligned}
A^* &= \zeta + \zeta^2 - \zeta^6, \\
B^* &= \zeta^2 + \zeta^4 - \zeta^5, \qquad (4) \\
C^* &= \zeta - \zeta^3 + \zeta^4.
\end{aligned}$$

3. Concurrent Simson lines

The Simson line of a point on the circumcircle of a triangle is the line containing the pedals of the point on the sidelines of the triangle.

Theorem 4. *The Simson lines of A', B', C' with respect to the heptagonal triangle $\mathbf{T}$ are concurrent.*

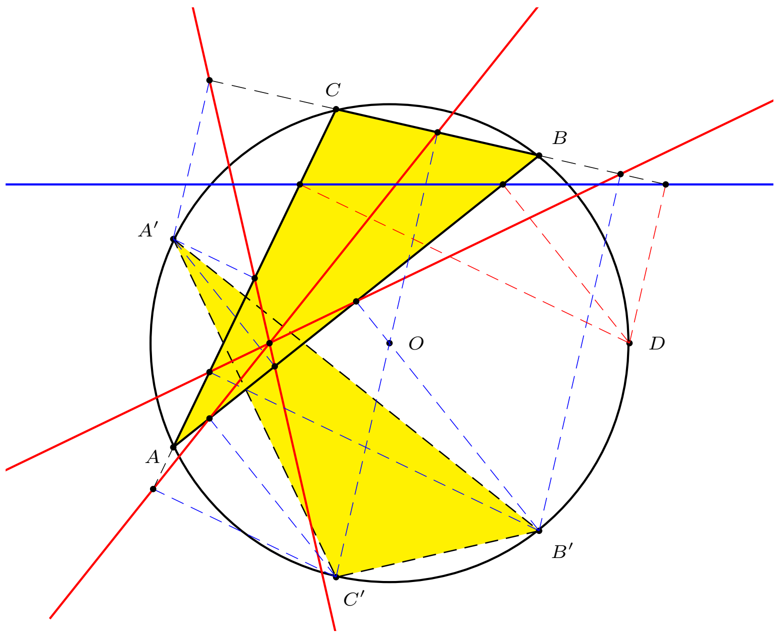

Figure 3. Simson lines

Proof. The pedals of A on BC is the midpoint A' of AA^*; similarly for those of B on CA and C on AB. We tabulate the coordinates of the pedals of A', B', C' on the sidelines BC, CA, AB respectively. These are easily calculated using Corollary 3.

	BC	CA	AB
A'	$\frac{1}{2}(-1+\zeta+\zeta^2+\zeta^3)$	$\frac{1}{2}(\zeta^2+\zeta^4)$	$\frac{1}{2}(\zeta-\zeta^2+\zeta^3+\zeta^4)$
B'	$\frac{1}{2}(\zeta+\zeta^2-\zeta^4+\zeta^6)$	$\frac{1}{2}(-1+\zeta^2+\zeta^4+\zeta^6)$	$\frac{1}{2}(\zeta+\zeta^4)$
C'	$\frac{1}{2}(\zeta+\zeta^2)$	$\frac{1}{2}(-\zeta+\zeta^2+\zeta^4+\zeta^5)$	$\frac{1}{2}(-1+\zeta+\zeta^4+\zeta^5)$

We check that the Simson lines of A', B', C' all contain the point $-\frac{1}{2}$. For these, it is enough to show that the complex numbers

$$(\zeta+\zeta^2+\zeta^3)(\overline{1+\zeta^2+\zeta^4}), \quad (\zeta^2+\zeta^4+\zeta^6)(\overline{1+\zeta+\zeta^4}), \quad (\zeta+\zeta^4+\zeta^5)(\overline{1+\zeta+\zeta^2})$$

are real. These are indeed $\zeta+\zeta^6$, $\zeta^2+\zeta^5$, $\zeta^3+\zeta^4$ respectively. $\square$

Remark. The Simson line of D, on the other hand, is parallel to OD (see Figure 3). This is because the complex number coordinates of the pedals of D, namely,

$$\frac{1+\zeta+\zeta^2-\zeta^3}{2}, \quad \frac{1+\zeta^2+\zeta^4-\zeta^6}{2}, \quad \frac{1+\zeta+\zeta^4-\zeta^5}{2},$$

all have the same imaginary part $\frac{1}{4}(\zeta-\zeta^6+\zeta^2-\zeta^5-\zeta^3+\zeta^4)$.

4. The nine-point circle

4.1. *A companion pair of heptagonal triangles on the nine-point circle.* As is well known, the nine-point circle is the circle through the vertices of the medial triangle and of the orthic triangle. The medial triangle of $\mathbf{T}$ clearly is heptagonal. It is known that $\mathbf{T}$ is the only obtuse triangle with orthic triangle similar to itself. [2] The medial and orthic triangles of $\mathbf{T}$ are therefore congruent. It turns out that they are companions.

Theorem 5. *The medial triangle and the orthic triangle of* $\mathbf{T}$ *are companion heptagonal triangles on the nine-point circle of* $\mathbf{T}$. *The residual vertex is the Euler reflection point* E (on the circumcircle of $\mathbf{T}$).

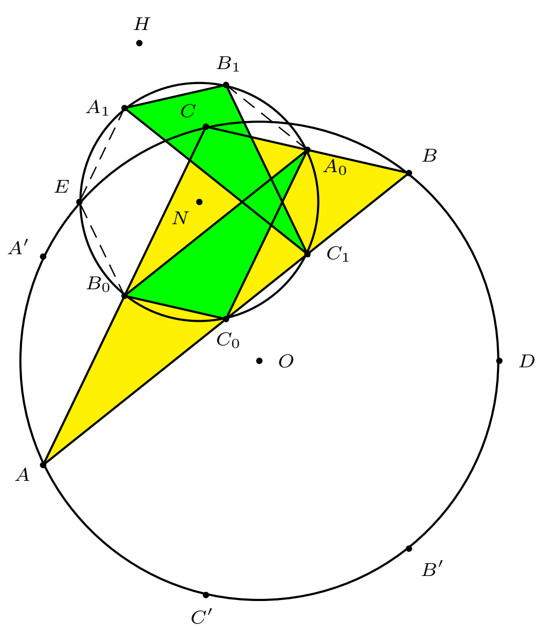

Figure 4. A companion pair on the nine-point circle

[2]If the angles of an obtuse angled triangle are $\alpha \le \beta < \gamma$, those of its orthic triangle are 2α, 2β, and $2\gamma-\pi$. The two triangles are similar if and only if $\alpha = 2\gamma-\pi$, $\beta = 2\alpha$ and $\gamma = 2\beta$. From these, $\alpha = \frac{\pi}{7}$, $\beta = \frac{2\pi}{7}$ and $\gamma = \frac{4\pi}{7}$. This shows that the triangle is heptagonal. The equilateral triangle is the only acute angled triangle similar to its own orthic triangle.

Proof. (1) The companionship of the medial and orthic triangles on the nine-point circle is clear from the table below.

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$E = \frac{1}{2}(-1 + \zeta + \zeta^2 + \zeta^4)$

Rotation	Medial triangle	Rotation	Orthic triangle
ζ^4	$A_0 = \frac{1}{2}(\zeta + \zeta^2)$	ζ^3	$C_1 = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4)$
ζ	$B_0 = \frac{1}{2}(\zeta^2 + \zeta^4)$	ζ^6	$A_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^6)$
ζ^2	$C_0 = \frac{1}{2}(\zeta + \zeta^4)$	ζ^5	$B_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^5)$

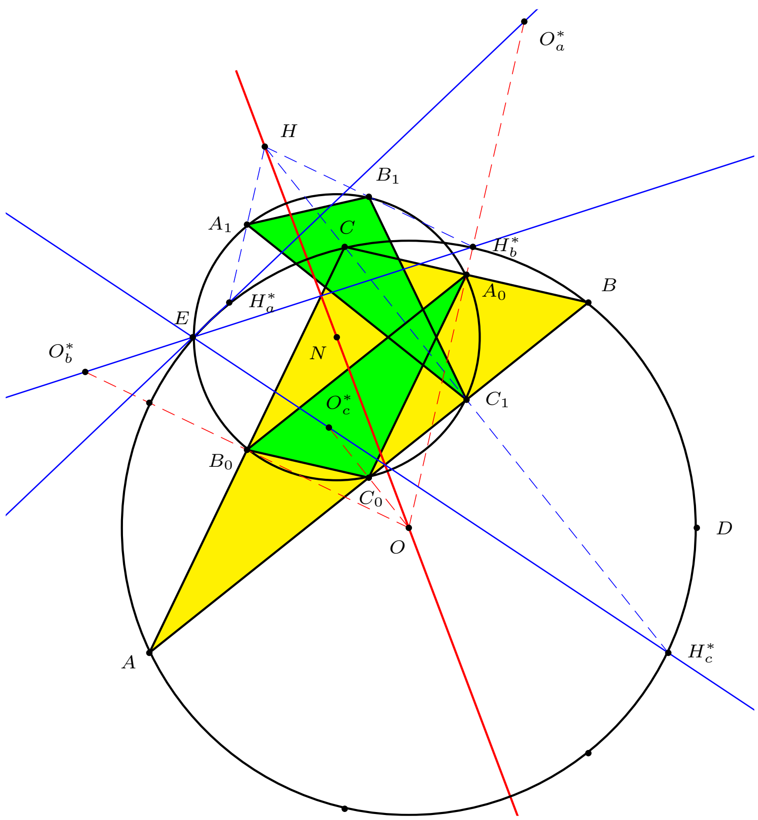

Figure 5. The Euler reflection point of $\mathbf{T}$

(2) We show that E is a point on the reflection of the Euler line in each of the sidelines of $\mathbf{T}$. In the table below, the reflections of O are computed from the simple fact that OBO_a^*C, OCO_b^*A, OAO_c^*B are rhombi. On the other hand, the reflections of H in the sidelines can be determined from the fact that HH_a^* and AA^* have the same midpoint, so do HH_b^* and BB^*, HH_c^* and CC^*. The various expressions for E given in the rightmost column can be routinely verified.

Line	Reflection of O	Reflection of H	$E =$
BC	$O_a^* = \zeta + \zeta^2$	$H_a^* = -\zeta^6$	$(-2c_1 - c_2 - c_3)O_a^* + (-c_2 - c_3)H_a^*$
CA	$O_b^* = \zeta^2 + \zeta^4$	$H_b^* = -\zeta^5$	$(-c_1 - 2c_2 - c_3)O_b^* + (-c_1 - c_3)H_b^*$
AB	$O_c^* = \zeta + \zeta^4$	$H_c^* = -\zeta^3$	$(-c_1 - c_2 - 2c_3)O_c^* + (-c_1 - c_2)H_c^*$

Thus, E, being the common point of the reflections of the Euler line of $\mathbf{T}$ in its sidelines, is the Euler reflection point of $\mathbf{T}$, and lies on the circumcircle of $\mathbf{T}$. $\square$

4.2. *The second intersection of the nine-point circle and the circumcircle.*

Lemma 6. *The distance between the nine-point center N and the A-excenter I_a is equal to the circumradius of the heptagonal triangle $\mathbf{T}$.*

Proof. Note that $I_a - N = \frac{2+\zeta+\zeta^2+\zeta^4}{2} = \frac{3+1+2(\zeta+\zeta^2+\zeta^4)}{4} = \frac{3+\sqrt{7}i}{4}$ is a unit complex number. $\square$

This simple result has a number of interesting consequences.

Proposition 7. (1) *The midpoint F_a of NI_a is the point of tangency of the nine-point circle and the A-excircle.*
(2) *The A-excircle is congruent to the nine-point circle.*
(3) *F_a lies on the circumcircle.*

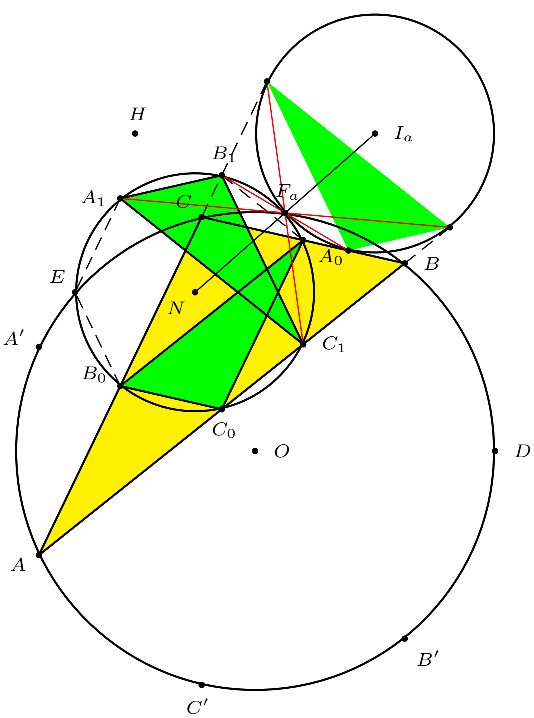

Figure 6. The A-Feuerbach point of $\mathbf{T}$

Proof. (1) By the Feuerbach theorem, the nine-point circle is tangent externally to each of the excircles. Since $NI_a = R$, the circumradius, and the nine-point circle has radius $\frac{1}{2}R$, the point of tangency with the A-excircle is the midpoint of NI_a, *i.e.*,

$$F_a = \frac{I_a + N}{2} = \frac{2 + 3(\zeta + \zeta^2 + \zeta^4)}{4}. \tag{5}$$

This proves (1).

(2) It also follows that the radius of the A-excircle is $\frac{1}{2}R$, and the A-excircle is congruent to the nine-point circle.

(3) Note that $F_a = \frac{1+3+6(\zeta+\zeta^2+\zeta^4)}{8} = \frac{1+3\sqrt{7}i}{8}$ is a unit complex number. □

Remark. The reflection of the orthic triangle in F_a is the A-extouch triangle, since the points of tangency are

$$-(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^3}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^5}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^6}{2}$$

(see Figure 6).

4.3. *Another companion pair on the nine-point circle.*

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$F_a = \frac{1}{4}(2 + 3(\zeta + \zeta^2 + \zeta^4))$

Rot.	Feuerbach triangle	Rot.	Companion
ζ^3	$F_b = \frac{1}{4}(\zeta + \zeta^2 + \zeta^3 + 2\zeta^4 - \zeta^6)$	ζ^4	$F'_a = \frac{1}{4}(3\zeta + 2\zeta^2 + 4\zeta^4 + \zeta^5 + \zeta^6)$
ζ^6	$F_e = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$	ζ	$F'_b = \frac{1}{4}(4\zeta + 3\zeta^2 + \zeta^3 + 2\zeta^4 + \zeta^5)$
ζ^5	$F_c = \frac{1}{4}(\zeta + 2\zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)$	ζ^2	$F'_c = \frac{1}{4}(2\zeta + 4\zeta^2 + \zeta^3 + 3\zeta^4 + \zeta^6)$

Proposition 8. *F_e, F_a, F_b, F_c are the points of tangency of the nine-point circle with the incircle and the A-, B-, C-excircles respectively* (see Figure 7).

Proof. We have already seen that $F_a = \frac{1}{2} \cdot N + \frac{1}{2} \cdot I_a$. It is enough to show that the points F_e, F_b, F_c lie on the lines NI, NI_b, NI_c respectively:

$$\begin{aligned} F_e &= -(c_1 - c_3) \cdot N + (-c_1 - 2c_2 - 3c_3) \cdot I, \\ F_b &= (c_2 - c_3) \cdot N + (-2c_1 - 3c_2 - c_3) \cdot I_b, \\ F_c &= (c_1 - c_2) \cdot N + (-3c_1 - c_2 - 2c_3) \cdot I_c. \end{aligned}$$

□

Proposition 9. *The vertices F'_a, F'_b, F'_c of the companion of $F_bF_eF_c$ are the second intersections of the nine-point circle with the lines joining F_a to A, B, C respectively.*

Proof.

$$\begin{aligned} F'_a &= -2c_2 \cdot F_a - 2(c_1 + c_3)A, \\ F'_b &= -2c_3 \cdot F_a - 2(c_1 + c_2)B, \\ F'_c &= -2c_1 \cdot F_a - 2(c_2 + c_3)C. \end{aligned}$$

□

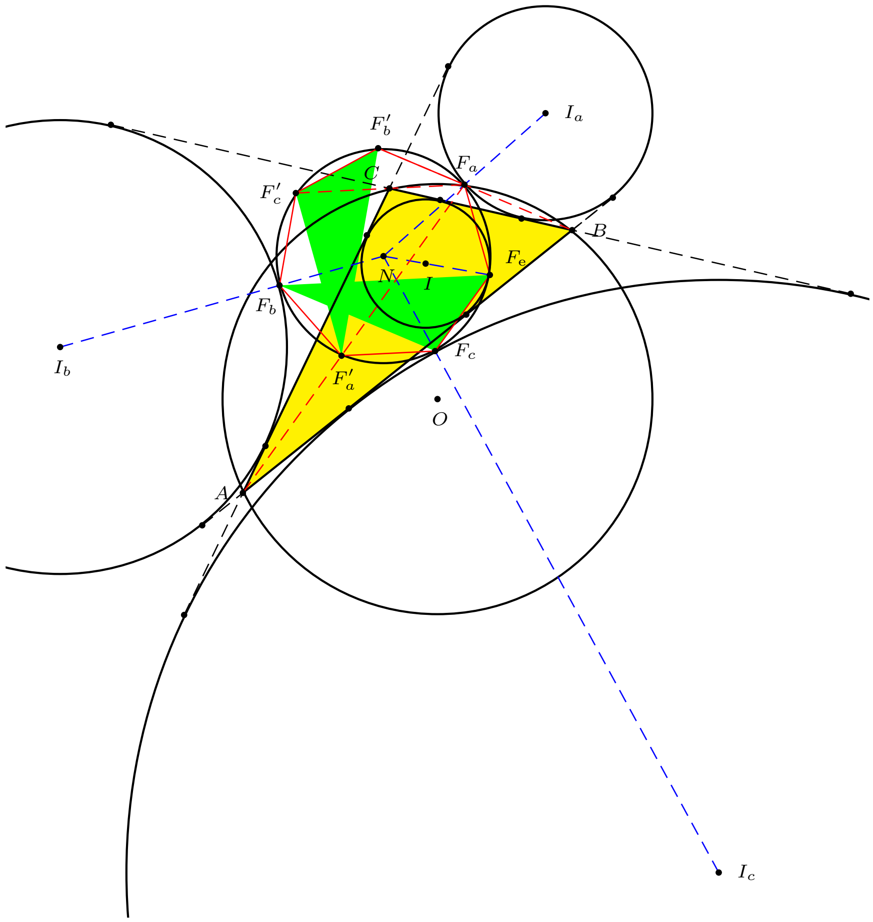

Figure 7. Another companion pair on the nine-point circle

5. The residual vertex as a Kiepert perspector

Theorem 10. *D is a Kiepert perspector of the heptagonal triangle ABC.*

Proof. What this means is that there are similar isosceles triangles $A''BC$, $B''CA$, $C''AB$ with the same orientation such that the lines AA'', BB'', CC'' all pass through the point D. Let A'' be the intersection of the lines AD and $A'B'$, B'' that of BD and $B'C'$, and C'' that of CD and $C'A'$ (see Figure 8). Note that $AC'B'A''$, $BAC'B''$, and $A'B'CC''$ are all parallelograms. From these,

$$A'' = \zeta^4 - \zeta^5 + \zeta^6,$$
$$B'' = \zeta - \zeta^3 + \zeta^5,$$
$$C'' = \zeta^2 + \zeta^3 - \zeta^6.$$

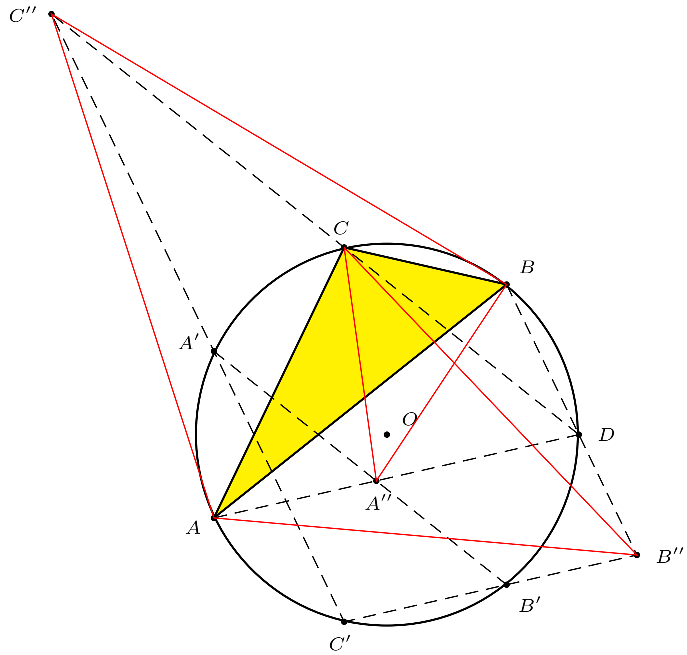

Figure 8. D as a Kiepert perspector of **T**

It is clear that the lines AA'', BB'' and CC'' all contain the point D. The coordinates of A'', B'', C'' can be rewritten as

$$A'' = \frac{\zeta^2+\zeta}{2} + \frac{\zeta^2-\zeta}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),$$
$$B'' = \frac{\zeta^4+\zeta^2}{2} + \frac{\zeta^4-\zeta^2}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),$$
$$C'' = \frac{\zeta+\zeta^4}{2} + \frac{\zeta-\zeta^4}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)).$$

Since $1+2(\zeta+\zeta^2+\zeta^4) = \sqrt{7}i$ (Gauss sum), these expressions show that the three isosceles triangles all have base angles $\arctan\sqrt{7}$. Thus, the triangles $A''BC$, $B''CA$, $C''AB$ are similar isosceles triangles of the same orientation. From these we conclude that D is a point on the Kiepert hyperbola. □

Corollary 11. *The center of the Kiepert hyperbola is the point*

$$K_\text{i} = -\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6). \tag{6}$$

Proof. Since D is the intersection of the Kiepert hyperbola and the circumcircle, the center of the Kiepert hyperbola is the midpoint of DH, where H is the orthocenter of triangle ABC (see Figure 9). This has coordinate as given in (6) above. □

Remark. K_i is also the midpoint of OI_a.

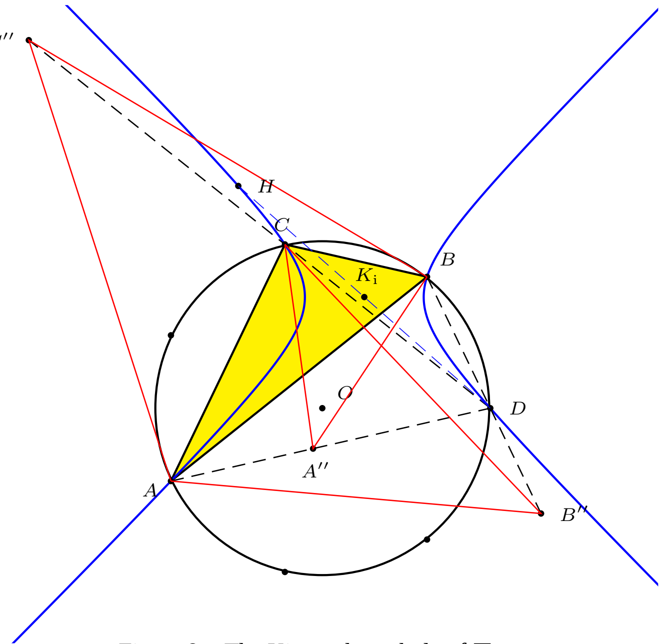

Figure 9. The Kiepert hyperbola of $\mathbf{T}$

Since $X = -1$ is antipodal to the Kiepert perspector $D = 1$ on the circumcircle, it is the Steiner point of $\mathbf{T}$, which is the fourth intersection of the Steiner ellipse with the circumcircle. The Steiner ellipse also passes through the circumcenter, the A-excenter, and the midpoint of HG. The tangents at I_a and X pass through H, and that at O passes through $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$ on the circumcircle such that $OXNY$ is a parallelogram (see Lemma 21).

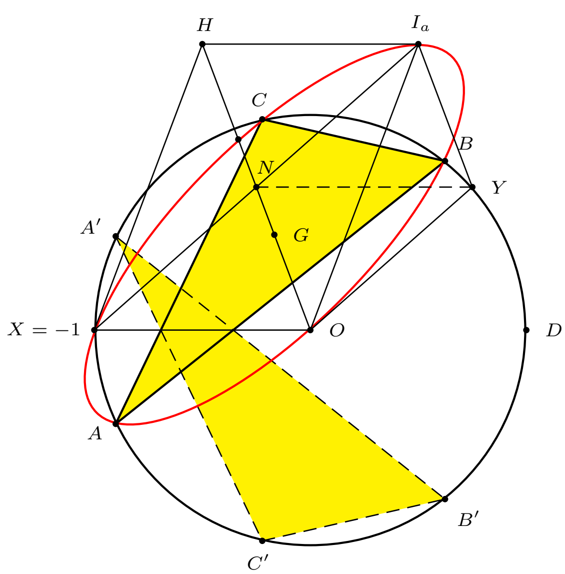

Figure 10. The Steiner ellipse of $\mathbf{T}$

6. The Brocard circle

6.1. *The Brocard points.*

Proposition 12 (Bankoff and Garfunkel)**.** *The nine-point center N is the first Brocard point.*

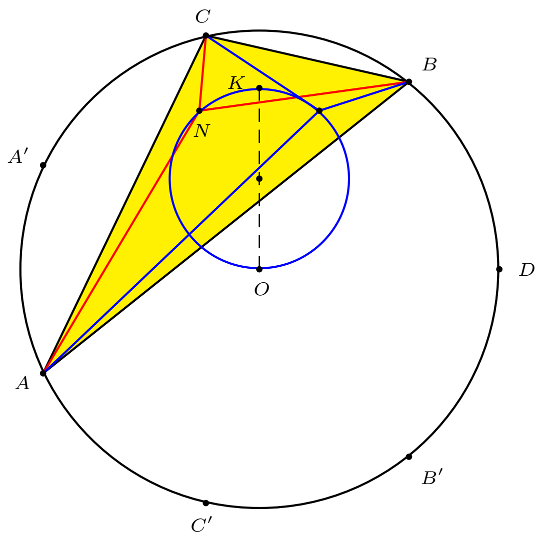

Figure 11. The Brocard points of the heptagonal triangle $\mathbf{T}$

Proof. The relations

$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^4=\frac{(-2c_1-3c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta-\zeta^4),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta=\frac{(-2c_1-2c_2-3c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^2-\zeta),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^2=\frac{(-3c_1-2c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^4-\zeta^2)$$

show that the lines NA, NB, NC are obtained by rotations of BA, CB, AC through the same angle (which is necessarily the Brocard angle ω). This shows that the nine-point center N is the first Brocard point of the heptagonal triangle $\mathbf{T}$. □

Remark. It follows that $4+\zeta+\zeta^2+\zeta^4=\sqrt{14}(\cos\omega+i\sin\omega)$.

Proposition 13. *The symmedian point K has coordinate* $\frac{2(1+2(\zeta+\zeta^2+\zeta^4))}{7}=\frac{2i}{\sqrt{7}}$.

Proof. It is known that on the Brocard circle with diameter OK, $\angle NOK = -\omega$. From this,

$$\begin{aligned}
K &= \frac{1}{\cos\omega}(\cos\omega - i\sin\omega)\cdot N\\
&= \left(1-\frac{i}{\sqrt{7}}\right)\cdot N\\
&= \frac{2(4+\zeta^3+\zeta^5+\zeta^6)}{7}\cdot\frac{\zeta+\zeta^2+\zeta^4}{2}\\
&= \frac{2}{7}(1+2(\zeta+\zeta^2+\zeta^4))\\
&= \frac{2i}{\sqrt{7}}
\end{aligned}$$

by Lemma 1. □

Corollary 14. *The second Brocard point is the Kiepert center* $K_{\rm i}$.

Proof. By Proposition 13, the Brocard axis OK is along the imaginary axis. Now, the second Brocard point, being the reflection of N in OK, is simply $-\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6)$. This, according to Corollary 11, is the Kiepert center $K_{\rm i}$. □

Since OD is along the real axis, it is tangent to the Brocard circle.

6.2. *A companion pair on the Brocard circle.*

Center:	$\frac{1}{7}(1+2(\zeta+\zeta^2+\zeta^4))$
Residual vertex:	$O=0$

Rot.	First Brocard triangle	Rot.	Companion
ζ^3	$A_{-\omega}=\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot(-\zeta^5)$	ζ^4	$\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot\zeta^2$
ζ^6	$B_{-\omega}=\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot(-\zeta^3)$	ζ	$\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot\zeta^4$
ζ^5	$C_{-\omega}=\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot(-\zeta^6)$	ζ^2	$\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot\zeta$

Since $-\zeta^5$ is the midpoint of the minor arc joining ζ and ζ^2, the coordinate of the point labeled $A_{-\omega}$ shows that this point lies on the perpendicular bisector of BC. Similarly, $B_{-\omega}$ and $C_{-\omega}$ lie on the perpendicular bisectors of CA and AB respectively. Since these points on the Brocard circle, they are the vertices of the first Brocard triangle.

The vertices of the companion are the second intersections of the Brocard circle with and the lines joining O to C, A, B respectively.

Proposition 15. *The first Brocard triangle is perspective with* ABC *at the point* $-\frac{1}{2}$ (see Figure 12).

Proof.

$$\begin{aligned}
-\frac{1}{2} &= (-3c_1-2c_2-2c_3)\cdot A_{-\omega}+c_1\cdot\zeta^4,\\
&= (-2c_1-3c_2-2c_3)\cdot B_{-\omega}+c_2\cdot\zeta,\\
&= (-2c_1-2c_2-3c_3)\cdot C_{-\omega}+c_3\cdot\zeta^2.
\end{aligned}$$

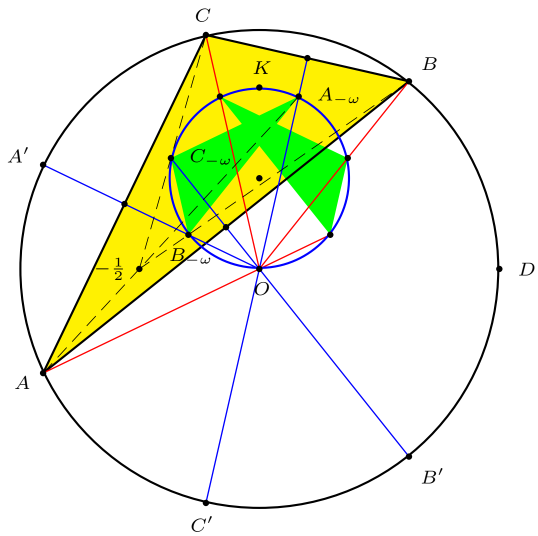

Figure 12. A regular heptagon on the Brocard circle

□

7. A companion of the triangle of reflections

We have computed the coordinates of the vertices of the triangle of reflections $A^*B^*C^*$ in (4). It is interesting to note that this is also a heptagonal triangle, and its circumcenter coincides with I_a. The residual vertex is the reflection of O in I_a.

Center:	$I_a = -(\zeta^3 + \zeta^5 + \zeta^6)$
Residual vertex:	$\overline{D} = -2(\zeta^3 + \zeta^5 + \zeta^6)$

Rotation	Triangle of reflections	Rotation	Companion
ζ^4	$A^* = \zeta + \zeta^2 - \zeta^6$	ζ^3	$\overline{B} = 1 + \zeta^4 - \zeta^6$
ζ	$B^* = \zeta^2 + \zeta^4 - \zeta^5$	ζ^6	$\overline{C} = 1 + \zeta - \zeta^5$
ζ^2	$C^* = \zeta - \zeta^3 + \zeta^4$	ζ^5	$\overline{A} = 1 + \zeta^2 - \zeta^3$

The companion has vertices on the sides of triangle ABC,

$$\overline{A} = (1 + 2c_1)\zeta - 2c_1 \cdot \zeta^2;$$
$$\overline{B} = (1 + 2c_2)\zeta^2 - 2c_2 \cdot \zeta^4;$$
$$\overline{C} = (1 + 2c_3)\zeta^4 - 2c_3 \cdot \zeta.$$

It is also perspective with $\mathbf{T}$. Indeed, the lines $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ are all perpendicular to the Euler line, since the complex numbers

$$\frac{1 + \zeta^2 - \zeta^3 - \zeta^4}{\zeta + \zeta^2 + \zeta^4}, \quad \frac{1 + \zeta^4 - \zeta^6 - \zeta}{\zeta + \zeta^2 + \zeta^4}, \quad \frac{1 + \zeta - \zeta^5 - \zeta^2}{\zeta + \zeta^2 + \zeta^4}$$

are all imaginary, being respectively $-\sqrt{2}(\zeta^2 - \zeta^5)$, $\sqrt{2}(\zeta^3 - \zeta^4)$, $-\sqrt{2}(\zeta - \zeta^6)$.

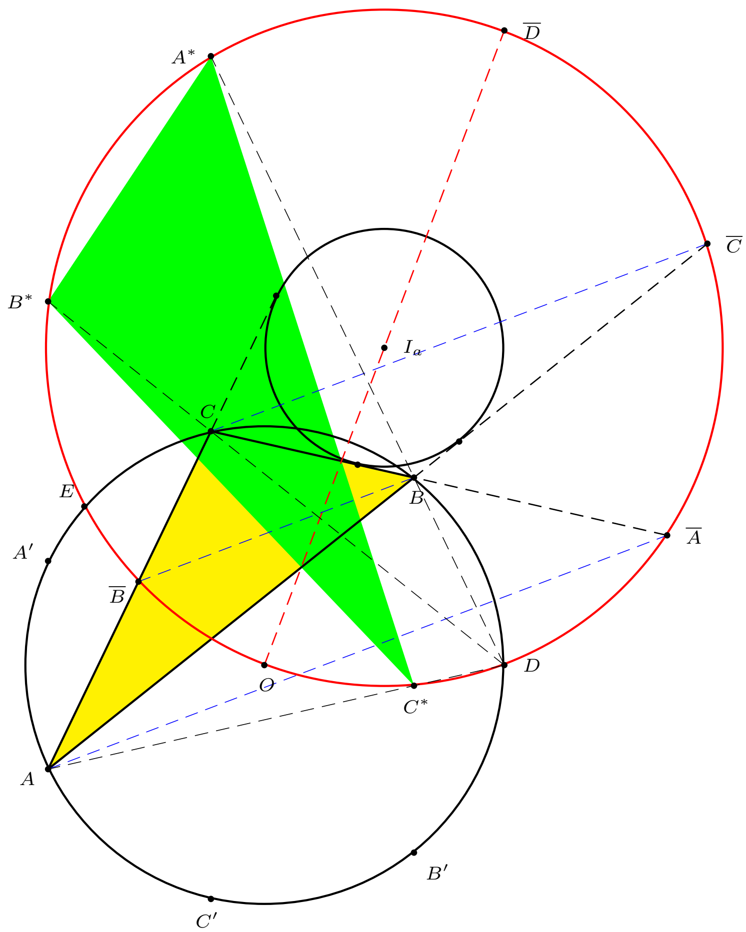

Figure 13. The triangle of reflections of $\mathbf{T}$

Proposition 16. *The triangle of reflections $A^*B^*C^*$ is triply perspective with* $\mathbf{T}$.

Proof. The triangle of reflection $A^*B^*C^*$ is clearly perspective with ABC at the orthocenter H. Since A^*C, B^*A, C^*B are all parallel (to the imaginary axis), the two triangles are triply perspective ([3, Theorem 381]). In other words, $A^*B^*C^*$ is also perspective with BCA. In fact, the perspector is the residual vertex D:

$$\begin{aligned} A^* &= -(1+2c_1)\cdot 1 + (2+2c_1)\zeta, \\ B^* &= -(1+2c_2)\cdot 1 + (2+2c_2)\zeta^2, \\ C^* &= -(1+2c_3)\cdot 1 + (2+2c_3)\zeta^4. \end{aligned}$$

□

Remark. The circumcircle of the triangle of reflections also contains the circumcenter O, the Euler reflection point E, and the residual vertex D.

8. A partition of T by the bisectors

Let $A_IB_IC_I$ be the cevian triangle of the incenter I of the heptagonal triangle $\mathbf{T} = ABC$. It is easy to see that triangles BCI, ACC_I and BB_IC are also heptagonal. Each of these is the image of the heptagonal triangle ABC under an affine mapping of the form $w = \alpha z + \beta$ or $w = \alpha\overline{z} + \beta$, according as the triangles have the same or different orientations. Note that the image triangle has circumcenter β and circumradius $|\alpha|$.

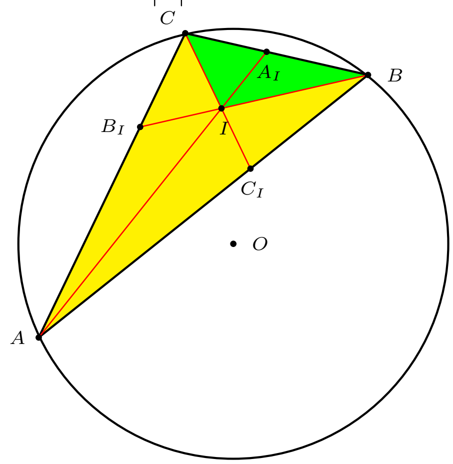

Figure 14. Partition of $\mathbf{T}$ by angle bisectors

Each of these mappings is determined by the images of two vertices. For example, since ABC and BCI have the same orientation, the mapping $f_1(z) = \alpha z + \beta$ is determined by the images $f_1(A) = B$ and $f_1(B) = C$; similarly for the mappings f_2 and f_3.

Affine mapping	A	B	C
$f_1(z) = (\zeta + \zeta^4)z - \zeta^5$	B	C	I
$f_2(z) = (1 + \zeta + \zeta^3 + \zeta^4)\overline{z} - (1 + \zeta^3 + \zeta^6)$	A	C	C_I
$f_3(z) = (1 + \zeta^2 + \zeta^4 + \zeta^5)\overline{z} - (1 + \zeta^3 + \zeta^5)$	B	B_I	C

Thus, we have

$$\begin{aligned} I =&\ f_1(C) = \zeta^3 - \zeta^5 + \zeta^6, \\ C_I =&\ f_2(C) = -1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5, \\ B_I =&\ f_3(B) = -1 + \zeta + \zeta^4 - \zeta^5 + \zeta^6. \end{aligned}$$

Note also that from $f_2(A_I) = I$, it follows that

$$A_I = 1 + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^6.$$

Remark. The affine mapping that associates a heptagonal triangle with circumcenter c and residual vertex d to its companion is given by

$$w = \frac{d - c}{\overline{d} - \overline{c}} \cdot \overline{z} + \frac{\overline{d}c - \overline{c}d}{\overline{d} - \overline{c}}.$$

8.1. *Four concurrent lines.* A simple application of the mapping f_1 yields the following result on the concurrency of four lines.

Proposition 17. *The orthocenter of the heptagonal triangle BCI lies on the line OC and the perpendicular from C_I to AC.*

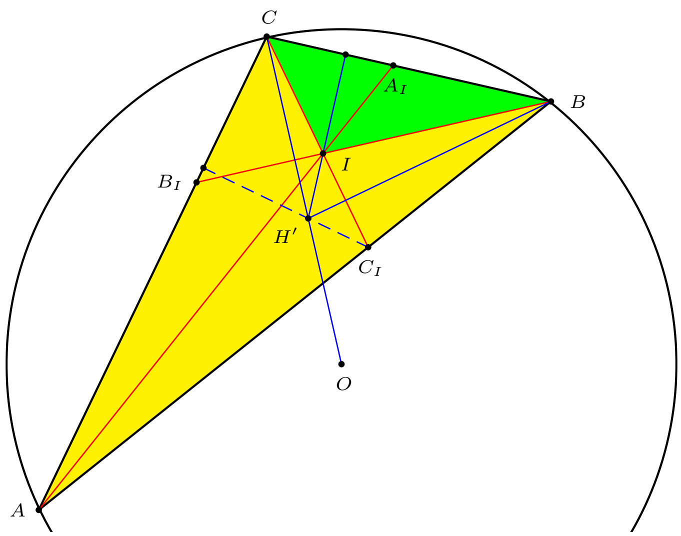

Figure 15. Four concurrent altitudes

Proof. Since ABC has orthocenter $H = \zeta + \zeta^2 + \zeta^4$, the orthocenter of triangle BCI is the point

$$H' = f_1(H) = -(1 + \zeta^4) = -(\zeta^2 + \zeta^5)\zeta^2.$$

This expression shows that H' lies on the radius OC. Now, the vector $H'C_I$ is given by

$$\begin{aligned} C_I - H' = & \ (-1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5) + (1 + \zeta^4) \\ = & \ \zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5. \end{aligned}$$

On the other hand, the vector AC is given by $\zeta^2 - \zeta^4$. To check that $H'C_I$ is perpendicular to AC, we need only note that

$$(\zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)\overline{(\zeta^2 - \zeta^4)} = -2(\zeta - \zeta^6) + (\zeta^2 - \zeta^5) + (\zeta^3 - \zeta^4)$$

is purely imaginary. □

Remark. Similarly, the orthocenter of ACC_I lies on the C-altiude of ABC, and that of BB_IC on the B-altitude.

8.2. *Systems of concurrent circles.*

Proposition 18. *The nine-point circles of ACC_I and* (the isosceles triangle) $B'A'C$ *are tangent internally at the midpoint of $B'C$.*

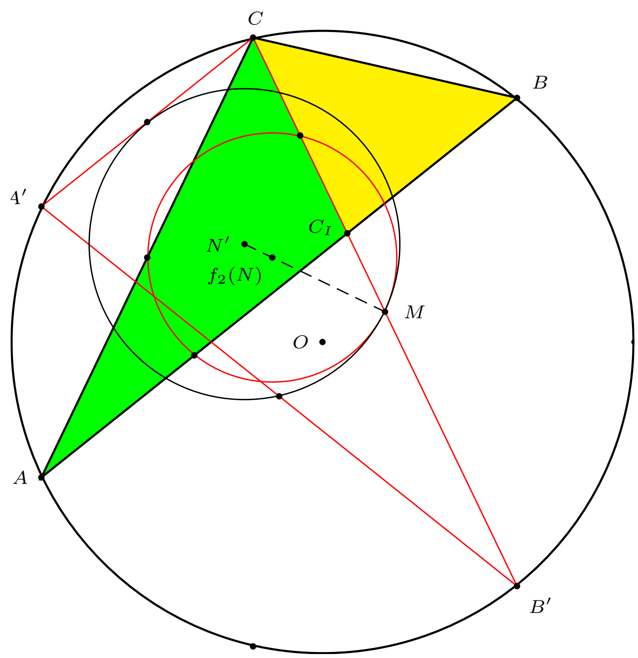

Figure 16. Two tangent nine-point circles

Proof. The nine-point circle of the isosceles triangle $B'A'C$ clearly contains the midpoint M of $B'C$. Since triangle $AB'C$ is also isosceles, the perpendicular from A to $B'C$ passes through M. This means that M lies on the nine-point circle of triangle ACC_I. We show that the two circles are indeed tangent at M.

The nine-point center of ACC_I is the point

$$f_2(N) = \frac{1}{2}(2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6).$$

On the other hand, the nine-point center of the isosceles triangle $B'A'C$ is the point

$$N' = \frac{1}{2}(\zeta^2 + \zeta^3 + \zeta^6).$$

Since

$$M = \frac{\zeta^2 + \zeta^6}{2} = (1 - 2c_2 - 4c_3)f_2(N) + (2c_2 + 4c_3)N'$$

as can be verified directly, we conclude that the two circles are tangent internally. □

Theorem 19. *The following circles have a common point.*
(i) *the circumcircle of* ACC_I,
(ii) *the nine-point circle of* ACC_I,
(iii) *the* A*-excircle of* ACC_I,
(iv) *the nine-point circle of* BB_IC.

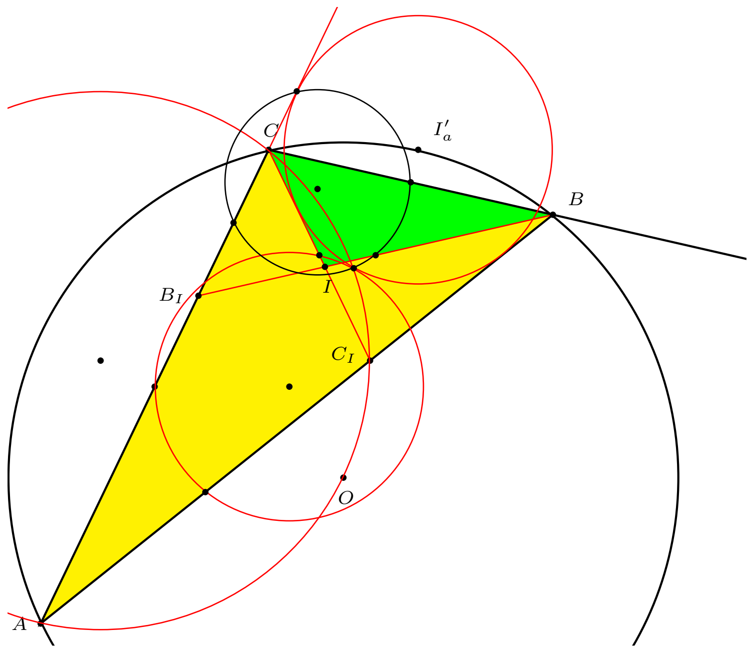

Figure 17. Four concurrent circles

Proof. By Proposition 7(3), the first three circles concur at the A-Feuerbach point of triangle ACC_I, which is the point

$$f_2(F_a) = \frac{1}{4}(\zeta + 2\zeta^2 + \zeta^4 - \zeta^5 + \zeta^6).$$

It is enough to verify that this point lies on the nine-point circle of BB_IC, which has center

$$f_3\left(\frac{\zeta + \zeta^2 + \zeta^4}{2}\right) = \frac{2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6}{2},$$

and square radius

$$\frac{1}{4}|1 + \zeta^2 + \zeta^4 + \zeta^5|^2 = -\frac{1}{4}\left(3(\zeta + \zeta^6) + (\zeta^2 + \zeta^5) + 2(\zeta^3 + \zeta^4)\right).$$

This is exactly the square distance between $f_2(F_a)$ and the center, as is directly verified. This shows that $f_2(F_a)$ indeed lies on the nine-point circle of BB_IC. $\square$

Theorem 20. *Each of the following circles contains the Feuerbach point F_{e} of* $\mathbf{T}$:
(i) *the nine-point circle of* $\mathbf{T}$,
(ii) *the incircle of* $\mathbf{T}$,
(iii) *the nine-point circle of the heptagonal triangle* BCI,
(iv) *the* C*-excircle of* BCI,
(v) *the* A*-excircle of the heptagonal triangle* ACC_I,
(vi) *the incircle of the isosceles triangle* BIC_I.

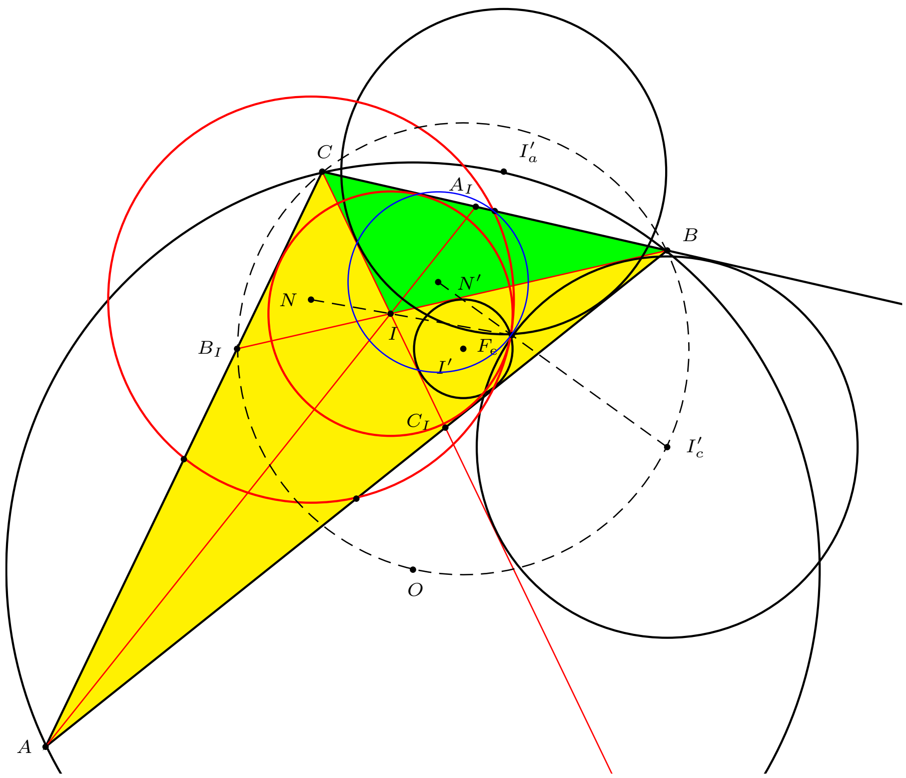

Figure 18. Six circles concurrent at the Feuerbach point of $\mathbf{T}$

Proof. It is well known that the nine-point circle and the incircle of $\mathbf{T}$ are tangent to each other internally at the Feuerbach point F_{e}. It is enough to verify that this point lies on each of the remaining four circles.

(iii) and (iv) The C-excircle of BCI is the image of the B-excircle of ABC under the affine mapping f_1. It is therefore enough to check that $f_1(F_b) = F_{\mathrm{e}}$:

$$\begin{aligned} f_1(F_b) = & \ \frac{1}{4}(\zeta+\zeta^4)(\zeta+\zeta^2+\zeta^3+2\zeta^4-\zeta^6)-\zeta^5 \\ = & \ \frac{1}{4}(2\zeta+\zeta^2+\zeta^4-\zeta^5+\zeta^6) = F_{\mathrm{e}}. \end{aligned}$$

(v) The heptagonal triangle ACC_I is the image of ABC under the mapping f_2. It can be verified directly that $W = -\frac{1}{4}(\zeta - \zeta^2 + 3\zeta^3 + 3\zeta^5) - \zeta^6$ is the point for which $f_2(W) = F_\mathrm{e}$. The square distance of W from the A-excenter $I_a = -(\zeta^3 + \zeta^5 + \zeta^6)$ is the square norm of $W - I_a = \frac{1}{4}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)$. An easy calculation shows that this is

$$\frac{1}{16}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)(\zeta^2 + \zeta^4 + \zeta^5 - \zeta^6) = \frac{1}{4} = r_a^2.$$

It follows that, under the mapping f_2, F_e lies on the A-excircle of ACC_I.

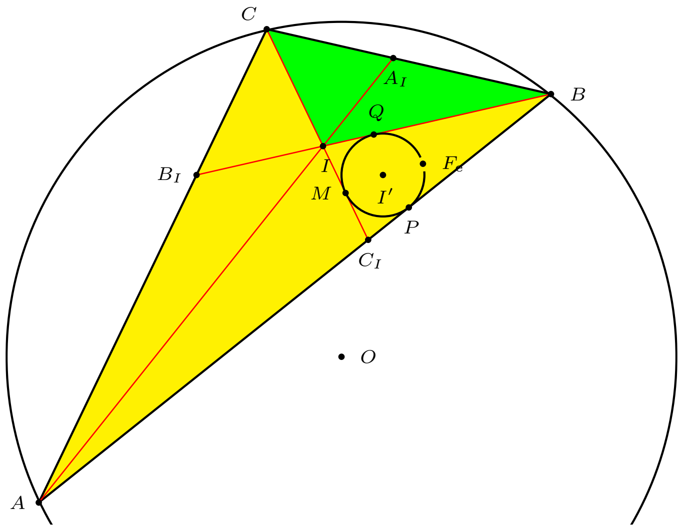

Figure 19. The incircle of an isosceles triangle

(vi) Since C_IBC and ICB_I are isosceles triangles, the perpendicular bisectors of BC and CB_I are the bisectors of angles IC_IB and C_IIB respectively. It follows that the incenter of the isosceles triangle BIC_I coincides with the circumcenter of triangle BB_IC, which is the point $I' = -(1 + \zeta^3 + \zeta^5)$ from the affine mapping f_3. This incircle touches the side IC_I at its midpoint M, the side IB at the midpoint Q of BB_I, and the side BC_I at the orthogonal projection P of C on AB (see Figure 19). A simple calculation shows that $\angle PMQ = \frac{3\pi}{7}$. To show that F_e lies on the same circle, we need only verify that $\angle PF_\mathrm{e}Q = \frac{4\pi}{7}$. To this end, we first determine some complex number coordinates:

$$P = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4),$$
$$Q = \frac{1}{2}(-1 + 2\zeta + \zeta^4 - \zeta^5 + \zeta^6).$$

Now, with $F_\mathrm{e} = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$, we have

$$Q - F_\mathrm{e} = (\zeta^4 + \zeta^6)(P - F_\mathrm{e}).$$

From the expression $\zeta^4 + \zeta^6 = \zeta^{-2}(\zeta + \zeta^6)$, we conclude that indeed $\angle PF_\mathrm{e}Q = \frac{4\pi}{7}$. $\square$

9. A theorem on the Fermat points

Lemma 21. *The perpendicular bisector of the segment ON is the line containing $X = -1$ and $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$.*

Proof. (1) Complete the parallelogram OI_aHX, then

$$X = O + H - I_a = (\zeta + \zeta^2 + \zeta^4) + (\zeta^3 + \zeta^5 + \zeta^6) = -1$$

is a point on the circumcircle. Note that N is the midpoint of I_aX. Thus, $NX = NI_a = R = OX$. This shows that X is on the bisector of ON.

(2) Complete the parallelogram ONI_aY, with $Y = O + I_a - N$. Explicitly, $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$. But we also have

$$X + Y = (O + H - I_a) + (O + I_a - N) = (2 \cdot N - I_a) + (O + I_a - N) = O + N.$$

This means that $OXNY$ is a rhombus, and $NY = OY$.

From (1) and (2), XY is the perpendicular bisector of ON. $\square$

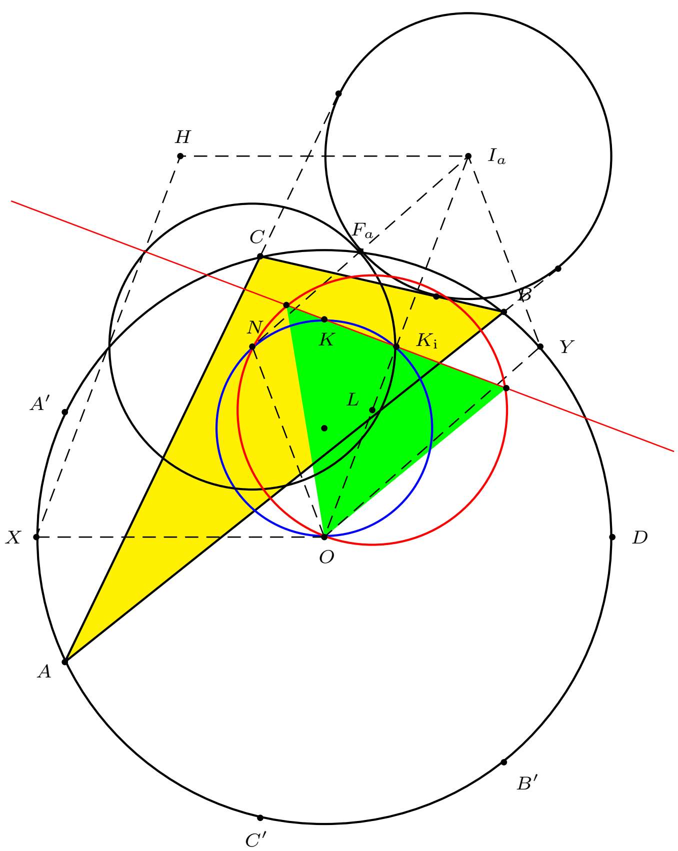

Figure 20. The circumcenter and the Fermat points form an equilateral triangle

Theorem 22. *The circumcenter and the Fermat points of the heptagonal triangle* **T** *form an equilateral triangle.*

Proof. (1) Consider the circle through O, with center at the point

$$L := -\frac{1}{3}(\zeta^3 + \zeta^5 + \zeta^6).$$

This is the center of the equilateral triangle with O as a vertex and $K_{\mathrm{i}} = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6)$ the midpoint of the opposite side. See Figure 20.

(2) With X and Y in Lemma 21, it is easy to check that $L = \frac{1}{3}(X + 2Y)$. This means that L lies on the perpendicular bisector of ON.

(3) Since K_{i} is on the Brocard circle (with diameter OK), OK_{i} is perpendicular to the line KK_{i}. It is well known that the line KK_{i} contains the Fermat points. [3] Indeed, K_{i} is the midpoint of the Fermat points. This means that L is lies on the perpendicular bisector of the Fermat points.

(4) By a well known theorem of Lester (see, for example, [5]), the Fermat points, the circumcenter, and the nine-point center are concyclic. The center of the circle containing them is necessarily L, and this circle coincides with the circle constructed in (1). The side of the equilateral triangle opposite to O is the segment joining the Fermat points. $\square$

Corollary 23. *The Fermat points of the heptagonal triangle* $\mathbf{T}$ *are the points*

$$F_+ = \frac{1}{3}(\lambda + 2\lambda^2)(\zeta^3 + \zeta^5 + \zeta^6),$$
$$F_- = \frac{1}{3}(\lambda^2 + 2\lambda)(\zeta^3 + \zeta^5 + \zeta^6),$$

where $\lambda = \frac{1}{2}(-1 + \sqrt{3}i)$ *and* $\lambda^2 = \frac{1}{2}(-1 - \sqrt{3}i)$ *are the imaginary cube roots of unity.*

Remarks. (1) The triangle with vertices I_a and the Fermat points is also equilateral.

(2) Since $OI_a = \sqrt{2}R$, each side of the equilateral triangle has length $\sqrt{\frac{2}{3}}R$.

(3) The Lester circle is congruent to the orthocentroidal circle, which has HG as a diameter.

(4) The Brocard axis OK is tangent to the A-excircle at the midpoint of I_aH.

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Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA

E-mail address: `yiu@fau.edu`

[3] The line joining the Fermat points contains K and K_{i}.

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The Symmedian Point and Concurrent Antiparallel Images

Shao-Cheng Liu

Abstract. In this note, we study the condition for concurrency of the GP lines of the three triangles determined by three vertices of a reference triangle and six vertices of the second Lemoine circle. Here G is the centroid and P is arbitrary triangle center different from G. We also study the condition for the images of a line in the three triangles bounded by the antiparallels through a given point to be concurrent.

1. Antiparallels through the symmedian point

Given a triangle ABC with symmedian point K, we consider the three triangles AB_aC_a, A_bBC_b, and A_cB_cC bounded by the three lines ℓ_a, ℓ_b, ℓ_c antiparallel through K to the sides BC, CA, AB respectively (see Figure 1). It is well known [4] that the 6 intercepts of these antiparallels with the sidelines are on a circle with center K. In other words, K is the common midpoint of the segments B_aC_a, C_bA_b and A_cB_c. The circle is called the second Lemoine circle.

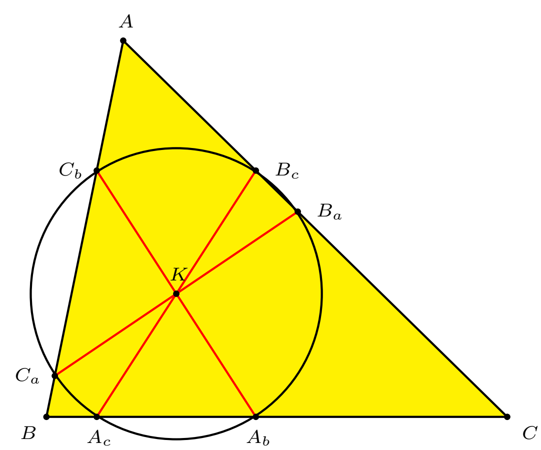

Figure 1.

Triangle AB_aC_a is similar to ABC, because it is the reflection in the bisector of angle A of a triangle which is a homothetic image of ABC. For an arbitrary triangle center P of ABC, denote by P_a the corresponding center in triangle AB_aC_a; similarly, P_b and P_c in triangles A_bBC_b and A_cB_cC.

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Now let P be distinct from the centroid G. Consider the line through A parallel to GP. Its reflection in the bisector of angle A intersects the circumcircle at a point Q', which is the isogonal conjugate of the infinite point of GP. So, the line G_aP_a is the image of AQ' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$, and it passes through a trisection point of the segment KQ' (see Figure 2).

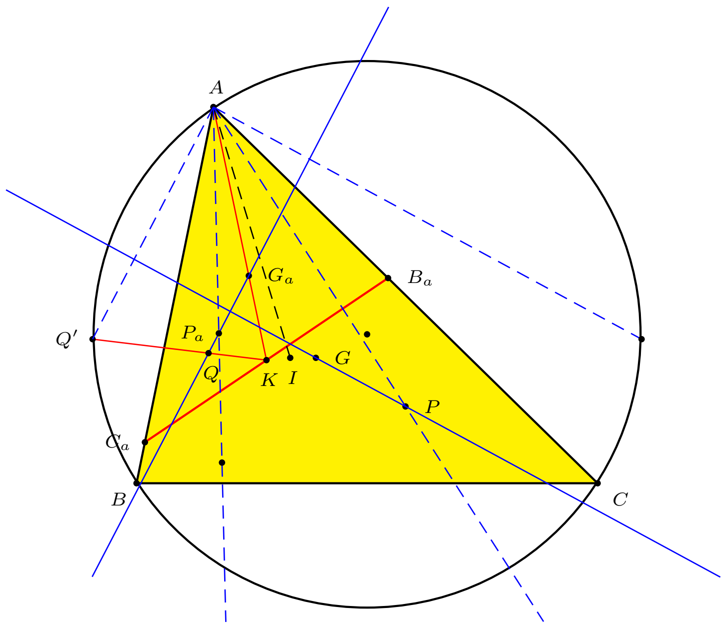

Figure 2.

In a similar manner, the reflections of the parallels to GP through B and C in the respective angle bisectors intersect the circumcircle at the same point Q'. Hence, the lines G_bP_b and G_cP_c also pass through the point Q, which is the image of Q' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$. It is clear that the point Q lies on the circumcircle of triangle $G_aG_bG_c$ (see Figure 3). We summarize this in the following theorem.

Theorem 1. *Let P be a triangle center of ABC, and P_a, P_b, P_c the corresponding centers in triangles AB_aC_a, BC_bA_b, CA_cB_c, which have centroids G_a, G_b, G_c respectively. The lines G_aP_a, G_bP_b, G_cP_c intersect at a point Q on the circumcircle of triangle $G_aG_bG_c$.*

Here we use homogeneous barycentric coordinates. Suppose $P = (u : v : w)$ with reference to triangle ABC.

(i) The isogonal conjugate of the infinite point of the line GP is the point

$$Q' = \left(\frac{a^2}{-2u+v+w} : \frac{b^2}{u-2v+w} : \frac{c^2}{u+v-2w}\right)$$

on the circumcircle.

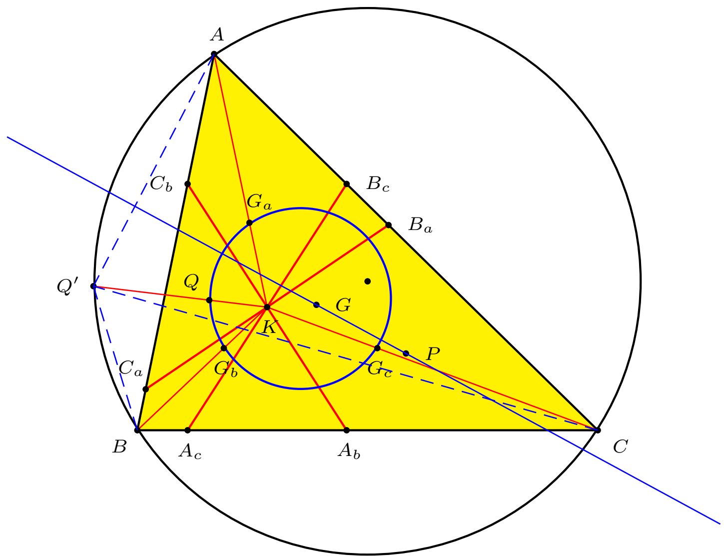

Figure 3.

(ii) The lines G_aP_a, G_bP_b, G_cP_c intersect at the point

$$Q = \left(\frac{a^2}{v+w-2u}\left(a^2 + \frac{b^2(w-u)}{w+u-2v} + \frac{c^2(v-u)}{u+v-2w}\right) : \cdots : \cdots\right).$$

which divides KQ' in the ratio $KQ : QQ' = 1 : 2$.

2. A generalization

More generally, given a point $T = (x : y : z)$, we consider the triangles intercepted by the antiparallels through T. These are the triangles AB_aC_a, A_bBC_b and A_cB_cC with coordinates (see [1, §3]):

$$\begin{aligned}
B_a &= (b^2x + (b^2-c^2)y : 0 : c^2y + b^2z),\\
C_a &= (c^2x - (b^2-c^2)z : c^2y + b^2z : 0),\\
C_b &= (a^2z + c^2x : c^2y + (c^2-a^2)z : 0),\\
A_b &= (0 : a^2y - (c^2-a^2)x : a^2z + c^2x),\\
A_c &= (0 : b^2x + a^2y : a^2z + (a^2-b^2)x),\\
B_c &= (b^2x + a^2y : 0 : b^2z - (a^2-b^2)y).
\end{aligned}$$

Now, for a point P with coordinates $(u : v : w)$ with reference to triangle ABC, the one with the same coordinates with reference to triangle AB_aC_a is

$$\begin{aligned}
P_a = \big(&b^2c^2(x+y+z)u + c^2(b^2x + (b^2-c^2)y)v + b^2(c^2x - (b^2-c^2)z)w :\\
&b^2(c^2y + b^2z)w : c^2(c^2y + b^2z)v\big).
\end{aligned}$$

By putting $u = v = w = 1$, we obtain the coordinates of the centroid

$$G_a = \big(3b^2c^2x + c^2(2b^2 - c^2)y - b^2(b^2 - 2c^2)z) : b^2(c^2y + b^2z) : c^2(c^2y + b^2z)\big)$$

of AB_aC_a. The equation of the line G_aP_a is

$$\begin{aligned}
&(c^2y + b^2z)(v - w)\mathbb{X}\\
&+ (c^2(x + y + z)u + (-2c^2x - c^2y + (b^2 - 2c^2)z)v + (c^2x - (b^2 - c^2)z)w)\mathbb{Y}\\
&- (b^2(x + y + z)u + (b^2x + (b^2 - c^2)y)v - (2b^2x + (2b^2 - c^2)y + b^2z)w)\mathbb{Z}\\
&= 0.
\end{aligned}$$

By cyclically replacing (a, b, c), (u, v, w), (x, y, z), and $(\mathbb{X}, \mathbb{Y}, \mathbb{Z})$ respectively by (b, c, a), (v, w, u), (y, z, x), and $(\mathbb{Y}, \mathbb{Z}, \mathbb{X})$, we obtain the equation of the line G_bP_b. One more applications gives the equation of G_cP_c.

Proposition 2. *The three lines G_aP_a, G_bP_b, G_cP_c are concurrent if and only if*

$$f(u, v, w)(x + y + z)^2 \left(b^2c^2(v - w)x + c^2a^2(w - u)y + a^2b^2(u - v)z\right) = 0,$$

where

$$f(u, v, w) = \sum_{\text{cyclic}} \left((2b^2 + 2c^2 - a^2)u^2 + (b^2 + c^2 - 5a^2)vw\right).$$

Computing the distance between G and P, we obtain

$$f(u, v, w) = 9(u + v + w)^2 \cdot GP^2.$$

This is nonzero for $P \neq G$. From this we obtain the following theorem.

Theorem 3. *For a fixed point $P = (u : v : w)$, the locus of a point T for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$b^2c^2(v - w)\mathbb{X} + c^2a^2(w - u)\mathbb{Y} + a^2b^2(u - v)\mathbb{Z} = 0.$$

Remarks. (1) The line clearly contains the symmedian point K and the point $(a^2u : b^2v : c^2w)$, which is the isogonal conjugate of the isotomic conjugate of P.

(2) The locus of the point of concurrency is the line

$$\sum_{\text{cyclic}} b^2c^2(v - w)((c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2)\mathbb{X} = 0.$$

This line contains the points

$$\left(\frac{a^2}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right)$$

and

$$\left(\frac{a^2u}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right).$$

Theorem 4. *For a fixed point $T = (x : y : z)$, the locus of a point P for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$\left(\frac{y}{b^2} - \frac{z}{c^2}\right)\mathbb{X} + \left(\frac{z}{c^2} - \frac{x}{a^2}\right)\mathbb{Y} + \left(\frac{x}{a^2} - \frac{y}{b^2}\right)\mathbb{Z} = 0.$$

Remark. This is the line containing the centroid G and the point $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$.

More generally, given a point $T = (x : y : z)$, we study the condition for which the images of the line

$$\mathcal{L} : \qquad u\mathbb{X} + v\mathbb{Y} + w\mathbb{Z} = 0$$

in the three triangles AB_aC_a, A_bBC_b and A_cB_cC are concurrent. Now, the image of the line $\mathcal{L}$ in AB_aC_a is the line

$$\begin{aligned}&-u(c^2y + b^2z)\mathbb{X} + ((c^2x - (b^2 - c^2)z)u - c^2(x + y + z)w)\mathbb{Y}\\ &+((b^2x + (b^2 - c^2)y)u - b^2(x + y + z)v)\mathbb{Z} = 0.\end{aligned}$$

Similarly, we write down the equations of the images in A_bBC_b and A_cB_cC. The three lines are concurrent if and only if

$$\begin{aligned}&((b^2 + c^2 - a^2)(v - w)^2 + (c^2 + a^2 - b^2)(w - u)^2 + (a^2 + b^2 - c^2)(u - v)^2)\\ &\cdot (x + y + z)^2\left(\sum_{\text{cyclic}} u \cdot a^2(c^2y + b^2z)\right) = 0.\end{aligned}$$

Since the first two factors are nonzero for nonzero (u, v, w) and (x, y, z), we obtain the following result.

Theorem 5. *Given $T = (x : y : z)$, the antiparallel images of a line are concurrent if and only if the line contains the point*

$$T' = \left(\frac{y}{b^2} + \frac{z}{c^2} : \frac{z}{c^2} + \frac{x}{a^2} : \frac{x}{a^2} + \frac{y}{b^2}\right).$$

Here are some examples of correspondence:

T	T'	T	T'	T	T'
X_1	X_{37}	X_{19}	X_{1214}	X_{69}	X_{1196}
X_2	X_{39}	X_{20}	X_{800}	X_{99}	X_{1084}
X_3	X_6	X_{40}	X_{1108}	X_{100}	X_{1015}
X_4	X_{216}	X_{55}	X_1	X_{110}	X_{115}
X_5	X_{570}	X_{56}	X_9	X_{111}	X_{2482}
X_6	X_2	X_{57}	X_{1212}	X_{887}	X_{888}

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Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: `liu471119@yahoo.com.tw`

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On Trigonometric Proofs of the Steiner-Lehmus Theorem

Róbert Oláh-Gál and József Sándor

Dedicated to the memory of Professor Ferenc Radó (*1921-1990*)

Abstract. We offer a survey of some lesser known or new trigonometric proofs of the Steiner-Lehmus theorem. A new proof of a recent refined variant is also given.

1. Introduction

The famous Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. For a recent survey of the Steiner-Lehmus theorem, see M. Hajja [8]. From the bibliography of [8] one can find many methods of proof, purely geometric, or trigonometric, of this theorem. Our aim in this note is to add some new references, and to draw attention to some little or unknown proofs, especially trigonometric ones. We shall also include a new trigonometric proof of a refined version of the Steiner-Lehmus theorem, published recently [9].

First, we want to point out some classical geometric proofs published in 1967 by A. Froda [4], attributed to W. T. Williams and G. T. Savage. Another interesting proof by A. Froda appears in his book [5] (see also the book of the second author [15]). Another purely geometric proof was published in 1973 by M. K. Sathya Narayama [16]. Other papers are by K. Seydel and C. Newman [17], or the more recent papers by D. Beran [1] or D. Rüthing [13]. None of the recent extensive surveys connected with the Steiner-Lehmus theorem mentions the use of complex numbers in the proof. Such a method appears in the paper by C. I. Lubin [11] in 1959.

Trigonometric proofs of Euclidean theorems have gained additional importance after the appearance of Ungar's book [18]. In this book, the author develops a kind of trigonometry that serves Hyperbolic Geometry in the same way our ordinary trigonometry does Euclidean Geometry. He calls it Gyrotrigonometry and proves that the ordinary trigonometric identities have counterparts in that trigonometry.

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The authors thank Professor D. Plachky for a reprint of [9] and Professor A. Furdek for providing a copy of [11]. They are also indebted to the referee for his many remarks and suggestions which helped improve the presentation of the paper, and for pointing out the new references [6, 7, 10, 18].

Consequently, he takes certain trigonometrical proofs of Euclidean theorems and shows that these proofs, hence also the corresponding theorems, remain valid in Hyperbolic Geometry. In this context, he includes the trigonometric proofs of the Urquhart and the Steiner-Lehmus theorems that appeared in [7] and [8]. Related to the question, first posed by Sylvester (also mentioned in [8]), whether there is a direct proof of the Steiner-Lehmus theorem, recently J. H. Conway (see [2]) has given an intriguing argument that there is no such proof. However, the validity of Conway's argument is debatable since a claim of the non-existence of a direct proof should be formulated in a more precise manner using, for example, the language of intuitionistic logic.

2. Trigonometric proofs of the Steiner-Lehmus theorem

2.1. Perhaps one of the shortest trigonometric proofs of the Steiner-Lehmus theorem one can find in a forgotten paper (written in Romanian) in 1916 by V. Cristescu [3]. Let BB' and CC' denote two angle bisectors of the triangle ABC (see Fig. 1). By using the law of sines in triangle $BB'C$, one gets

$$\frac{BB'}{\sin C} = \frac{BC}{\sin\left(C + \frac{B}{2}\right)}.$$

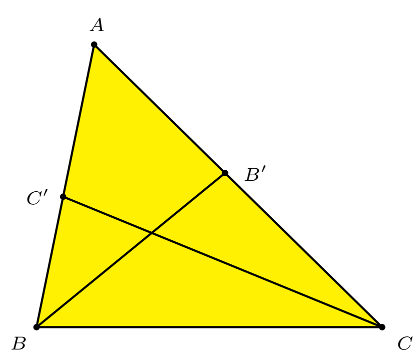

Figure 1.

As $C + \frac{B}{2} = C + \frac{180^\circ - C - A}{2} = 90^\circ - \frac{A-C}{2}$, one has

$$BB' = a \cdot \frac{\sin C}{\cos \frac{A-C}{2}}.$$

Similarly,

$$CC' = a \cdot \frac{\sin B}{\cos \frac{A-B}{2}}.$$

Assuming $BB' = CC'$, and using the identities $\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}$, and $\sin \frac{C}{2} = \cos \frac{A+B}{2}$, $\sin \frac{B}{2} = \cos \frac{A+C}{2}$, we have

$$\cos \frac{C}{2} \cdot \cos \frac{A+B}{2} \cos \frac{A-B}{2} = \cos \frac{B}{2} \cos \frac{A+C}{2} \cos \frac{A-C}{2}. \tag{1}$$

Now from the identity

$$\cos(x+y)\cdot\cos(x-y)=\cos^2 x+\cos^2 y-1,$$

relation (1) becomes

$$\cos\frac{C}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-1\right)=\cos\frac{B}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{C}{2}-1\right).$$

This simplifies into

$$\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(\sin^2\frac{A}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right)=0.$$

As the second paranthesis of (6) is strictly positive, this implies $\cos\frac{B}{2}-\cos\frac{C}{2}=0$, so $B=C$.

2.2. In 2000, respectively 2001, the German mathematicians D. Plachky [12] and D. Rüthing [14] have given other trigonometric proofs of the Steiner-Lehmus theorem, based on area considerations. We present here the method by Plachky. Denote the angles at B and C respectively by β and γ, and the angle bisectors BB' and AA' by w_b and w_a (see Figure 2).

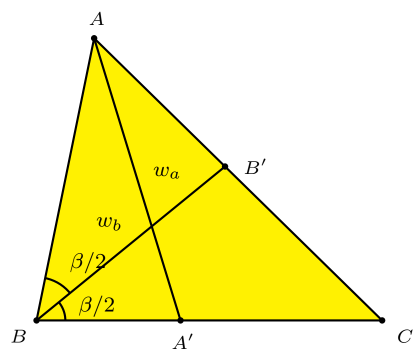

Figure 2.

By using the trigonometric form $\frac{1}{2}ab\sin\gamma$ of the area of triangle ABC, and decomposing the initial triangle in two triangles, we get

$$\frac{1}{2}aw_\beta\sin\frac{\beta}{2}+\frac{1}{2}cw_\beta\sin\frac{\beta}{2}=\frac{1}{2}bw_\alpha\sin\frac{\alpha}{2}+\frac{1}{2}cw_\alpha\sin\frac{\alpha}{2}.$$

By the law of sines we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin(\pi-(\alpha+\beta))}{c},$$

so assuming $w_\alpha=w_\beta$, we obtain

$$\frac{c\sin\alpha}{\sin(\alpha+\beta)}\sin\frac{\beta}{2}+c\sin\frac{\beta}{2}=\frac{c\sin\beta}{\sin(\alpha+\beta)}\sin\frac{\alpha}{2}+c\sin\frac{\alpha}{2},$$

or

$$\sin(\alpha+\beta)\left(\sin\frac{\alpha}{2}-\sin\frac{\beta}{2}\right)+\sin\frac{\alpha}{2}\sin\beta-\sin\alpha\sin\frac{\beta}{2}=0. \qquad (2)$$

Writing $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ *etc*, and using the formulae

$$\sin u - \sin v = 2\sin\frac{u-v}{2}\cos\frac{u+v}{2}, \tag{3}$$

$$\cos u - \cos v = -2\sin\frac{u-v}{2}\sin\frac{u+v}{2}, \tag{4}$$

we rewrite (2) as

$$2\sin\frac{\alpha-\beta}{4}\left(\sin(\alpha+\beta)\cos\frac{\alpha+\beta}{4} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\alpha+\beta}{2}\right) = 0.$$

Since $\alpha+\beta < \pi$, the expression inside the parenthesis is strictly positive. It follows that $\alpha = \beta$.

2.3. The following trigonometric proof seems to be much simpler. It can found in [10, pp. 194-196]. According to Honsberger, this proof was rediscovered by M. Hajja who later came across in it some obscure Russian book. The authors rediscovered again this proof, and wish to thank the referee for this information. Writing the area of triangle ABC in two different ways (using triangles ABB' and $BB'C$) we get immediately

$$w_b = \frac{2ac}{a+c}\cos\frac{\beta}{2}. \tag{5}$$

Similarly,

$$w_a = \frac{2bc}{b+c}\cos\frac{\alpha}{2}. \tag{6}$$

Suppose now that, $a > b$. Then $\alpha > \beta$, so $\frac{\alpha}{2} > \frac{\beta}{2}$. As $\frac{\alpha}{2}, \frac{\beta}{2} \in (0, \frac{\pi}{2})$, one gets $\cos\frac{\alpha}{2} < \cos\frac{\beta}{2}$. Also, $\frac{bc}{b+c} < \frac{ac}{a+c}$ is equivalent to $b < a$. Thus (5) and (6) imply $w_a > w_b$. This is indeed a proof of the Steiner-Lehmus theorem, as supposing $w_a = w_b$ and letting $a > b$, we would lead to the contradiction $w_a > w_b$, a contradiction; similarly with $a < b$.

For another trigonometric proof of a generalized form of the theorem, we refer the reader to [6].

3. A new trigonometric proof of a refined version

Recently, M. Hajja [9] proved the following stronger version of the Steiner-Lehmus theorem. Let BY and CZ be the angle bisectors and let $BY = y$, $CZ = z$, $YC = v$, $BZ = V$ (see Figure 3).

Then

$$c > b \Rightarrow y + v > z + V. \tag{7}$$

As $V = \frac{ac}{a+b}$, $v = \frac{ab}{a+c}$, it is immediate that $c > b \Rightarrow V > v$. Thus, assuming $c > b$, and using (7) we get $y > z$, i.e. the Steiner-Lehmus theorem (see §2.3). In [9], the proof of (7) made use of a nice lemma by R. Breusch. We offer here a new trigonometric proof of (7), based only on the law of sines, and simple trigonometric facts.

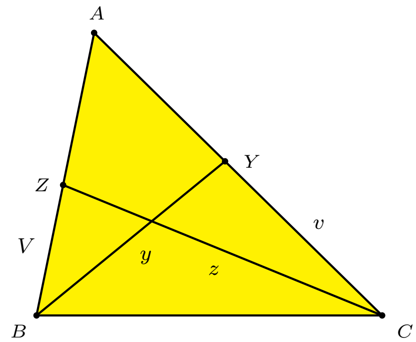

Figure 3.

In triangle BCY one can write

$$\frac{a}{\sin\left(C+\frac{B}{2}\right)}=\frac{CY}{\sin\frac{B}{2}}=\frac{BY}{\sin C},$$

so

$$\frac{y+v}{\sin C+\sin\frac{B}{2}}=\frac{a}{\sin\left(C+\frac{B}{2}\right)},$$

implying

$$y+v=\frac{a\left(\sin C+\sin\frac{B}{2}\right)}{\sin\left(C+\frac{B}{2}\right)}. \tag{8}$$

Similarly,

$$z+V=\frac{a\left(\sin B+\sin\frac{C}{2}\right)}{\sin\left(B+\frac{C}{2}\right)}. \tag{9}$$

Assume now that $y+v>z+V$. Applying $\sin u+\sin v=2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$ and using the facts that $\cos\left(\frac{C}{2}+\frac{B}{4}\right)>0$, $\cos\left(\frac{B}{2}+\frac{C}{4}\right)>0$, after simplification, from (8) and (9) we get the inequality

$$\cos\left(\frac{C}{2}-\frac{B}{4}\right)\cos\left(\frac{B}{2}+\frac{C}{4}\right)>\cos\left(\frac{B}{2}-\frac{C}{4}\right)\cos\left(\frac{C}{2}+\frac{B}{4}\right).$$

Using $2\cos u\cos v=\cos\frac{u+v}{2}+\cos\frac{u-v}{2}$, this implies

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)+\cos\left(\frac{C}{4}-\frac{3B}{4}\right)>\cos\left(\frac{3B}{4}+\frac{C}{4}\right)+\cos\left(\frac{B}{4}-\frac{3C}{4}\right),$$

or

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)-\cos\left(\frac{3B}{4}+\frac{C}{4}\right)>\cos\left(\frac{B}{4}-\frac{3C}{4}\right)-\cos\left(\frac{C}{4}-\frac{3B}{4}\right).$$

Now applying (4), we get

$$-\sin\frac{B}{2}\sin\frac{3C}{2}>-\sin\frac{C}{2}\sin\frac{3B}{2}. \tag{10}$$

By $\sin 3u=3\sin u-4\sin^3 u$ we get immediately from (10) that

$$-3+4\sin^2\frac{C}{2}>-3+4\sin^2\frac{B}{2}. \tag{11}$$

Since the function $x \mapsto \sin^2 x$ is strictly increasing in $x \in \left(0, \frac{\pi}{2}\right)$, the inequality (11) is equivalent to $C > B$. We have actually shown that $y+v > z+V \Leftrightarrow C > B$, as desired.

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Róbert Oláh-Gál: Department of Mathematics and Informatics, Babeş–Bolyai University, Extension of Miercurea–Ciuc, Str. Topliţa Nr. 20. Miercurea–Ciuc, Romania
E-mail address: `olah_gal@topnet.ro`

József Sándor: Department of Mathematics and Informatics, Babeş–Bolyai University, Extension of Miercurea–Ciuc, Str. Topliţa Nr. 20. Miercurea–Ciuc, Romania
E-mail address: `jsandor@math.ubbcluj.ro`

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An Angle Bisector Parallel Applied to Triangle Construction

Harold Connelly and Beata Randrianantoanina

Abstract. We prove a theorem describing a line parallel to an angle bisector of a triangle and passing through the midpoint of the opposite side. We then apply this theorem to the solution of four triangle construction problems.

Consider the triangle ABC with angle bisector AT_a, altitude AH_a, midpoint M_a of side BC and Euler point E_a (see Figure 1). Let the circle with center at E_a and passing through M_a intersect AH_a at P. Draw the line M_aP. We prove the following theorem.

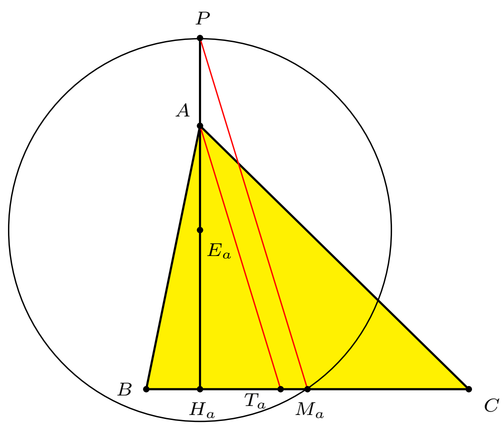

Figure 1.

Theorem 1. *In any triangle ABC with H_a not coinciding with M_a, the line M_aP is parallel to the angle bisector AT_a.*

Proof. Let O be the circumcenter of ABC (see Figure 2). The perpendicular bisector M_aO and the angle bisector AT_a intersect the circumcircle (O) at S. Let the midpoint of E_aO be R, and reflect the entire figure through R. Let the reflection of ABC be $A'B'C'$. Since E_aM_a is equal to the circumradius, the circle $E_a(M_a)$ is the reflection of (O) and is the circumcircle of $A'B'C'$. Since segments AE_a and M_aO are equal and parallel, A is the reflection of M_a and is therefore the midpoint of $B'C'$. Thus, AH_a is the perpendicular bisector of $B'C'$. Finally, AH_a intersects circle $E_a(M_a)$ at P, therefore M_aP is the bisector of angle $B'A'C'$ and parallel to AT_a. $\square$

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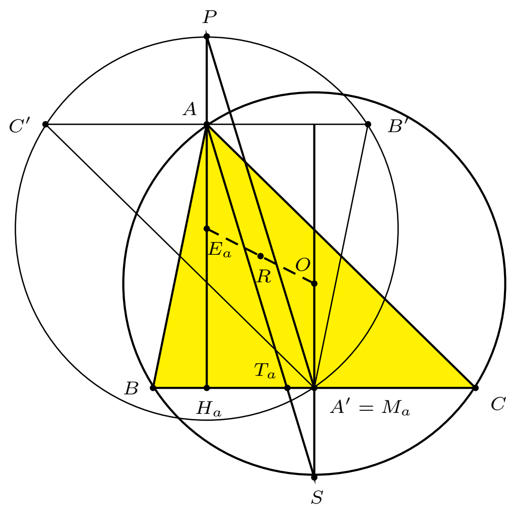

Figure 2.

Remark. For the case where H_a and M_a coincide, triangle ABC is isosceles (with apex at A) or equilateral. The lines M_aP and AT_a will coincide.

Wernick [3] presented 139 problems of constructing a triangle with ruler and compass given the location of three points associated with the triangle and chosen from a list of sixteen points. See Meyers [2] for updates on the status of the problems from this list. Connelly [1] extended this work by adding four more points to the list and 140 additional problems, many of which were designated as unresolved. We now apply Theorem 1 to solve four of these previously unresolved problems. The problem numbers are those given by Connelly.

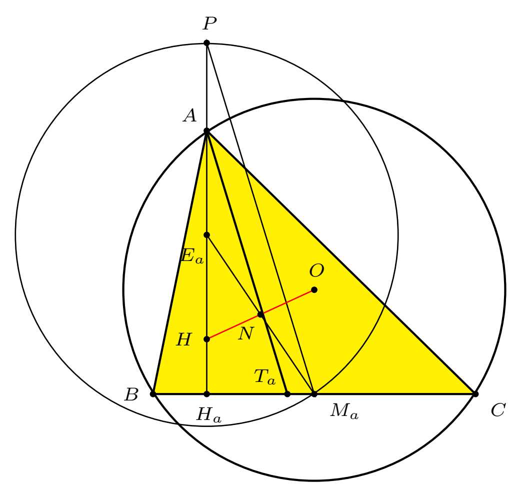

Figure 3.

Problem 99. Given E_a, M_a and T_a construct triangle ABC.
Solution. Draw line M_aT_a containing the side BC and then the altitude E_aH_a

to this side (see Figure 3). The circle with center E_a and passing through M_a intersects the altitude at P. Draw M_aP. By Theorem 1, the line through T_a parallel to M_aP intersects the altitude at A. Reflect A through E_a to get the orthocenter H. The midpoint of E_aM_a is the nine-point center N. Reflect H through N to obtain O. Draw the circumcircle through A, intersecting M_aT_a at B and C.

Number of Solutions. Depending on the relative positions of the three points, there are two solutions, no solution or an infinite number of solutions. We start by locating E_a and M_a. Then the segment E_aM_a is a diameter of the nine-point circle (N). Since, for any triangle, angle $E_aT_aM_a$ must be greater than $90°$, T_a must be inside (N), or coincide with M_a, to have a valid solution. For the case with T_a inside (N), we have two solutions since the circle $E_a(M_a)$ intersects the altitude twice and each intersection leads to a distinct solution. If the three points are collinear, the two triangles are congruent reflections of each other through the line. If T_a is outside or on (N), except at M_a, there is no solution. If T_a coincides with M_a, there are an infinite number of solutions. In this case, the vertex A can be chosen anywhere on the open segment M_aM_a' (where M_a' is the reflection of M_a in E_a), and there is a resultant isosceles triangle.

Problem 108. Given E_a, N and T_a construct triangle ABC.

Problem 137. Given M_a, N and T_a construct triangle ABC.
Solution. Since N is the midpoint of E_aM_a, both of these problems reduce to Problem 99.

Problem 130. Given H_a, N and T_a construct triangle ABC.
Solution. The nine-point circle, with center N and passing through H_a, intersects line H_aT_a again at M_a, also reducing this problem to Problem 99.

Related to these, the solutions of the following two problems are locus restricted:
(i) Problem 78: given E_a, H_a, T_a;
(ii) Problem 99 in Wernick's list [3]: given M_a, H_a, T_a.

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Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: `cherylandharold@roadrunner.com`

Beata Randrianantoanina: Department of Mathematics, Miami University, Oxford, Ohio 45056, USA
E-mail address: `randrib@muohio.edu`

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A Family of Quartics Associated with a Triangle

Peter Yff

Abstract. It is known [1, p.115] that the envelope of the family of pedal lines (Simson or Wallace lines) of a triangle ABC is Steiner's deltoid, a three-cusped hypocycloid that is concentric with the nine-point circle of ABC and touches it at three points. Also known [2, p.249] is that the nine-point circle is the locus of the intersection point of two perpendicular pedal lines. This paper considers a generalization in which two pedal lines form any acute angle θ. It is found that the locus of their intersection point, for any value of θ, is a quartic curve with the same axes of symmetry as the deltoid. Moreover, the deltoid is the envelope of the family of quartics. Finally, it is shown that all of these quartics, as well as the deltoid and the nine-point circle, may be simultaneously generated by points on a circular disk rolling on the inside of a fixed circle.

1. Sketching the loci

Consider two pedal lines of triangle ABC which intersect and form an angle θ. It is required to find the locus of the intersection point for all such pairs of pedal lines for any fixed value of θ. There are infinitely many loci as θ varies between 0 and $\frac{\pi}{2}$. By plotting points, some of the loci are sketched in Figure 1. These include the cases $\theta = \frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{5\pi}{12}$, and $\frac{\pi}{2}$, the curves have been colored. As $\theta \to 0$, the locus approaches Steiner's deltoid. It will be shown later that in general the locus is a quartic curve. As $\theta \to \frac{\pi}{2}$, the quartic merges into two coincident circles (the nine-point circle). Otherwise each curve has three double points, which seem to merge into a triple point when $\theta = \frac{\pi}{3}$. This case resembles the familiar trefoil, or "three-leaved rose" of polar coordinates.

2. A conjecture

Figure 1 seems to suggest that all of the loci might be generated simultaneously by points on a circular disk that rolls inside a fixed circle concentric with the nine-point circle. For example, the deltoid could be generated by a point on the circumference of the disk, provided that the radius of the disk is one third that of the circle. The other curves might be hypotrochoids generated by interior points of the disk. However, this fails because, for example, there is no generating point for the nine-point circle.

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Another possible approach is given by Zwikker [2, pp.248–249], who shows that the same hypocycloid of three cusps may be generated when the radius of the rolling circle is two thirds of the radius of the fixed circle. In this case the deltoid is generated in the opposite sense, and two circuits of the rolling circle are required to generate the entire curve. Simultaneously the nine-point circle is generated by the center of the rolling disk. It is now necessary to prove that every locus in the family is generated by a point on the rolling disk.

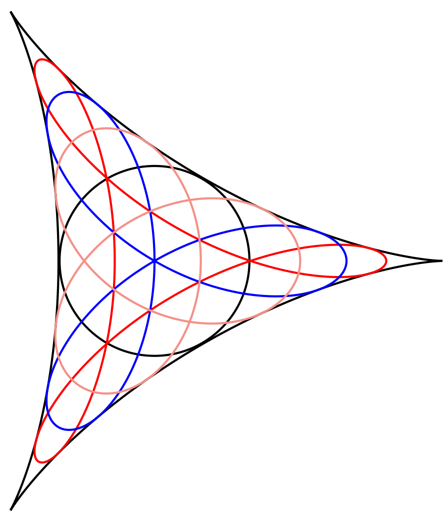

Figure 1

3. Partial proof of the conjecture

In Figure 2 the nine-point circle is placed with its center at the origin of an xy-plane. Its radius is $\frac{R}{2}$, R being the radius of the circumcircle of ABC. The radius of the fixed circle, also with center at the origin, is $\frac{3R}{2}$. The rolling disk has radius R, and initially it is placed so that it is touching the fixed circle at a cusp of the deltoid. Let the x-axis pass through this point of tangency. The center of the rolling disk is designated by Q, so that $OQ = \frac{R}{2}$. ST is a diameter of the rolling disk, with T initially at its starting point $\left(\frac{3R}{2}, 0\right)$. Let $P = \left(\frac{R}{2} + u,\ 0\right)$ be any point on the radius $QT (0 \leq u \leq R)$. Then, as the disk rotates clockwise about its center, it rolls counterclockwise along the circumference of the fixed circle, and the locus of P is represented parametrically by

$$\begin{aligned} x &= \frac{R}{2}\cos t + u\cos\frac{t}{2}, \\ y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}. \end{aligned} \tag{1}$$

In these equations u is the parameter of the family of hypotrochoids, while t is the running parameter on each curve. When $u = 0$, the locus is the nine-point circle $x^2 + y^2 = \frac{R^2}{4}$. When $u = R$, the parametric equations become

$$x = \frac{R}{2}\left(\cos t + 2\cos\frac{t}{2}\right),$$
$$y = \frac{R}{2}\left(\sin t - 2\sin\frac{t}{2}\right), \qquad (2)$$

which are well known to represent a deltoid.

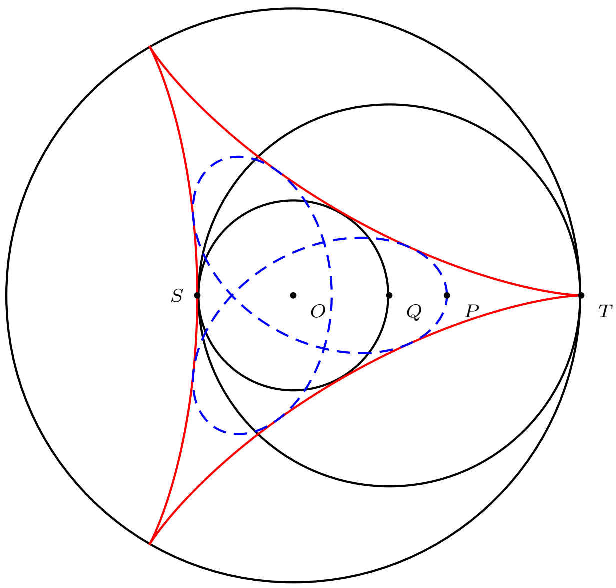

Figure 2

One further example is the case $u = \frac{R}{2}$, for which

$$x = \frac{R}{2}\left(\cos t + \cos\frac{t}{2}\right) = R\cos\frac{3t}{4}\cos\frac{t}{4},$$
$$y = \frac{R}{2}\left(\sin t - \sin\frac{t}{2}\right) = R\cos\frac{3t}{4}\sin\frac{t}{4}. \qquad (3)$$

These equations represent a trefoil, for which the standard equation in polar coordinates is

$$r = a\cos 3\theta,$$

from which $x = a\cos 3\theta\cos\theta$ and $y = a\cos 3\theta\sin\theta$. This result is identical with (3) when $t = 4R$ and $R = a$. Hence $u = \frac{R}{2}$ gives a trefoil (see Figure 3).

The foregoing is not a complete proof of the conjecture, because it is necessary to establish a connection with the loci of Figure 1. These are the curves generated by the intersection points of pedals lines that form a constant angle.

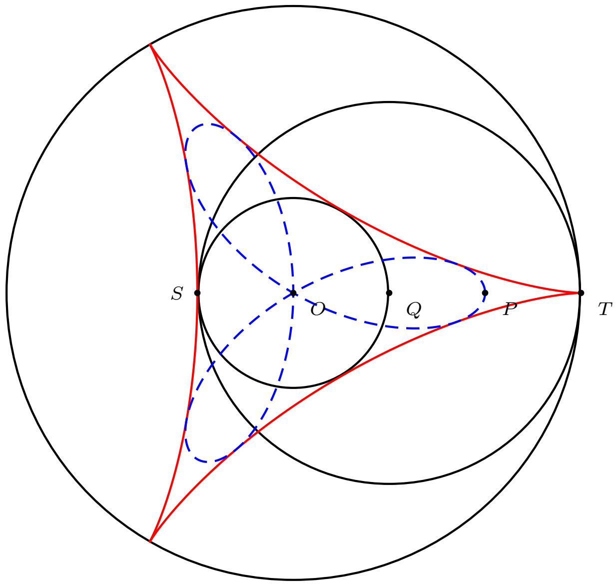

Figure 3

4. A family of quartics

By means of elementary algebra and trigonometric identities, the parameter t may be eliminated from equations (1) to obtain

$$\begin{aligned}&(4R^2(x^2+y^2)+24u^2Rx+8u^4+2u^2R62-T^4)^2\\ &=4u^2(4Rx+4u^2-R^2)^2(4Rx+u^2+2R^2).\end{aligned} \tag{4}$$

Thus, (1) is transformed into an equation of degree 4 in x and y. The only exceptional case is $u=0$, which reduces to $(4x^2+4y^2-R^2)^2=0$. This represents the nine-point circle, taken twice.

5. Envelope of the family

In order to find the envelope of (4), it is more practical to use the parametric form (1). The parameter u will be eliminated by using the partial differential equation

$$\frac{\partial x}{\partial t}\frac{\partial y}{\partial u}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial t},$$

or

$$-\left(\frac{R}{2}\sin t+\frac{u}{2}\sin\frac{t}{2}\right)\left(-\sin\frac{t}{2}\right)=\left(\cos\frac{t}{2}\right)\left(\frac{R}{2}\cos t-\frac{u}{2}\cos\frac{t}{2}\right).$$

This reduces to $u=R\cos\frac{3t}{2}$, and substitution in (1) results in the equations

$$\begin{aligned}x&=\frac{R}{2}\left(2\cos t+\cos 2t\right),\\ y&=\frac{R}{2}\left(2\sin t-\sin 2t\right).\end{aligned} \tag{5}$$

Replacing t by $-\frac{t}{2}$ transforms (5) to (2), showing that the envelope of (1) or (4) is the deltoid, which is itself a member of the family.

6. A "rolling" diameter

At the point given by (2) the slope of the deltoid is easily found to be $\tan\frac{t}{4}$. Hence the equation of the tangent line may be calculated to be

$$y-\frac{R}{2}\left(\sin t-2\sin\frac{t}{2}\right)=\tan\frac{t}{4}\left(x-\frac{R}{2}\left(\cos t+2\cos\frac{t}{2}\right)\right),$$

or

$$y=\tan\frac{t}{4}\left(x-\frac{R}{2}\left(1+2\cos\frac{t}{2}\right)\right). \tag{6}$$

Since the deltoid is a quartic curve, and since the point of tangency may be regarded as a double intersection with the tangent line (6), the tangent must meet the curve at two other points. Let $\left(\frac{R}{2}\left(\cos v+2\cos\frac{v}{2}\right),\ \frac{R}{2}\left(\sin v-2\sin\frac{v}{2}\right)\right)$ be any point on the curve, and substitute this for (x,y) in (6). The result is

$$\frac{R}{2}\left(\sin v-2\sin\frac{v}{2}\right)=\tan\frac{t}{4}\left(\frac{R}{2}\left(\cos v+2\cos\frac{v}{2}\right)-\frac{R}{2}\left(1+2\cos\frac{t}{2}\right)\right),$$

which becomes

$$\begin{aligned}&2\sin\frac{v}{2}\cos\frac{v}{2}-2\sin\frac{v}{2}\\ =&\ \tan\frac{t}{4}\left(\cos^2\frac{v}{2}-\sin^2\frac{v}{2}+2\cos\frac{v}{2}-1-2\cos\frac{t}{2}\right).\end{aligned} \tag{7}$$

In order to rewrite this as a homogeneous quartic equation, we make use of the identities

$$\begin{aligned}\sin\frac{v}{2}&=2\sin\frac{v}{4}\cos\frac{v}{4},\\ \cos\frac{v}{2}&=\ \cos^2\frac{v}{4}-\sin^2\frac{v}{4},\\ 1&=\ \sin^2\frac{v}{4}+\cos^2\frac{v}{4}.\end{aligned}$$

Then (7) becomes

$$\begin{aligned}&4\sin\frac{v}{4}\cos\frac{v}{4}\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)-4\sin\frac{v}{4}\cos\frac{v}{4}\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)\\ =&\ \tan\frac{t}{4}\left[\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)^2-\left(2\sin\frac{v}{4}\cos\frac{v}{4}\right)^2\right.\\ &+2\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)-\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)^2\\ &\left.-2\cos\frac{t}{2}\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)^2\right].\end{aligned}$$

The terms are then arranged according to descending powers of $\sin\frac{v}{4}$ to obtain

$$2\tan\frac{t}{4}\left(1+\cos\frac{t}{2}\right)\sin^4\frac{v}{4}-8\sin^3\frac{v}{4}\cos\frac{v}{4}$$
$$+\,4\tan\frac{t}{4}\left(2+\cos\frac{t}{2}\right)\sin^2\frac{v}{4}\cos^2\frac{v}{4}-2\tan\frac{t}{4}\left(1-\cos\frac{t}{2}\right)\cos^4\frac{v}{4}=0.$$

Dividing by $2\tan\frac{t}{4}\cos^4\frac{v}{4}$ and letting $V:=\tan\frac{v}{4}$ simplifies this to

$$\left(1+\cos\frac{t}{2}\right)V^4-4\cot\frac{t}{4}\cdot V^3+2\left(2+\cos\frac{t}{2}\right)V^2-\left(1-\cos\frac{t}{2}\right)=0.$$

Since the tangent line touches the deltoid where $v=t$, the quartic expression must contain the double factor $\left(V-\tan\frac{t}{4}\right)^2$. The factored result is

$$\left(1+\cos\frac{t}{2}\right)\left(V-\tan\frac{t}{4}\right)^2\left[V^2-2\cot\frac{t}{4}\cdot V-1\right]=0.$$

Hence the other solutions are found by solving

$$V^2-2\cot\frac{t}{4}\cdot V-1=0,$$

which yields $V=\cot\frac{t}{4}\pm\csc\frac{t}{4}=\cot\frac{t}{8}$ or $-\tan\frac{t}{8}$. Since $V=\tan\frac{v}{4}$, these may be expressed as $v=2\pi-\frac{t}{2}$ and $v=-\frac{t}{2}$ respectively. Because of periodicity there are other solutions to the quadratic equation, but geometrically there are only two, and the ones found here are distinct. The first one, substituted in (2), gives

$$(x,y)=\left(\frac{R}{2}\left(\cos\frac{t}{2}-2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}-2\sin\frac{t}{4}\right)\right).$$

Let this be the point T, shown in Figures 2 and 3. The point S at the other end of the diameter is given by the second solution $v=-\frac{t}{2}$:

$$S=\left(\frac{R}{2}\left(\cos\frac{t}{2}+2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}+2\sin\frac{t}{4}\right)\right).$$

The usual distance formula shows that the length of ST is $2R$. Moreover, the midpoint of ST is $\left(\frac{R}{2}\cos\frac{t}{2},\ -\frac{R}{2}\sin\frac{t}{2}\right)$, which is on the nine-point circle. Therefore it is the center of the rolling disk, and ST is a diameter. Since both S and T generate the deltoid, this confirms the fact that, for any line tangent to the deltoid, the segment within the curve is of constant length. See [2, p.249].

In order for the point T to trace one arch of the deltoid, the rolling disk travels through $\frac{4\pi}{3}$ radians on the fixed circle. Simultaneously the diameter ST rolls end over end to generate (as a tangent) the other two arches of the deltoid.

7. Proof of the conjecture

It remains to be shown that every locus defined by the intersection point of two pedal lines meeting at a fixed angle is a hypotrochoid defined by (1). Let one pedal line be given by (6), with slope $\tan\frac{t}{4}$. A second pedal line, forming the angle θ with the first, is obtained by replacing $\frac{t}{4}$ by $\frac{t}{4}+\theta$. (There is no need to include

$-\theta$, because this will be taken care of while t ranges over all of its values). The equation of the second pedal line will therefore be

$$y = \tan\left(\frac{t}{4}+\theta\right)\left[x-\frac{R}{2}\left(1+2\cos\left(\frac{t}{2}+2\theta\right)\right)\right]. \tag{8}$$

Simultaneous solution of (6) and (7), after manipulation with trigonometrical identities, gives the result

$$\begin{aligned} x &= \frac{R}{2}\left[\cos(t+2\theta)+2\cos\theta\cos\left(\frac{t}{2}+\theta\right)\right], \\ y &= \frac{R}{2}\left[\sin(t+2\theta)-2\cos\theta\sin\left(\frac{t}{2}+\theta\right)\right]. \end{aligned} \tag{9}$$

Finally, replacing $t+2\theta$ by t and $R\cos\theta$ by u, we transform (9) into

$$\begin{aligned} x &= \frac{R}{2}\cos t + u\cos\frac{t}{2}, \\ y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}, \end{aligned}$$

precisely equal to (1), the parametric equations of the family of hypotrochoids. Thus the result is established.

Remark. The family of quartics contains loci which are outside the deltoid, but these correspond to values of $u > R$, in which case θ would be imaginary.

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[2] C. Zwikker, *The Advanced Geometry of Plane Curves and Their Applications*, Dover, 1963.

Peter Yff: 10840 Cook Ave., Oak Lawn, Illinois 60453, USA
E-mail address: pjyff@aol.com

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Circle Chains Inside a Circular Segment

Giovanni Lucca

Abstract. We consider a generic circles chain that can be drawn inside a circular segment and we show some geometric properties related to the chain itself. We also give recursive and non recursive formulas for calculating the centers coordinates and the radius of the circles.

1. Introduction

Consider a circle with diameter AB, center C, and a chord GH perpendicular to AB (see Figure 1). Point O is the intersection between the diameter and the chord. Inside the circular segment bounded by the chord GH and the arc GBH, it is possible to construct a doubly infinite chain of circles each tangent to the chord, and to its two immediate neighbors.

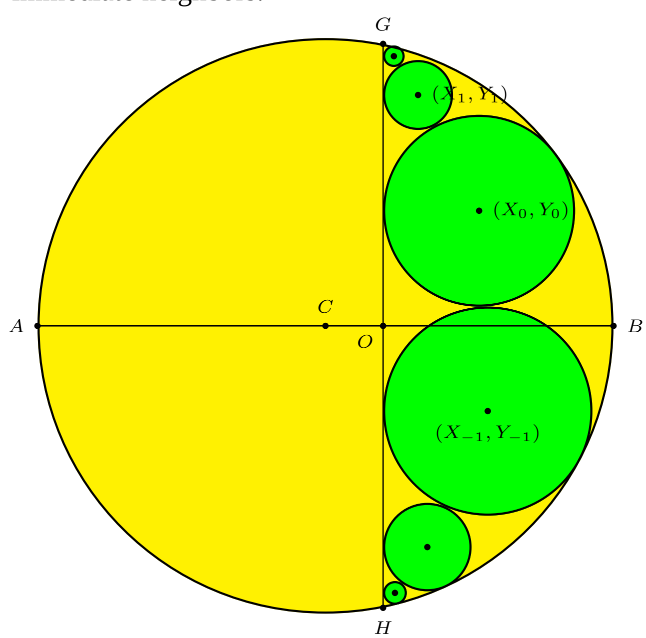

Figure 1. Circle chain inside a circular segment

Let $2(a + b)$ be the diameter of the circle and $2b$ the length of the segment OB. We set up a cartesian coordinate system with origin at O. Beginning with a circle with center (X_0, Y_0) and radius r_0 tangent to the chord GH and the arc GBH, we construct a doubly infinite chain of tangent circles, with centers (X_i, Y_i) and radius r_i for integer values of i, positive and negative.

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2. Some geometric properties of the chain

We first demonstrate some basic properties of the doubly infinite chain of circles.

Proposition 1. *The centers of the circles lie on the parabola with axis along AB, focus at C, and vertex the midpoint of OB.*

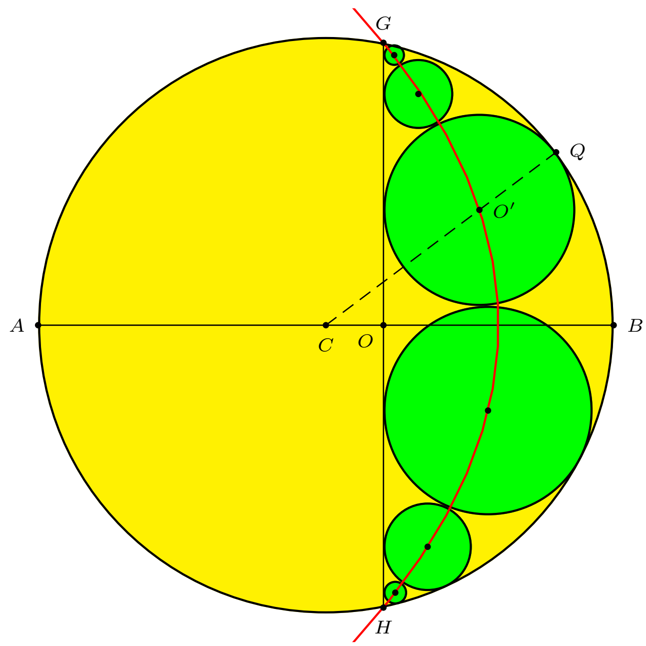

Figure 2. Centers of circles in chain on a parabola

Proof. Consider a circle of the chain with center $O'(x, y)$, radius r, tangent to the arc GBH at Q. Since the segment CQ contains O' (see Figure 2), we have, by taking into account that C has coordinates $(b - a, 0)$ and

$$\begin{aligned} CQ &= a + b, \\ CO' &= \sqrt{(x - b + a)^2 + y^2}, \\ O'Q &= r = x, \\ CO' &= CQ - O'Q. \end{aligned}$$

From these, we have

$$\sqrt{(x - b + a)^2 + y^2} = a + b - x,$$

which simplifies into

$$y^2 = -4a(x - b). \tag{1}$$

This clearly represents the parabola symmetric with respect to the x-axis, vertex $(b, 0)$, the midpoint of OB, and focus $(b - a, 0)$, which is the center C of the given circle. $\square$

Proposition 2. *The points of tangency between consecutive circles of the chain lie on the circle with center A and radius AG.*

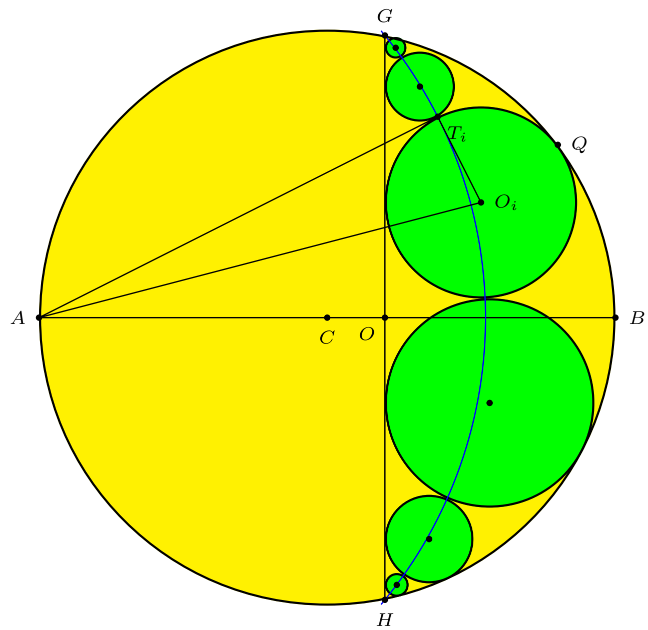

Figure 3. Points of tangency on a circular arc

Proof. Consider two neighboring circles with centers (X_i, Y_i), (X_{i+1}, Y_{i+1}), radii r_i, r_{i+1} respectively, tangent to each other at T_i (see Figure 3). By using Proposition 1 and noting that A has coordinates $(-2a, 0)$, we have

$$\begin{aligned} AO_i^2 = & \ (X_i + 2a)^2 + Y_i^2 = \left(-\frac{Y_i^2}{4a} + b + 2a\right)^2 + Y_i^2, \\ r_i^2 = & \ X_i^2 = \left(-\frac{Y_i^2}{4a} + b\right)^2. \end{aligned}$$

Applying the Pythagorean theorem to the right triangle AO_iT_i, we have

$$AT_i^2 = AO_i^2 - r_i^2 = 4a(a + b) = AO \cdot AB = AG^2.$$

It follows that T_i lies on the circle with center A and radius AG. □

Proposition 3. *If a circle of the chain touches the chord GH at P and the arc GBH at Q, then the points A, P, Q are collinear.*

Proof. Suppose the circle has center O'. It touches GH at P and the arc GBH at Q (see Figure 4). Note that triangles CAQ and $O'PQ$ are isosceles triangles with $\angle ACQ = \angle PO'Q$. It follows that $\angle CQA = \angle O'QP$, and the points A, P, Q are collinear. □

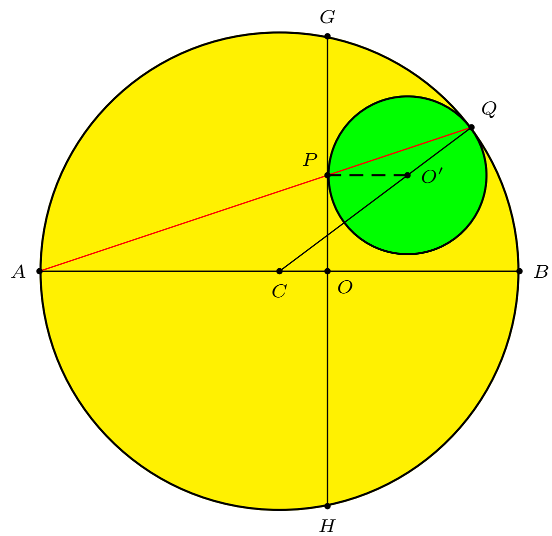

Figure 4. Line joining points of tangency

Remark. Proposition 3 gives an easy construction of the circle given any one of the points of tangency. The center of the circle is the intersection of the line CQ and the perpendicular to GH at P.

3. Coordinates of centers and radii

Figure 5 shows a right triangle $O_iO_{i-1}K_i$ with the centers O_{i-1} and O_i of two neighboring circles of the chain. Since these circles have radii $r_{i-1} = X_{i-1}$ and $r_i = X_i$ respectively, we have

$$\begin{aligned}(X_{i-1} - X_i)^2 + (Y_i - Y_{i-1})^2 =&\ (r_i + r_{i-1})^2 = (X_i + X_{i-1})^2,\\ (Y_i - Y_{i-1})^2 =&\ 4X_iX_{i-1}.\end{aligned}$$

Making use of (1), we rewrite this as

$$(Y_i - Y_{i-1})^2 = 4\left(b - \frac{Y_i^2}{4a}\right)\left(b - \frac{Y_{i-1}^2}{4a}\right),$$

or

$$\frac{4a(a+b) - Y_{i-1}^2}{4a^2} \cdot Y_i^2 - 2Y_{i-1}Y_i + \frac{(a+b)Y_{i-1}^2 - 4ab^2}{a} = 0. \tag{2}$$

If we index the circles in the chain in such a way that the ordinate Y_i increases with the index i, then from (2) we have

$$Y_i = \frac{2Y_{i-1} - \left(\frac{Y_{i-1}^2}{a} - 4b\right)\sqrt{1 + \frac{b}{a}}}{2\left(1 + \frac{b}{a} - \frac{Y_{i-1}^2}{4a^2}\right)}. \tag{3a}$$

This is a recursive formula that can be applied provided that the ordinate Y_0 of the first circle is known. Note that Y_0 must be chosen in the interval $(-2\sqrt{ab},\ 2\sqrt{ab})$.

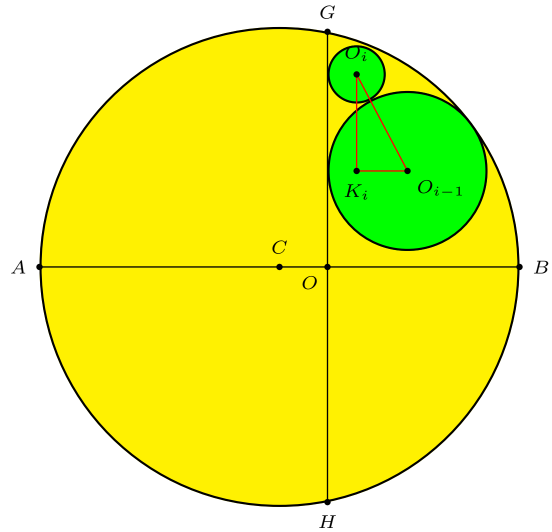

Figure 5. Construction for determination of recursive formula

Formulas for the abscissa of the centers and radii are immediately derived from (1), *i.e.*:

$$X_i = r_i = -\frac{Y_i^2}{4a} + b. \tag{4}$$

Now, it is possible to transform the recursion formula (**??**) into a continued fraction. In fact, after some simple algebraic steps (which we omit for brevity), we have

$$Y_i = 2a\left(\sqrt{1+\frac{b}{a}} - \frac{1}{\frac{Y_{i-1}}{2a} + \sqrt{1+\frac{b}{a}}}\right). \tag{5}$$

Defining

$$\alpha = 2\sqrt{1+\frac{b}{a}}, \quad \text{and} \quad Z_i = \frac{Y_i}{2a} - \sqrt{1+\frac{b}{a}} \tag{6}$$

for $i = 1,\ 2,\ \ldots,$ we have

$$Z_i = -\frac{1}{\alpha + Z_{i-1}}.$$

Thus, for positive integer values of i,

$$Z_i = -\cfrac{1}{\alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots - \frac{1}{\alpha + Z_{0+}}}}},$$

where we have used Z_{0+} in place of Z_0

$$Z_{0+} = \frac{Y_0}{2a} - \sqrt{1+\frac{b}{a}}.$$

This is to distinguish from the extension of the chain by working the recursion (3a) backward: [1]

$$Y_{i-1} = \frac{2Y_i + \left(\frac{Y_i^2}{a} - 4b\right)\sqrt{1+\frac{b}{a}}}{2\left(1+\frac{b}{a}-\frac{Y_i^2}{4a^2}\right)}. \qquad (3b)$$

Thus, for negative integer values of i, with

$$Z_{-i} = \frac{Y_{-i}}{2a} + \sqrt{1+\frac{b}{a}},$$

we have

$$Z_{-i} = -\cfrac{1}{-\alpha - \cfrac{1}{-\alpha - \cfrac{1}{\ddots - \frac{1}{-\alpha + Z_{0-}}}}},$$

where

$$Z_{0-} = \frac{Y_0}{2a} + \sqrt{1+\frac{b}{a}}.$$

It is possible to give nonrecursive formulas for calculating Y_i and Y_{-i}. For brevity, in the following, we shall consider only Y_i for positive integer indices because, as far as Y_{-i} is concerned, it is enough to change, in all the formulae involved, α into $-\alpha$, Z_i into Z_{-i}, and Z_{0+} into Z_{0-}. Starting from (5), and by considering its particular structure, one can write, for $i = 1,\ 2,\ 3,\ \ldots,$

$$Z_i = -\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}$$

where $Q_i(\alpha)$ are polynomials with integer coefficients. Here are the first ten of them.

$Q_0(\alpha)$	1
$Q_1(\alpha)$	$\alpha + Z_{0+}$
$Q_2(\alpha)$	$(\alpha^2-1) + \alpha Z_{0+}$
$Q_3(\alpha)$	$(\alpha^3-2\alpha) + (\alpha^2-1)Z_{0+}$
$Q_4(\alpha)$	$(\alpha^4-3\alpha^2+1) + (\alpha^3-2\alpha)Z_{0+}$
$Q_5(\alpha)$	$(\alpha^5-4\alpha^3+3\alpha) + (\alpha^4-3\alpha^2+1)Z_{0+}$
$Q_6(\alpha)$	$(\alpha^6-5\alpha^4+6\alpha^2-1) + (\alpha^5-4\alpha^3+3\alpha)Z_{0+}$
$Q_7(\alpha)$	$(\alpha^7-6\alpha^5+10\alpha^3-4\alpha) + (\alpha^6-5\alpha^4+6\alpha^2-1)Z_{0+}$
$Q_8(\alpha)$	$(\alpha^8-7\alpha^6+15\alpha^4-10\alpha^2+1) + (\alpha^7-6\alpha^5+10\alpha^3-4\alpha)Z_{0+}$
$Q_9(\alpha)$	$(\alpha^9-8\alpha^7+21\alpha^5-20\alpha^3+5\alpha) + (\alpha^8-7\alpha^6+15\alpha^4-10\alpha^2+1)Z_{0+}$

According to a fundamental property of continued fractions [1], these polynomials satisfy the second order linear recurrence

$$Q_i(\alpha) = \alpha Q_{i-1}(\alpha) - Q_{i-2}(\alpha). \qquad (7)$$

[1] Equation (3b) can be obtained by solving equation (2) for Y_{i-1}.

We can further write

$$Q_i(\alpha) = P_i(\alpha) + P_{i-1}(\alpha)Z_{0+}, \tag{8}$$

for a sequence of simpler polynomials $P_i(\alpha)$, each either odd or even. In fact, from (7) and (8), we have

$$P_{i+2}(\alpha) = \alpha P_{i+1}(\alpha) - P_i(\alpha).$$

Explicitly,

$$P_i(\alpha) = \begin{cases} 1, & i = 0, \\ \sum_{k=0}^{\frac{i}{2}} (-1)^{\frac{i}{2}+k} \binom{\frac{i}{2}+k}{2k} \alpha^{2k}, & i = 2, 4, 6, \dots, \\ \sum_{k=1}^{\frac{i+1}{2}} (-1)^{\frac{i+1}{2}+k} \binom{\frac{i-1}{2}+k}{2k-1} \alpha^{2k-1}, & i = 1, 3, 5, \dots. \end{cases}$$

From (6), we have

$$Y_i = a\left(\alpha - 2\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}\right),$$

for $i = 1,\ 2,\ \dots.$

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Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy
E-mail address: vanni_lucca@inwind.it

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On Three Circles

David Graham Searby

Abstract. The classical Three-Circle Problem of Apollonius requires the construction of a fourth circle tangent to three given circles in the Euclidean plane. For circles in general position this may admit as many as eight solutions or even no solutions at all. Clearly, an "experimental" approach is unlikely to solve the problem, but, surprisingly, it leads to a more general theorem. Here we consider the case of a chain of circles which, starting from an arbitrary point on one of the three given circles defines (uniquely, if one is careful) a tangent circle at this point and a tangency point on another of the given circles. Taking this new point as a base we construct a circle tangent to the second circle at this point and to the third circle, and repeat the construction cyclically. For any choice of the three starting circles, the tangency points are concyclic and the chain can contain at most six circles. The figure reveals unexpected connections with many classical theorems of projective geometry, and it admits the Three-Circle Problem of Apollonius as a particular case.

In the third century B.C., Apollonius of Perga proposed (and presumably solved, though the manuscript is now lost) the problem of constructing a fourth circle tangent to three given circles. A partial solution was found by Jean de la Viète around 1600, but here we shall make use of Gergonne's extremely elegant solution, which covers all cases. The closure theorem presented here is a generalization of this classical problem, and it reveals somewhat surprising connections with theorems of Monge, D'Alembert, Pascal, Brianchon, and Desargues.

Unless the three given circles are tangent at a common point, the Problem of Apollonius may have no solutions at all or it may have as many as eight – a Cartesian formulation would have to take into consideration the coordinates of the three centers as well as the three radii, and even after normalization we would still be left with an eighth degree polynomial. Algebraic and geometrical considerations lead us to consider points as circles with radius zero, and lines as circles with infinite radius. Inversion will, of course, permit us to eliminate lines altogether, however

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A biographical note is included at the end of the paper by the Editor, who also provides the annotations.

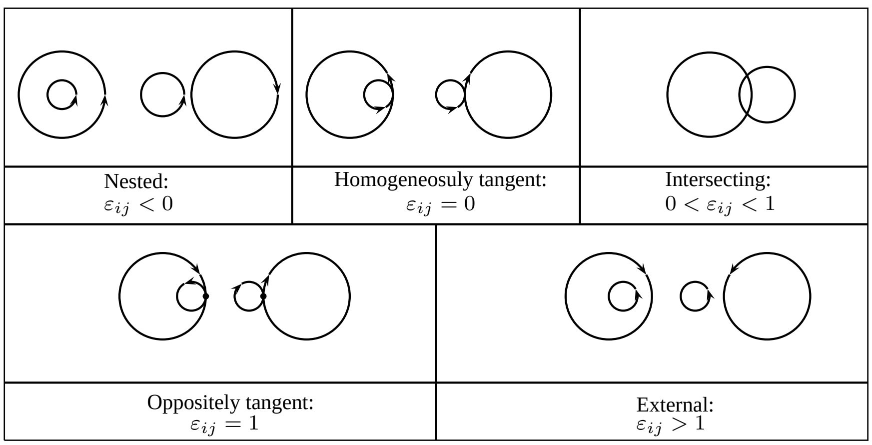
we must take into account the possibility of negative radii[1]. This apparent complication in reality allows us to define general parameters to describe the relationship between pairs of circles:

Notation and Definitions (*Circular Excess*) Let $\mathcal{C}_i = \mathcal{C}_i(x_i, y_i; r_i)$ be the circle with center (x_i, y_i) and radius r_i; define

$$e_i = x_i^2+y_i^2-r_i^2, \quad e_{ij} = (x_i-x_j)^2+(y_i-y_j)^2-(r_i-r_j)^2, \quad \text{and} \quad \epsilon_{ij} = \frac{e_{ij}}{4r_ir_j}.$$

The usefulness of the "excess" quantities e and ϵ will be evident from the following definitions.

Definition. We distinguish five types of relationships between pairs of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ with nonzero radii, as illustrated in the accompanying table.



These descriptions are preserved by inversion – specifically,

Theorem 1. (Inversive Invariants). *The parameter ϵ_{ij} is invariant under inversion in any circle whose center does not lie on either of the two given circles.*

Proof. The circle $\mathcal{C}_0(x_0, y_0; r_0)$ inverts $\mathcal{C}(x, y; r)$ to $\mathcal{C}'(x', y'; r')$, where if d is the Euclidean distance between the centers of $\mathcal{C}$ and $\mathcal{C}_0$, and $I_0 = \frac{r_0^2}{d^2-r^2}$, we find

$$\begin{aligned} x' =&\ x_0 + I_0(x - x_0), \\ y' =&\ y_0 + I_0(y - y_0), \\ r' =&\ rI_0. \end{aligned}$$

[1]There are two common ways to interpret signed radii. They provide an orientation to the circles (as in [6]), so that $r > 0$ would indicate a counterclockwise orientation, $r < 0$ clockwise, and $r = 0$ an unoriented point. In the limit $r = \pm\infty$, and one obtains oriented lines. This seems to be Searby's interpretation. Alternatively, as in [11], one can assume a circle for which $r > 0$ to be a disk (that is, a circle with its interior), while $r < 0$ indicates a circle with its exterior; a line for which $r = \infty$ determines one half plane and $r = -\infty$ the other. This interpretation works especially well in the inversive plane (called the *circle plane* here) which, in the model that fits best with this paper, is the Euclidean plane extended by a single point at infinity that is incident with every line of the plane.

(See [5, p. 79]). Upon applying the formula for ϵ_{ij} to $\mathcal{C}'_i$ and $\mathcal{C}'_j$ then simplifying, we obtain the theorem. $\square$

Theorem 1 permits us to work in a *circle plane* using Cartesian coordinates, the Euclidean definition of circles being extended to admit negative, infinite, and zero radii.

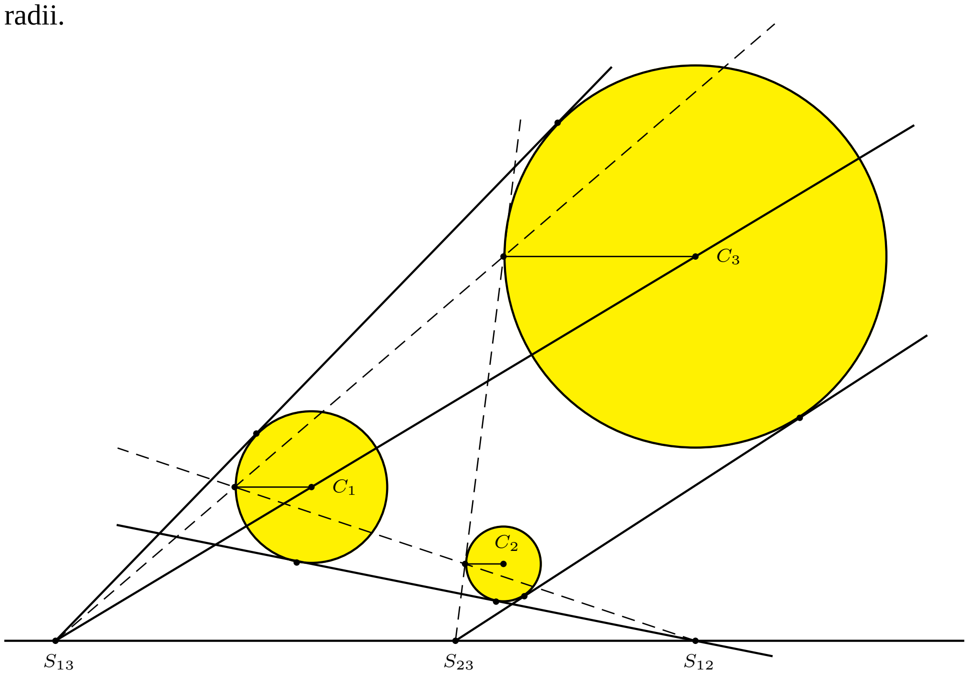


Figure 1. The centers of similitude S_{ij} of three circles lie on the axis of similitude.

Observation (*D'Alembert-Monge*). The centers of similitude S_{ij} of two circles $\mathcal{C}_i$ and $\mathcal{C}_j$ are the points on the line of centers where the common tangents (when they exist) intersect. In Cartesian coordinates we have [7, Art. 114, p.105]

$$S_{ij} = \left(\frac{r_i x_j - r_j x_i}{r_i - r_j}, \frac{r_i y_j - r_j y_i}{r_i - r_j} \right).$$

Note that if the radii are of the same sign these coordinates correspond to the *external* center of similitude; if the signs are opposite the center is *internal*. Moreover, three circles with signed radii generate three collinear points that lie on a line called the *axis of similitude* (or *Monge Line*) σ, whose equation is [7, Art.117, p.107]

$$\sigma = \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \end{vmatrix}.$$

As similar determinants appear frequently, we shall write them as Δ_{abc} if the rows are a_i, b_i, c_i; or simply Δ_{ab} should $c_i = 1$.

Lemma 2 (Second tangency point)**.** *If $P(x_0, y_0)$ is a point on a circle $\mathcal{C}_i$ while $\mathcal{C}_j$ is a second circle, then there exists exactly one circle $\mathcal{C}_a(x_a, y_a; r_a)$ that is homogeneously tangent to $\mathcal{C}_i$ at P and to $\mathcal{C}_j$ at some point $P'(x'_0, y'_0)$. Moreover $\mathcal{C}_a$ has parameters*

$$x_a = x_i + \frac{(x_0 - x_i)e_{ij}}{2f^0_{ij}}, \quad y_a = y_i + \frac{(y_0 - y_i)e_{ij}}{2f^0_{ij}}; \quad r_a = -r_i - \frac{(r_0 - r_i)e_{ij}}{2f^0_{ij}},$$

where

$$r_0 := 0 \quad \textit{and} \quad f^0_{ij} := r_i r_j - (x_0 - x_i)(x_0 - x_j) - (y_0 - y_i)(y_0 - y_j);$$

and the coordinates of P' are

$$\begin{aligned} x'_0 = &\ x_i + \frac{r_i e_{0j}(x_j - x_i) + r_j e_{ij}(x_0 - x_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})}, \\ y'_0 = &\ y_i + \frac{r_i e_{0j}(y_j - y_i) + r_j e_{ij}(y_0 - y_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})}, \end{aligned}$$

where

$$e_{0j} = (x_0 - x_j)^2 + (y_0 - y_j)^2 - r_j^2$$

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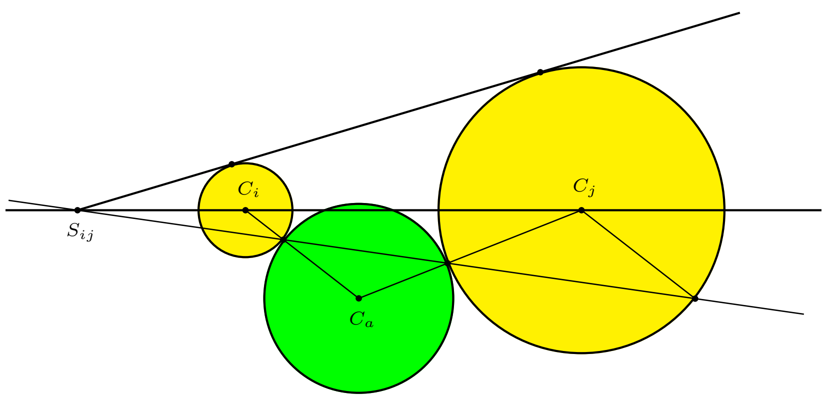

Figure 2. The second tangency point of Lemma 2.

Proof. (Outline)[2] The two tangency points P and P' are collinear with a center of similitude S_{ij}, which will be external or internal according as the radii have the same or different signs [7, Art. 117, p. 108]. It is then sufficient to find the intersections of $S_{ij}P$ with $\mathcal{C}_j$. One of the roots of the resulting quadratic equation

[2]The existence and uniqueness of $\mathcal{C}$ is immediate to anybody familiar with inversive geometry: inversion in a circle with center P sends P to infinity and $\mathcal{C}_i$ to an oriented line; the image of $\mathcal{C}_a$ under that inversion is then the unique parallel oriented line that is homogeneously tangent to the image of $\mathcal{C}_j$. Searby's intent here was to provide explicit parameters, which were especially useful to him for producing accurate figures in the days before the graphics programs that are now common. I, however, drew the figures using *Cinderella*. Searby did all calculations by hand, but they are too lengthy to include here; I confirmed the more involved formulas using *Mathematica*.

represents the point on $\mathcal{C}_j$ whose radius is parallel to that of P on $\mathcal{C}_i$; the other yields the coordinates of P', and the rest follows. □

We are now ready for the main theorem. The first part of the theorem – the closure of the chain of circles – was first proved by Tyrrell and Powell [10], having been conjectured earlier from a drawing.

Theorem 3 (Apollonius Closure)**.** *Let $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ be three circles in the Circle Plane, and choose a point P_1 on $\mathcal{C}_1$. Define $\mathcal{C}_{12}$ to be the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_1 and to $\mathcal{C}_2$, thus defining $P_2 \in \mathcal{C}_2$. Continue with $\mathcal{C}_{23}$ homogeneously tangent to $\mathcal{C}_2$ at P_2 and to $\mathcal{C}_3$ at P_3, then $\mathcal{C}_{34}$ homogeneously tangent to $\mathcal{C}_3$ at P_3 and to $\mathcal{C}_1$ at P_4, ..., and $\mathcal{C}_{67}$ homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$ at P_7. Then this chain closes with $\mathcal{C}_{78} = \mathcal{C}_{12}$ or, more simply, $P_7 = P_1$. Moreover, the points $P_1, \ldots, P_6$ are cyclic* (see Figure 3).

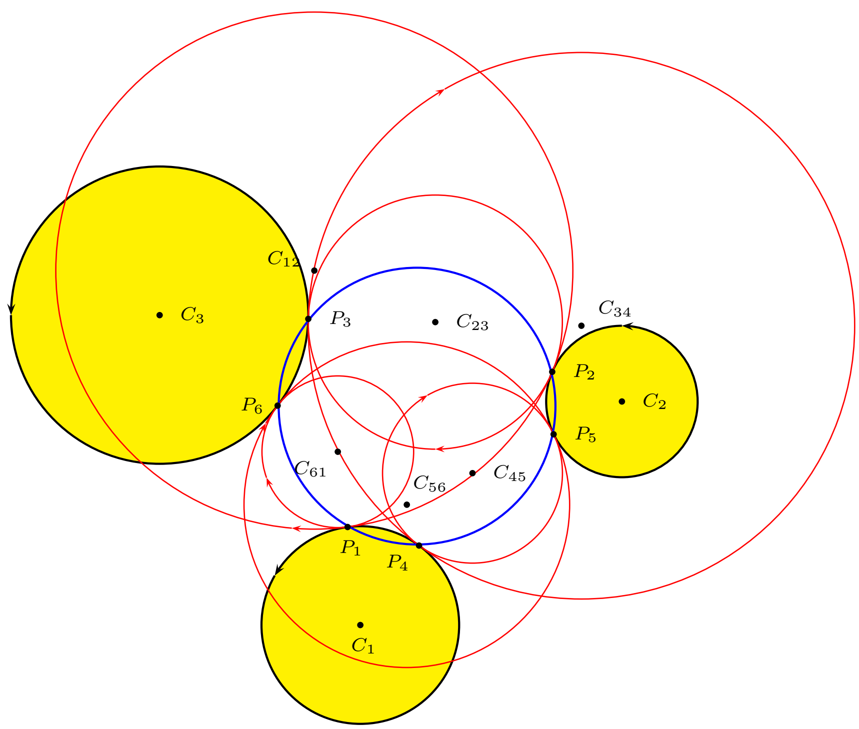

Figure 3. For $i = 1, 2$, and 3 the given circle $\mathcal{C}_i$ (in yellow) is homogeneously tangent at P_i to $\mathcal{C}_{i(i+1)}$ and $\mathcal{C}_{(i+5)i}$, and at P_{i+3} to $\mathcal{C}_{(i+3)(i+4)}$ and $\mathcal{C}_{(i+2)(i+3)}$ (where the subscripts $6 + \ell$ of $\mathcal{C}_{jk}$ are reduced to ℓ).

Proof.[3] We first show that four consecutive P_i's lie on a circle, taking P_1, P_2, P_3, P_4 as a typical example. See Figure 4.

[3]Rigby provides two proofs of this theorem in [6]. Searby independently rediscovered the result around 1987; he showed it to me at that time and I provided yet another proof in [3]. Searby's approach has the virtue of being entirely elementary.

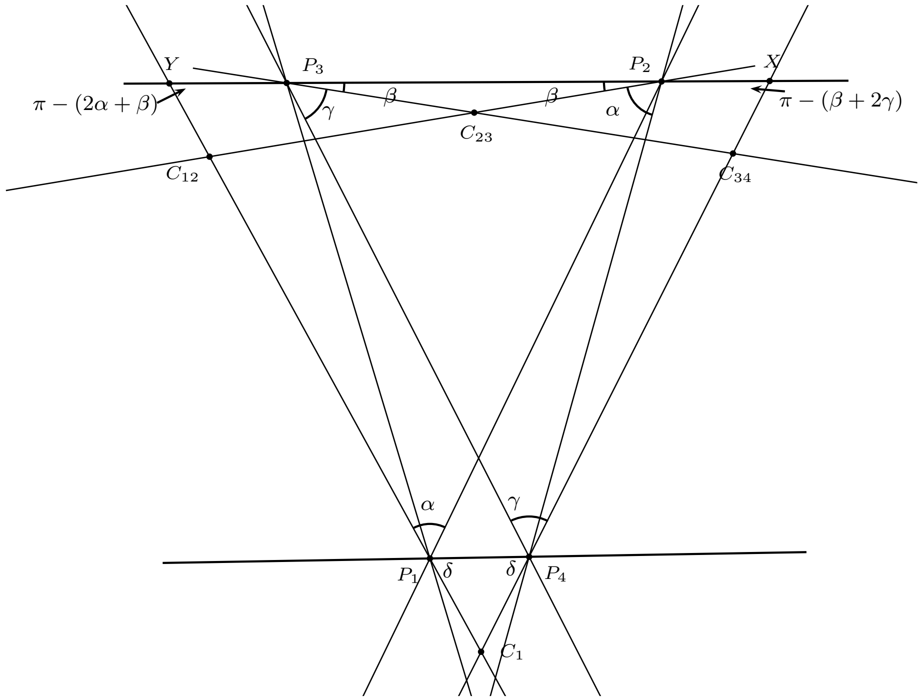

Figure 4. Proof of the Main Theorem 3

Special cases are avoided by using directed angles (so that $\angle ABC$ is the angle between 0 and π through which the line BA must be rotated counterclockwise about B to coincide with BC). Denote by C_i and C_{ij} the centers of the circles $\mathcal{C}_i$ (where $i = 1, 2, 3$) and $\mathcal{C}_{ij}$ (where $1 \leq i < j$). By hypothesis P_i is on the lines joining $C_{(i-1)i}$ to $C_{i(i+1)}$ and C_i to both $C_{i(i+1)}$ and $C_{(i-1)i}$, where we use the convention that $C_{3+k} = C_k$ as shown in Figure 4. In that figure we denote the base angles of the isosceles triangles $\triangle C_{i(i+1)}P_iP_{i+1}$ by α, β, and γ, while δ is the base angle of $\triangle C_1P_4P_1$. Consider $\triangle C_1XY$ formed by the lines $C_1P_4C_{34},\ P_2P_3$, and $C_{12}P_1C_1$. In $\triangle XP_3P_4$, $\angle P_4 = \gamma$ and $\angle P_3 = \angle P_2P_3C_{23} + \angle C_{23}P_3P_4 = \beta + \gamma$, whence $\angle X = \pi - (\beta + 2\gamma)$. In $\triangle YP_1P_2$, $\angle P_1 = \alpha$ and $\angle P_2 = \angle P_1P_2C_{12} + \angle C_{12}P_2P_3 = \alpha + \beta$, whence $\angle Y = \pi - (2\alpha + \beta)$. Consequently, $\angle C_1 = \pi - (\angle X + \angle Y) = 2(\alpha + \beta + \gamma) - \pi$. But in $\triangle C_1P_4P_1$, $\angle C_1 = \pi - 2\delta$; whence, $2(\alpha + \beta + \gamma) - \pi = \pi - 2\delta$, or

$$\alpha + \beta + \gamma + \delta = \pi.$$

Because $\angle P_2P_3P_4 = \beta + \gamma$ and $\angle P_2P_1P_4 = \alpha + \delta$, we conclude that these angles are equal and the points P_1, P_2, P_3, P_4 lie on a circle. By cyclically permuting the indices we deduce that P_5 and P_6 lie on that same circle, which proves the claim in the final statement of the theorem. This new circle already intersects $\mathcal{C}_1$ at P_1 and P_4, so that the sixth circle of the chain, namely the unique circle $\mathcal{C}_{67}$ that is homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$, would necessarily be tangent to $\mathcal{C}_1$ at P_1 or P_4. Should the tangency point be P_4, recalling that $\mathcal{C}_{34}$ is the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_4 and to $\mathcal{C}_3$ at P_3, we would necessarily have $P_3 = P_6$. In that case we would necessarily have also $P_1 = P_4$ and $P_2 = P_5$, and the circle $P_1P_2P_3$ would be one of the pair of Apollonius Circles mentioned earlier. In any case, $P_7 = P_1$, and the sixth circle $\mathcal{C}_{67}$ touches $\mathcal{C}_1$ at P_1, closing the chain, as claimed. □

This Euclidean proof is quite general: if any of the circles were straight lines we could simply invert the figure in any appropriate circle to obtain a configuration of ten proper circles. We shall call the circle through the six tangency points the *six-point circle*, and denote it by $\mathcal{S}$. Note the symmetric relationship among the nine circles – any set of three non-tangent circles chosen from the circles of the configuration aside from $\mathcal{S}$ will generate the same figure. Indeed, the names of the circles can be arranged in an array

	P_1	P_5	P_3
P_4	C_1	C_{45}	C_{34}
P_2	C_{12}	C_2	C_{23}
P_6	C_{61}	C_{56}	C_3

so that the circles in any row or column homogeneously touch one another at the point that heads the row or column. Given the configuration of these nine circles without any labels, there are six ways to choose the initial three non-tangent circles. This observation should make clear that the closure of the chain is guaranteed even when the Apollonius Problem has no solution.

Observation (*Apollonius Axis*). The requirement that a circle $\mathcal{C}(x, y; r)$ be tangent to three circles $\mathcal{C}_i(x_i, y_i; r_i)$ yields a system of three quadratic equations which can be simplified to a linear equation in x and y, and which will be satisfied by the coordinates of the centers of two of the solutions of the Apollonius Problem. (The other six solutions are obtained by taking one of the radii to be negative.) We shall call the line through those two centers (whose points satisfy the resulting linear equation) the *Apollonius Axis* and denote it by α; its equation [7, Art. 118, pp.108-110] is

$$\alpha : x\Delta_{xr} + y\Delta_{yr} = \frac{\Delta_{er}}{2}.$$

Note that the two lines σ and α are perpendicular; they are defined even when the corresponding Apollonius Circles fail to exist (or, more precisely, are not real).

Observation (*Radical Center*). The locus of all points having the same power (that is, the square of the distance from the center minus the square of the radius) with respect to two circles is a straight line, the *radical axis* [7, Art. 106, 107, pp.98–99]:

$$\rho_{ij} : 2(x_i - x_j)x + 2(y_i - y_j)y = e_i - e_j.$$

The axes determined by three circles are concurrent at their ***radical center***

$$C_R = \left(\frac{\Delta_{ey}}{2\Delta_{xy}}, \frac{\Delta_{xe}}{2\Delta_{xy}} \right).$$

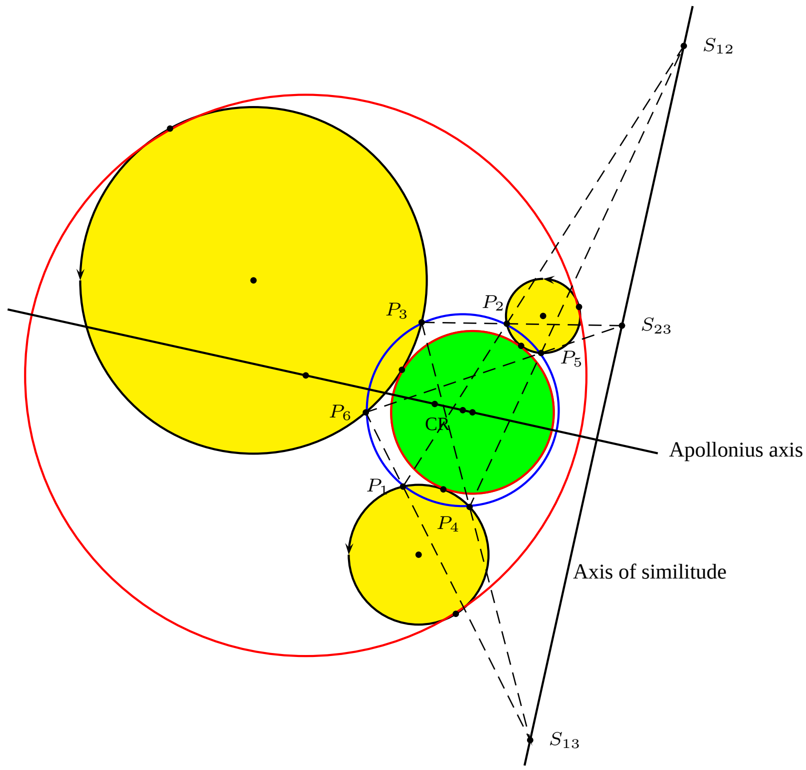

Figure 5. The Apollonius Axis of three oriented circles contains the centers of the two Apollonius circles, the radical center $\mathcal{C}_R$, and the centers S of all six-point circles. It is perpendicular to the axis of similitude.

This point is also known as the *Monge Point*, as it is the center of the circle, called the *Monge Circle*, that is orthogonal to all three given circles whenever such a circle exists. By substitution one sees that C_R lies on α. In summary,

Theorem 4 (Monge Circle)**.** *The Apollonius Axis α of three given circles is the line through their radical center C_R that is perpendicular to the axis of similitude σ; furthermore the Monge circle, if it exists, is a six-point circle that inverts the nine-circle configuration of Theorem 3 into itself.*

Theorem 5 (Centers of Six-Point Circles)**.** *For any three given non-tangent circles, as P_1 moves around $\mathcal{C}_1$ the locus of the center S of the corresponding six-point circle is either the entire Apollonius Axis α, the segment of α between the centers of the two Apollonius Circles (homogeneously tangent to all three of the given circles), or that segment's complement in α.*

Proof. Let $P_1 = (x_0, y_0)$. We saw (while finding the second tangency point) that the line P_1P_2 coincides with $S_{12}P_1$, which (by the formula for S_{12}) has gradient

$$\frac{r_1(y_0 - y_2) - r_2(y_0 - y_1)}{r_1(x_0 - x_2) - r_2(x_0 - x_1)};$$

because the perpendicular bisector β of P_1P_2 passes through C_{12}, its equation must therefore be

$$\beta: \ \frac{y - y_1 - \frac{(y_0-y_1)e_{12}}{2f_{12}^0}}{x - x_1 - \frac{(x_0-x_1)e_{12}}{2f_{12}^0}} = -\frac{r_1(x_0 - x_2) - r_2(x_0 - x_1)}{r_1(y_0 - y_2) - r_2(y_0 - y_1)}.$$

We can then use Cramer's rule to find the point where β intersects the Apollonius Axis α, which entails the arduous but rewarding calculation of the denominator,

$$(r_2 - r_1)\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ r_0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

It requires only a little more effort to find the coordinates of the desired intersection point, which we claim to be S, namely

$$S = \left(\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}, -\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}} \right), \tag{1}$$

where we have used e_0 to stand for $x_0^2 + y_0^2$ (with $r_0 = 0$). Of course, the same calculation could be applied to $\mathcal{C}_{61}$, and we would obtain the same point (1). In other words, the perpendicular bisectors of the chords of $\mathcal{S}$ formed by the tangency points of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$, and of $\mathcal{C}_{61}$ with $\mathcal{C}_1$ and $\mathcal{C}_3$, must intersect at S, which is necessarily the center of $\mathcal{S}$. As a byproduct of the way its coordinates were calculated, we must have S on α, as claimed. Finally, the Main Theorem guarantees the existence of S, and (1) shows that its coordinates are continuous functions of x_0 and y_0. Since a solution circle to the Problem of Apollonius is obviously a (degenerate) six-point circle, the second part of the theorem is also proved. $\square$

Setting $S = (s_1, s_2)$ and rewriting the first coordinate of (1) as

$$\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 2s_1 \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \tag{2}$$

we readily see that this is an equation of the form

$$x_0^2 - 2s_1x_0 + y_0^2 - 2s_2y_0 + \text{ (terms involving neither } x_0 \text{ nor } y_0\text{) } = 0;$$

the only step that cannot be done in one's head is checking that the coefficient of y_0 necessarily equals the second coordinate of (1). In particular, we see that the point (x_0, y_0) satisfies the equation of a circle with center $S = (s_1, s_2)$. But, the unique circle with center S that passes through (x_0, y_0) is our six-point circle $\mathcal{S}$. Because both equation (2) and the corresponding equation using the second coordinate of (1) hold for any point P_1 in the plane, even if P_1 does not lie on $\mathcal{C}_1$, we see that $\mathcal{S}$ is part of a larger family of circles that cover the plane. We therefore deduce that

Theorem 6 (Six-point Pencil)**.** *The equations* (2) *represent the complete set of six-point circles, which is part of a pencil of circles whose radical axis is σ. When the pencil consists of intersecting circles, σ might itself be a six-point circle.*[4]

Proof. S_{12} has the same power[5], namely $\frac{e_{12}r_1r_2}{r_1^2-r_2^2}$, with respect to all circles tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$; but, for any point $P_1 \in \mathcal{C}_1$, the quantity $S_{12}P_1 \times S_{12}P_2$ is also the power of S_{12} with respect to the six-point circle determined by P_1. Since similar claims hold for S_{23} and S_{31}, it follows that σ (the line containing the centers of similitude) is the required radical axis. The rest follows quickly from the definitions. □

Since the tangency points P_1 and P_2 of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$ are collinear with S_{12}, and similarly for the other pairs, we see immediately that (as in Figure 6)

Theorem 7 (Pascal)**.** *The points where the six-point circle $\mathcal{S}$ meets the given circles form a Pascal hexagon $P_1P_2P_3P_4P_5P_6$ whose axis is the axis of similitude σ.*

Again, the pair of Apollonius circles deriving from Gergonne's construction and (if they are real) delimiting the pencil of Theorem 6 are special positions of the $\mathcal{C}_{ij}$, whence (as in Figure 7)

Theorem 8 (Gergonne-Desargues)**.** *For any given triple of circles, the six tangency points of a pair of Apollonius Circles, the three centers of similitude S_{ij}, and the radical center C_R are ten points of a Desargues Configuration.*

Proof. We should mention for completeness that by Gergonne's construction[6], the poles (x_i', y_i') of σ with respect to $\mathcal{C}_i$ are collinear with the radical center C_R and the tangency points of the two Apollonius Circles with $\mathcal{C}_i$. For those who prefer the use of coordinates,

$$x_i' = x_i + r_i \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix}, \qquad y_i' = y_i + r_i \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix},$$

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[4]One easily sees that each six-point circle cuts the three given circles at equal angles. Salmon [7, Art. 118] derives the same conclusion as our Theorem 6 while determining the locus of the center of a circle cutting three given circles at equal angles.

[5]I wonder if Searby used the definition of *power* that he gave earlier (in the form $d^2 - r^2$), which seems quite awkward for the calculations needed here. The claim about the constant power of S_{12} is clear, however, without such a calculation: the circle, or circles, of inversion that interchange $\mathcal{C}_1$ with $\mathcal{C}_2$, called the *mid-circles* in [2, Sections 5.7 and 5.8] (see, especially, Exercise 5.8.1 on p.126), is the locus of points P such that two circles, tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, are tangent to each other at P. The center of this mid-circle is S_{12}, and the square of its radius $S_{12}P$ is the power of S_{12} with respect to any of these common tangent circles.

[6]Details concerning Gergonne's construction can be found in many of the references that deal with the Problem of Apollonius such as [1, Section 10.11.1, p.318], [4, Section 1.10, pp.22–23], or [7, Art. 119 to 121, pp.110–113].

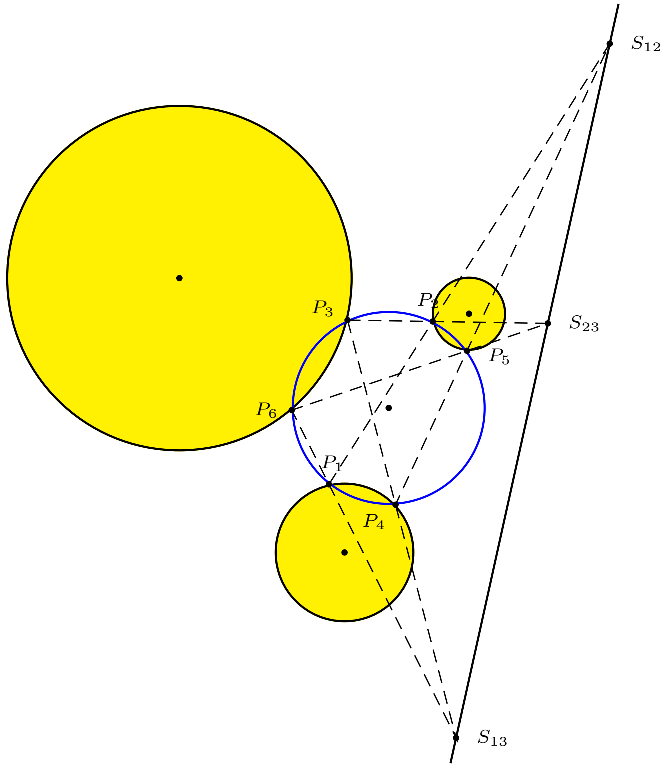

Figure 6. The points where the six-point circle $\mathcal{S}$ (in blue) meets the given circles (in yellow) form a Pascal hexagon whose axis is the axis of similitude

and the equation of the line joining C_R to the points where $\mathcal{C}_i$ is tangent to the Apollonius Circles is

$$(x-x_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} + (y-y_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} + r_i\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Gergonne's construction yields the tangency points in three pairs collinear with C_R, which is, consequently, the center of perspectivity of the triangles inscribed in the Apollonius Circles. The axis is clearly σ because, as with the circles $\mathcal{C}_{ij}$, an Apollonius circle is tangent to the given circles $\mathcal{C}_i$ and $\mathcal{C}_j$ at points whose joining line passes through S_{ij}. □

Finally, on inverting the intersection point of σ and α in $\mathcal{S}$ and tracing the six tangent lines to $\mathcal{S}$ at the points P_i where it meets the given circles, after much routine algebra (which we leave to the reader)[7] we obtain

Theorem 9 (Brianchon)**.** *The inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$ is the Brianchon Point of the hexagon circumscribing $\mathcal{S}$ and tangent to it at the six points where it intersects the given circles $\mathcal{C}_i$, taken in the order indicated by the labels.*

[7]There is no need for any calculation here: Theorem 9 is the projective dual of Theorem 7 – the polarity defined by $\mathcal{S}$ takes each point P_i to the line tangent there to $\mathcal{S}$, while (because $\sigma \perp \alpha$) it interchanges the axis of similitude σ with the inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$.

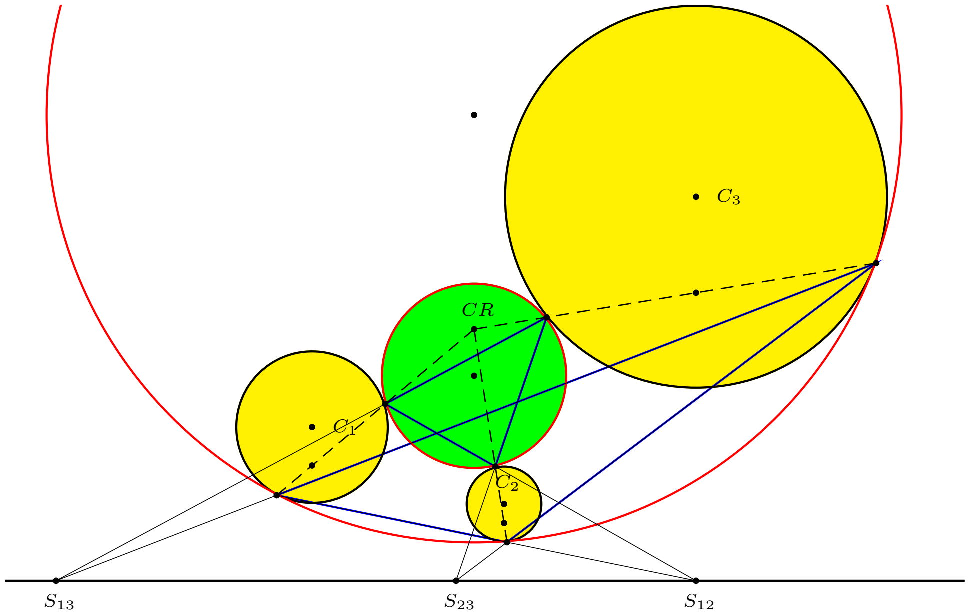

Figure 7. The triangles (shown in blue) whose vertices are the points where the Apollonius circles (red) are tangent to the three given oriented circles (yellow) are perspective from the radical center $\mathcal{C}_R$; the axis of the perspectivity is the axis of similitude of the given circles.

Conclusion. Uniting as it does the classical theorems of Monge, D'Alembert, Desargues, Pascal, and Brianchon together with the problem of Apollonius, we feel that this figure merits to be better known. The ubiquitous and extremely useful e and ϵ symbols take their name from the Einstein-Minkowski metric: in fact, the circle plane (or its three-dimensional analogue) is a vector space which, on substitution of the last coordinate (that is, the radius) by the imaginary distance cti (where $i^2 = -1$) yields interesting analogies with relativity theory. [8]

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[8]See, for example, [11, Sections 11 and 15] where there is a discussion of the Lorentz group and further references can be found.

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David Graham Searby (1948 - 1998): David Searby was born on October 30, 1948 in Melbourne, Australia, and lived in both Melbourne and Adelaide while growing up. He graduated from Flinders University of South Australia with a Bachelor of Science degree (1970) and a Bachelor of Science Honours (1972). He began graduate work there under the supervision of John Wamsley, with whom he published his first paper [8]. He moved to Italy in 1974, and for three years was supported by a foreign-student scholarship at the University of Bologna, funded by the National Research Council (CNR) of Italy. He lived in Bologna for the remainder of his life; although he maintained a tenuous relationship with the university, he survived by giving private lessons in English and in mathematics. He was an effective and inspiring teacher (although he taught relatively little) and scholar – in addition to his mathematics, he knew every corner of Italy, its history and literature; he spoke flawless Italian and even mastered the local Bolognese dialect. He was driven by boundless curiosity and intellectual excitement, and loved to spend long evenings in local bars, sustained by soup, beer and ideas. His lifestyle, however, was unsustainable, and he died on August 19, 1998, just short of his 50th birthday. He left behind a box full of his notes and computations in no apparent order. One of his research interests concerned configurations in projective planes, both classical and finite. An early paper [9] on the existence of Pappus configurations in planes of order nine indicated the direction his research was to take: he found that in the Hughes Plane of order nine there exist triangles which fail to contain a Pappus configuration that has three points on each of its sides. Were coordinates introduced using such a triangle as the triangle of reference, the imposed algebraic structure would be nearly trivial. From this Searby speculated on the existence of finite projective planes whose order is not the power of a prime, and whose coordinates have so little structure that the plane could be discovered only by computer. He collected configuration theorems throughout most of his life with a goal toward finding configurations on which he could base a computer search. Unfortunately, he never had access to a suitable computer. Among his papers was the first draft of a monograph (in Italian) that brought together many of his elementary discoveries on configurations; it is highly readable, but a long way from being publishable. There was also the present paper, almost ready to submit for publication, which brings together several of his discoveries involving configuration theorems. It has been lightly edited by me; I added the footnotes, references, and figures. I would like to thank David's brother Michael and his friends David Glynn, Ann Powell, and David Tiley for their help in preparing the biographical note. (J. Chris Fisher)

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Class Preserving Dissections of Convex Quadrilaterals

Dan Ismailescu and Adam Vojdany

Abstract. Given a convex quadrilateral Q having a certain property $\mathcal{P}$, we are interested in finding a dissection of Q into a finite number of smaller convex quadrilaterals, each of which has property $\mathcal{P}$ as well. In particular, we prove that every cyclic, orthodiagonal, or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal, or circumscribed quadrilaterals, respectively. The problem becomes much more interesting if we restrict our study to a particular type of partition we call *grid dissection*.

1. Introduction

The following problem represents the starting point and the motivation of this paper.

Problem. Find all convex polygons which can be dissected into a finite number of pieces, each similar to the original one, but not necessarily congruent.

It is easy to see that all triangles and parallelograms have this property (see e.g. [1, 7]).

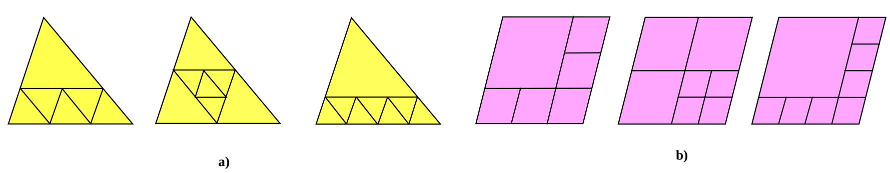

Figure 1. (a) Triangle dissection into similar triangles.
(b) Parallelogram dissection into similar parallelograms.

Indeed, every triangle can be partitioned into 6, 7 or 8 triangles, each similar to the initial one (see Figure 1 a). Simple inductive reasoning shows that for every $k \geq 6$, any triangle T can be dissected into k triangles similar to T. An analogous statement is true for parallelograms (see Figure 1 b). Are there any other polygons besides these two which have this property?

The origins of Problem 1 can be traced back to an early paper of Langford [10]. More then twenty years later, Golomb [8] studied the same problem without notable success. It was not until 1974 when the first significant results were published by Valette and Zamfirescu.

Theorem 1 (Valette and Zamfirescu, [13])**.** *Suppose a given convex polygon P can be dissected into four congruent tiles, each of which similar to P. Then P is either a triangle, a parallelogram or one of the three special trapezoids shown in Figure 2 below.*

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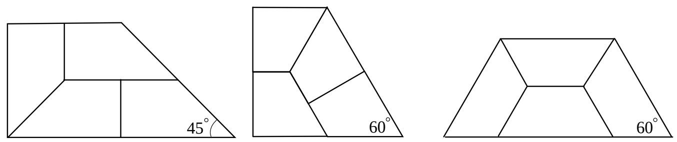

Figure 2. Trapezoids which can be partitioned into four congruent pieces.

Notice that the hypothesis of the above theorem is much more restrictive: the number of pieces must be exactly four and the small polygons must all be congruent to each other, not only similar. However, as of today, the convex polygons presented in figures 1 and 2 are the only known solutions to the more general problem 1.

From a result of Bleicher [2], it is impossible to dissect a convex n-gon (a convex polygon with n vertices) into a finite number of convex n-gons if $n \geq 6$. The same result was proved by Bernheim and Motzkin [3] using slightly different techniques.

Although any convex pentagon can be partitioned into any number $k \geq 6$ of convex pentagons, a recent paper by Ding, Schattschneider and Zamfirescu [4] shows that it is impossible to dissect a convex pentagon into similar replicas of itself.

Given the above observations, it follows that for solving problem 1 we can restrict ourselves to convex quadrilaterals. It is easy to prove that a necessary condition for a quadrilateral to admit a dissection into similar copies of itself is that the measures of its angles are linearly dependent over the integers. Actually, a stronger statement holds true: if the angles of a convex quadrilateral Q do not satisfy this dependence condition, then Q cannot be dissected into a finite number of smaller similar convex polygons which are not necessarily similar to Q (for a proof one may consult [9]). Nevertheless, in spite of all the above simplifications and renewed interest in the geometric dissection topic (see e. g. [6, 12, 15]), problem 1 remains open.

2. A Related Dissection Problem

Preserving similarity under dissection is difficult: although all triangles have this property, there are only a handful of known quadrilaterals satisfying this condition (parallelograms and some special trapezoids), while no n-gon can have this property if $n \geq 5$. In the sequel, we will try to examine what happens if we weaken the similarity requirement.

Problem. Suppose that a given polygon P has a certain property $\mathcal{C}$. Is it possible to dissect P into smaller polygons, each having property $\mathcal{C}$ as well?

For instance, suppose $\mathcal{C}$ means "convex polygon with n sides". As we have mentioned in the previous section, in this particular setting Problem 2 has a positive answer if $3 \leq n \leq 5$ and a negative answer for all $n \geq 6$. Before we proceed we need the following:

Definition. a) A convex quadrilateral is said to be *cyclic* if there exists a circle passing through all of its vertices.
b) A convex quadrilateral is said to be *orthodiagonal* if its diagonals are perpendicular.
c) A convex quadrilateral is said to be *circumscribed* if there exists a circle tangent to all of its sides.
d) A convex quadrilateral is said to be a *kite* if it is both orthodiagonal and circumscribed.

The following theorem provides characterizations for all of the quadrilaterals defined above and will be used several times throughout the remainder of the paper.

Theorem 2. *Let $ABCD$ be a convex quadrilateral.*
(a) *$ABCD$ is cyclic if and only if opposite angles are supplementary – say, $\angle A + \angle C = 180°$.*
(b) *$ABCD$ is orthodiagonal if and only if the sum of squares of two opposite sides is equal to the sum of the squares of the remaining opposite sides – that is, $AB^2 + CD^2 = AD^2 + BC^2$.*
(c) *$ABCD$ is circumscribed if and only if the two pairs of opposite sides have equal total lengths – that is, $AB + CD = AD + BC$.*
(d) *$ABCD$ is a kite if and only if (after an eventual relabeling) $AB = BC$ and $CD = DA$.*

A comprehensive account regarding cyclic, orthodiagonal and circumscribed quadrilaterals and their properties, including proofs of the above theorem, can be found in the excellent collection of geometry notes [14]. An instance of Problem 2 we will investigate is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal or circumscribed quadrilaterals, respectively?

It has been shown in [1] and [11] that every cyclic quadrilateral can be dissected into **four** cyclic quadrilaterals two of which are isosceles trapezoids (see Figure 3 a).

Another result is that every cyclic quadrilateral can be dissected into **five** cyclic quadrilaterals one of which is a rectangle (see Figure 3 b). This dissection is based on the following property known as *The Japanese Theorem* (see [5]).

Theorem 3. *Let $ABCD$ be a cyclic quadrilateral and let M, N, P and Q be the incenters of triangles ABD, ABC, BCD and ACD, respectively. Then $MNPQ$ is a rectangle and quadrilaterals $AMNB$, $BNPC$, $CPQD$ and $DQMA$ are cyclic* (see Figure 3).

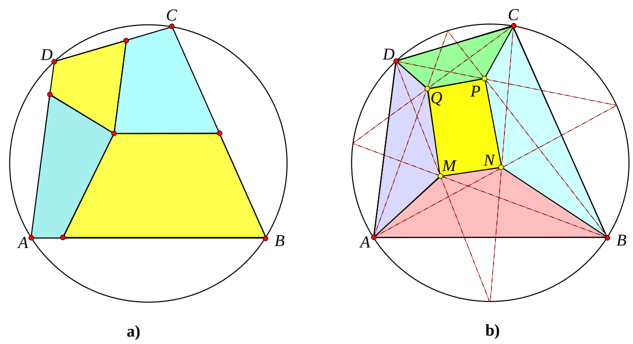

Figure 3. (a) Cyclic quadrilateral = 2 isosceles trapezoids + 2 cyclic quadrilaterals.
(b) Cyclic quadrilateral = one rectangle + four cyclic quadrilaterals.

Since every isosceles trapezoid can be dissected into an arbitrary number of isosceles trapezoids, it follows that every cyclic quadrilateral can be dissected into k cyclic quadrilaterals, for every $k \geq 4$.

It is easy to dissect an orthodiagonal quadrilateral into four smaller orthodiagonal ones.

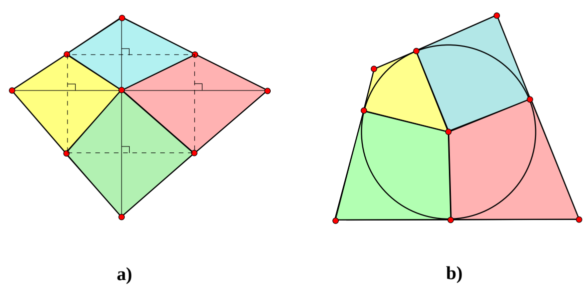

Figure 4. (a) Orthodiagonal quadrilateral = four orthodiagonal quadrilaterals.
(b) Circumscribed quadrilateral = four circumscribed quadrilaterals.

Consider for instance the quadrilaterals whose vertex set consists of one vertex of the initial quadrilateral, the midpoints of the sides from that vertex and the intersection point of the diagonals (see Figure 4 a). It is easy to prove that each of these quadrilaterals is orthodiagonal.

A circumscribed quadrilateral can be dissected into four quadrilaterals with the same property by simply taking the radii from the incenter to the tangency points (see Figure 4 b).

Actually, it is easy to show that each of these smaller quadrilaterals is not only circumscribed but cyclic and orthodiagonal as well.

The above discussion provides a positive answer to problem 2. In fact, much more is true.

Theorem 4 (Dissecting arbitrary polygons)**.** *Every convex n-gon can be partitioned into $3\,(n-2)$ cyclic kites* (see Figure 5).

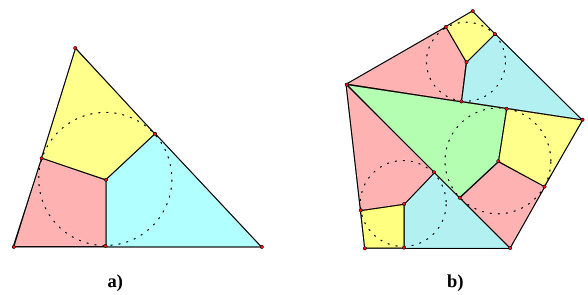

Figure 5. Triangle = three cyclic kites; Pentagon = nine cyclic kites.

Proof. Notice first that every triangle can be dissected into three cyclic kites by dropping the radii from the incenter to the tangency points (see Figure 5 a). Partition the given n-gon into triangles. For instance, one can do this by drawing all the diagonals from a certain vertex. We obtain a triangulation consisting of $n-2$ triangles. Dissect then each triangle into cyclic kites as indicated in Figure 5 b). $\square$

3. Grid Dissections of Convex Quadrilaterals

We have seen that the construction used in theorem 4 renders problem 2 almost trivial. The problem becomes much more challenging if we do restrict the type of dissection we are allowed to use. We need the following

Definition. Let $ABCD$ be a convex quadrilateral and let m and n be two positive integers. Consider two sets of segments $\mathcal{S} = \{s_1, s_2, \ldots, s_{m-1}\}$ and $\mathcal{T} = \{t_1, t_2, \ldots, t_{n-1}\}$ with the following properties:
a) If $s \in \mathcal{S}$ then the endpoints of s belong to the sides AB and CD. Similarly, if $t \in \mathcal{T}$ then the endpoints of t belong to the sides AD and BC.
b) Every two segments in $\mathcal{S}$ are pairwise disjoint and the same is true for the segments in $\mathcal{T}$.
We then say that segments $s_1, s_2, \ldots, s_{m-1}, t_1, t_2, \ldots, t_{n-1}$ define an m-by-n *grid dissection* of $ABCD$ (see Figure 6).

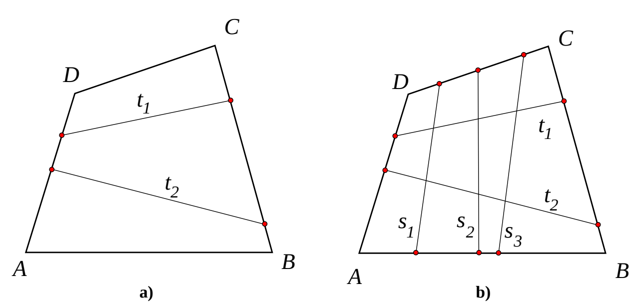

Figure 6. A 3-by-1 and a 3-by-4 grid dissection of a convex quadrilateral

The really interesting problem is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be partitioned into cyclic, orthodiagonal, or circumscribed quadrilaterals,respectively, **via a grid dissection?** Such dissections shall be referred to as *class preserving grid dissections*, or for short *CPG* dissections (or CPG partitions).

3.1. *Class Preserving Grid Dissections of Cyclic Quadrilaterals.* In this section we study whether cyclic quadrilaterals have class preserving grid dissections. We start with the following

Question. Under what circumstances does a cyclic quadrilateral admit a 2-by-1 grid dissection into cyclic quadrilaterals? What about a 2-by-2 grid dissection with the same property?

The answer can be readily obtained after a straightforward investigation of the sketches presented in Figure 7.

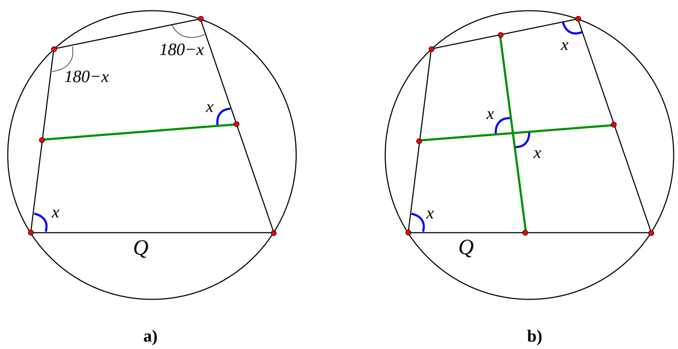

Figure 7. (a) 2-by-1 CPG dissection of cyclic Q exists iff $Q =$ trapezoid.
(b) 2-by-2 CPG dissection of cyclic Q exists iff $Q =$ rectangle.

A quick analysis of the angles reveals that a 2-by-1 CPG partition is possible if only if the initial cyclic quadrilateral is an isosceles trapezoid - see Figure 7 a). A similar reasoning leads to the conclusion that a 2-by-2 CPG partition exists if and only if the original quadrilateral is a rectangle – Figure 7 b). These observations can be easily extended to the following:

Theorem 5. *Suppose a cyclic quadrilateral Q has an m-by-n grid partition into mn cyclic quadrilaterals. Then:*
a) If m and n are both even, Q is necessarily a rectangle.
b) If m is odd and n is even, Q is necessarily an isosceles trapezoid.

We leave the easy proof for the reader. It remains to see what happens if both m and n are odd. The next two results show that in this case the situation is more complex.

Theorem 6 (A class of cyclic quadrilaterals which have 3-by-1 CPG dissections)**.** *Every cyclic quadrilateral all of whose angles are greater than* $\arccos \frac{\sqrt{5}-1}{2} \approx 51.83^\circ$ *admits a* 3*-by-*1 *grid dissection into three cyclic quadrilaterals.*

Proof. If $ABCD$ is an isosceles trapezoid, then any two segments parallel to the bases will give the desired dissection. Otherwise, assume that $\angle D$ is the largest angle (a relabeling of the vertices may be needed). Since $\angle B + \angle D = \angle A + \angle C = 180^\circ$ it follows that $\angle B$ is the smallest angle of $ABCD$. We therefore have:

$$\angle B < \min\{\angle A,\ \angle C\} \leq \max\{\angle A,\ \angle C\} < \angle D. \tag{1}$$

Denote the measures of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ on the circumcircle of $ABCD$ by $2a$, $2b$, $2c$ and $2d$ respectively (see Figure 8 a). Inequalities (1) imply that $c + d < \min\{b + c,\ a + d\} \leq \max\{b + c,\ a + d\} < a + b$, that is, $c < a$ and $d < b$.

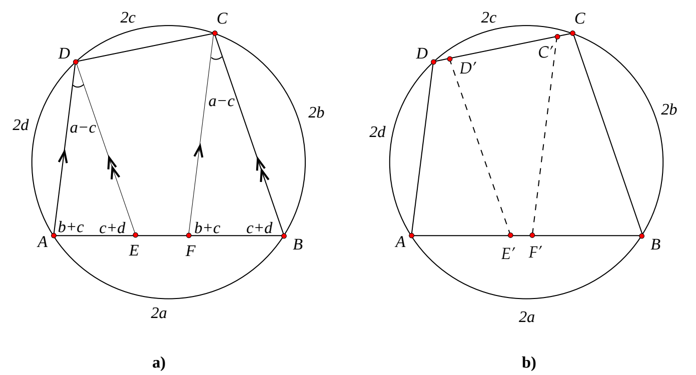

Figure 8. (a) $DE \parallel BC$, $CF \parallel AD$, E and F between A and B due to $c < a$.
(b) 3-by-1 grid dissection into cyclic quads if DE and CF do not intersect.

Through vertex D construct a segment $DE \parallel BC$ with E on line AB. Since $c < a$, point E is going to be between A and B. Similarly, through vertex C construct a segment $CF \parallel AD$ with F on AB. As above, since $c < a$, point F will lie between A and B.

If segments DE and CF do not intersect then a 3-by-1 grid dissection of $ABCD$ into cyclic quadrilaterals can be obtained in the following way:
Choose two points C' and D' on side CD, such that C' is close to C and D' is close to D. Construct $D'E' \parallel DE$ and $C'F' \parallel CF$ as shown in figure 8 b). Since segments DE and CF do not intersect it follows that for choices of C' and D' sufficiently close to C and D respectively, the segments $D'E'$ and $C'F'$ will not intersect. A quick verification shows that each of the three quadrilaterals into which $ABCD$ is dissected ($AE'D'D$, $D'E'F'C'$ and $C'F'BC$) is cyclic.

It follows that a sufficient condition for this grid dissection to exist is that points $A - E - F - B$ appear exactly in this order along side AB, or equivalently, $AE + BF < AB$.

The law of sines in triangle ADE gives that $AE \sin(c + d) = AD \sin(a - c)$ and since $AD = 2R \sin d$ we obtain

$$AE = \frac{2R \sin d\ \sin(a - c)}{\sin(c + d)}, \tag{2}$$

where R is the radius of the circumcircle of $ABCD$.

Similarly, using the law of sines in triangle BCF we have $BF\,\sin(b+c) = BC\,\sin(a-c)$ and since $BC = 2R\,\sin b$ it follows that

$$BF = \frac{2R\sin b\,\sin(a-c)}{\sin(b+c)}. \tag{3}$$

Using equations (2), (3) and the fact that $AB = 2R\,\sin a$, the desired inequality $AE + BF < AB$ becomes equivalent to

$$\begin{aligned} &\frac{\sin d\,\sin(a-c)}{\sin(c+d)} + \frac{\sin b\,\sin(a-c)}{\sin(b+c)} < \sin a \\ \Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \frac{\sin(b+c-c)}{\sin(b+c)} < \frac{\sin(a-c+c)}{\sin(a-c)} \\ \Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \cos c - \sin c\,\cot(b+c) < \cos c + \sin c\,\cot(a-c), \end{aligned}$$

and after using $a+b+c+d = 180^\circ$ and simplifying further,

$$AE + BF < AB \Leftrightarrow \sin(a-c)\,\sin(b+c)\,\sin d < \sin^2(c+d)\,\sin(c). \tag{4}$$

Recall that points E and F belong to AB as a result of the fact that $c < a$. A similar construction can be achieved using the fact that $d < b$.

Let $AG \parallel CD$ and $DH \parallel AB$ as shown in Figure 9 a). Since $d < b$, points G and H will necessarily belong to side BC. As in the earlier analysis, if segments AG and DH do not intersect, small parallel displacements of these segments will produce a 3-by-1 grid partition of $ABCD$ into 3 cyclic quadrilaterals: $ABG'A'$, $A'G'H'D''$ and $H'D''DC$ (see Figure 9 b).

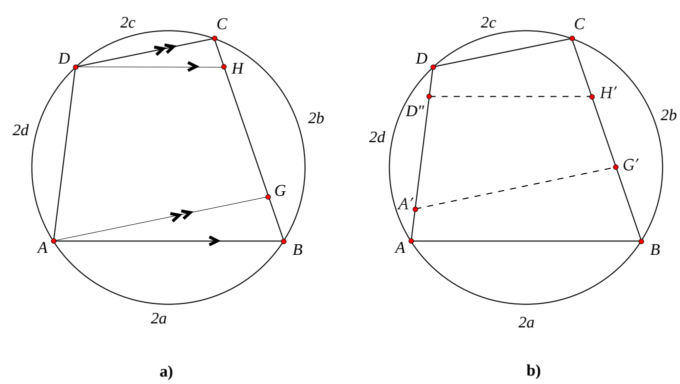

Figure 9. a) $AG \parallel CD$, $DH \parallel AB$, G and H between B and C since $d < b$
b) 3-by-1 CPG grid dissection if AG and DH do not intersect.

The sufficient condition for this construction to work is that points $B{-}G{-}H{-}C$ appear in this exact order along side BC, or equivalently, $BG + CH < BC$.

Using similar reasoning which led to relation (4) we obtain that

$$BG + CH < BC \Leftrightarrow \sin(b-d)\,\sin(a+d)\,\sin c < \sin^2(a+b)\,\sin d. \tag{5}$$

The problem thus reduces to proving that if $\min\{\angle A, \angle B, \angle C, \angle D\} > \arccos\frac{\sqrt{5}-1}{2}$ then at least one of the inequalities that appear in (4) and (5) will hold.
To this end, suppose none of these inequalities is true. We thus have:

$$\begin{aligned}\sin(a-c)\,\sin(b+c) &\geq \sin^2(c+d)\cdot\frac{\sin c}{\sin d} \qquad \text{and,}\\ \sin(b-d)\,\sin(a+d) &\geq \sin^2(a+b)\cdot\frac{\sin d}{\sin c}.\end{aligned}$$

Recall that $a+b+c+d = 180^\circ$ and therefore $\sin(a+d) = \sin(b+c)$ and $\sin(a+b) = \sin(c+d)$. Adding the above inequalities term by term we obtain

$$\begin{aligned}&\sin(b+c)\cdot(\sin(a-c)+\sin(b-d)) \geq \sin^2(c+d)\cdot\left(\frac{\sin c}{\sin d}+\frac{\sin c}{\sin d}\right)\\ \Rightarrow\ &\sin(b+c)\cdot 2\cdot\sin(90^\circ-c-d)\cdot\cos(90^\circ-b-d) \geq \sin^2(c+d)\cdot 2\\ \Leftrightarrow\ &\sin(b+c)\cdot\sin(c+d)\cdot\cos(c+d) \geq \sin^2(c+d)\\ \Rightarrow\ &\cos(c+d) \geq 1-\cos^2(c+d)\\ \Rightarrow\ &\cos(c+d) = \cos(\angle B) \geq \frac{\sqrt{5}-1}{2}, \text{contradiction.}\end{aligned}$$

This completes the proof. Notice that the result is the best possible in the sense that $\arccos\frac{\sqrt{5}-1}{2} \approx 51.83^\circ$ cannot be replaced by a smaller value. Indeed, it is easy to check that a cyclic quad whose angles are $\arccos\frac{\sqrt{5}-1}{2}$, 90°, 90° and $180^\circ - \arccos\frac{\sqrt{5}-1}{2}$ does not have a 3-by-1 grid partition into cyclic quadrilaterals. $\square$

The following result can be obtained as a corollary of Theorem 6.

Theorem 7. (A class of cyclic quadrilaterals which have 3-by-3 grid dissections) *Let $ABCD$ be a cyclic quadrilateral such that the measure of each of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ determined by the vertices on the circumcircle is greater than 60°. Then $ABCD$ admits a 3-by-3 grid dissection into nine cyclic quadrilaterals.*

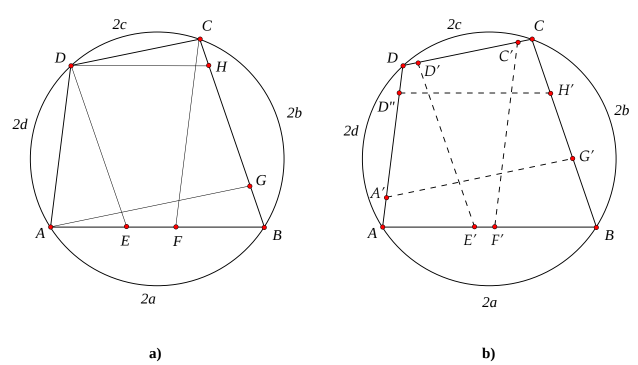

Figure 10. a) $DE \parallel BC$, $CF \parallel AD$, $AG \parallel CD$, $DH \parallel AB$
b) 3-by-3 grid dissection into nine cyclic quadrilaterals.

Proof. Notice that the condition regarding the arc measures is stronger than the requirement that all angles of $ABCD$ exceed $60°$. We will use the same assumptions and notations as in Theorem 6. The idea is to overlay the two constructions in Theorem 6 (see Figure 10).

It is straightforward to check that each of the nine quadrilaterals shown in figure 10 b) is cyclic. The problem reduces to proving that $\min\{a,\, b,\, c,\, d\} > 30°$ implies that both inequalities in (4) and (5) hold simultaneously. Due to symmetry it is sufficient to prove that (5) holds. Indeed,

$$
\begin{aligned}
BG + CH < BC &\Leftrightarrow\ \sin(b-d)\,\sin(b+c)\,\sin c < \sin^2(c+d)\,\sin d\\
&\Leftrightarrow\ \cos(c+d) - \cos(2b+c-d) < \frac{2\,\sin^2(c+d)\,\sin d}{\sin c}\\
&\Leftarrow\ \cos(c+d) + 1 < \frac{2\sin^2(c+d)\sin d}{\sin c}\\
&\Leftrightarrow\ 2\cos^2\frac{c+d}{2}\sin c < 8\sin^2\frac{c+d}{2}\cos^2\frac{c+d}{2}\sin d\\
&\Leftrightarrow\ \sin c < 2\,\sin d\cdot(1-\cos(c+d))\\
&\Leftrightarrow\ \sin c < 2\,\sin d - 2\,\sin d\,\cos(c+d)\\
&\Leftrightarrow\ \sin c < 2\,\sin d + \sin c - \sin(c+2d))\\
&\Leftrightarrow\ \sin(c+2d) < 2\sin d\\
&\Leftarrow\ 1 < 2\sin d.
\end{aligned}
$$

The last inequality holds true since we assumed $d > 30°$. This completes the proof.

□

3.2. *Class Preserving Grid Dissections of Orthodiagonal Quadrilaterals.* It is easy to see that an orthodiagonal quadrilateral cannot have a 2-by-1 grid dissection into orthodiagonal quadrilaterals. Indeed, if say we attempt to dissect the quadrilateral $ABCD$ with a segment MP, where M is on AB and P is on CD, then the diagonals of $ADPM$ are forced to intersect in the interior of the right triangle AOD, preventing them from being perpendicular to each other (see Figure 11 a).

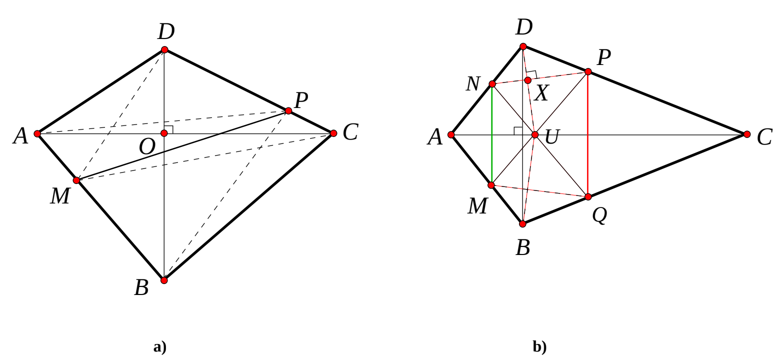

Figure 11. a) Orthodiagonal quadrilaterals have no 2-by-1 CPG dissections
b) A kite admits infinitely many 2-by-2 CPG dissections

The similar question concerning the existence of 2-by-2 CPG dissections turns out to be more difficult. We propose the following:

Conjecture 1. *An orthodiagonal quadrilateral has a* 2*-by-*2 *grid dissection into four orthodiagonal quadrilaterals if and only if it is a kite.*

The "only if" implication is easy to prove. We can show that every kite has infinitely many 2-by-2 CPG grid dissections. Indeed, let $ABCD$ be a kite ($AB = AD$ and $BC = CD$) and let $MN \parallel BD$ with M and N fixed points on sides AB and respectively AD. Consider then a variable segment $PQ \parallel BD$ as shown in figure 11 b). Denote $U = NQ \cap MP$; due to symmetry $U \in AC$. Consider the grid dissection generated by segments MP and NQ. Notice that quadrilaterals $ANUM$ and $CPUQ$ are orthodiagonal independent of the position of PQ. Also, quadrilaterals $DNUP$ and $BMUQ$ are congruent and therefore it is sufficient to have one of them be orthodiagonal.

Let point P slide along CD. If P is close to vertex C, it follows that Q and U are also close to C and therefore the measure of angle $\angle DXN$ is arbitrarily close to the measure of $\angle DCN$, which is acute. On the other hand, when P is close to vertex D, Q is close to B and the angle $\angle DXN$ becomes obtuse.

Since the measure of $\angle DXN$ depends continuously on the position of point P it follows that for some intermediate position of P on CD we will have $\angle DXN = 90^\circ$. For this particular choice of P both $DNUP$ and $BMUQ$ are orthodiagonal. This proves the "only if" part of the conjecture.

Extensive experimentation with Geometer's Sketchpad strongly suggests the direct statement also holds true. We used MAPLE to verify the conjecture in several particular cases - for instance, the isosceles orthodiagonal trapezoid with base lengths of 1 and $\sqrt{7}$ and side lengths 2 does not admit a 2-by-2 dissection into orthodiagonal quadrilaterals.

3.3. *Class Preserving Grid Dissections of Circumscribed Quadrilaterals.* After the mostly negative results from the previous sections, we discovered the following surprising result.

Theorem 8. *Every circumscribed quadrilateral has a* 2*-by-*2 *grid dissection into four circumscribed quadrilaterals.*

Proof. (Sketch) This is in our opinion a really unexpected result. It appears to be new and the proof required significant amounts of inspiration and persistence. We approached the problem analytically and used MAPLE extensively to perform the symbolic computations. Still, the problem presented great challenges, as we will describe below.

Let $MNPQ$ be a circumscribed quadrilateral with incenter O. With no loss of generality suppose the incircle has unit radius. Let O_i, $1 \leq i \leq 4$ denote projections of O onto the sides as shown in Figure 12 a). Denote the angles $\angle O_4OO_1 = 2a$, $\angle O_1OO_2 = 2b$, $\angle O_2OO_3 = 2c$ and $\angle O_3OO_4 = 2d$. Clearly, $a + b + c + d = 180^\circ$ and $\max\{a, b, c, d\} < 90^\circ$. Consider a coordinate system centered at O such that the coordinates of O_4 are $(1, 0)$.

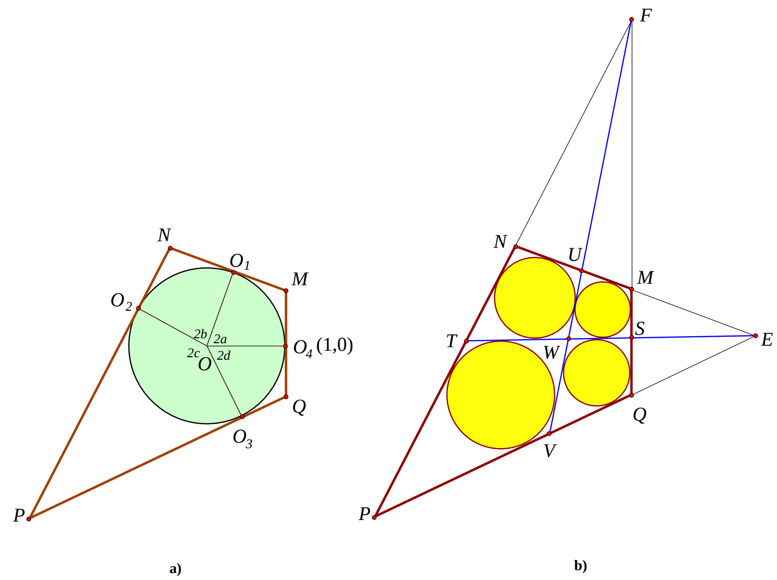

Figure 12. a) A circumscribed quadrilateral
b) Attempting a 2-by-2 CPG dissection with lines through E and F

We introduce some more notation: $\tan a = A$, $\tan b = B$, $\tan c = C$, $\tan d = D$. Notice that quantities A, B, C and D are not independent. Since $a + b + c + d = 180°$ it follows that $A + B + C + D = ABC + ABD + ACD + BCD$. Moreover, since $\max\{a, b, c, d\} < 90°$ we have that A, B, C and D are all positive.

It is now straightforward to express the coordinates of the vertices M, N, P and Q in terms of the tangent values A, B, C and D. Two of these vertices have simple coordinates: $M(1, A)$ and $Q(1, -D)$. The other two are

$$N\left(\frac{1 - A^2 - 2AB}{1 + A^2}, \frac{2A + B - A^2B}{1 + A^2}\right) \quad \text{and} \quad P\left(\frac{1 - D^2 - 2CD}{1 + D^2}, \frac{C + 2D - CD^2}{1 + D^2}\right).$$

The crux of the proof lies in the following idea. Normally, we would look for four points (one on each side), which create the desired 2-by-2 grid partition. We would thus have four degrees of freedom (choosing the points) and four equations (the conditions that each of the smaller quadrilaterals formed is circumscribed).

However, the resulting algebraic system is extremely complicated. Trying to eliminate the unknowns one at a time leads to huge resultants which even MAPLE cannot handle.

Instead, we worked around this difficulty. Extend the sides of $MNPQ$ until they intersect at points E and F as shown in Figure 12 b). (Ignore the case when $MNPQ$ is a trapezoid for now). Now locate a point U on side MN and a point S on side QM such that when segments FU and ES are extended as in the figure, the four resulting quadrilaterals are all circumscribed. This reduces the number of variables from four to two and thus the system appears to be over-determined. However, extended investigations with Geometer's Sketchpad indicated that this construction is possible. At this point we start computing the coordinates of the newly introduced points. We have

$$E\left(\frac{1+AD}{1-AD}, \frac{A-D}{1-AD}\right) \quad \text{and} \quad F\left(1, \frac{A+B}{1-AB}\right).$$

Denote

$$m = \frac{MU}{MN} \qquad \text{and} \qquad q = \frac{QS}{QM}.$$

Clearly, the coordinates of U and V are rational functions on m, A, B, C and D while the coordinates of S and T depend in a similar manner on q, A, B, C and D. These expressions are quite complicated; for instance, each one of the coordinates of point T takes five full lines of MAPLE output. The situation is the same for the coordinates of point V.

Define the following quantities:

$$\begin{aligned} Z_1 &= MU + WS - WU - MS, \\ Z_2 &= NT + WU - WT - NU, \\ Z_3 &= PV + WT - WV - PT, \\ Z_4 &= QS + WV - WS - QV. \end{aligned}$$

By Theorem 2 b), a necessary and sufficient condition for the quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ to be cyclic is that $Z_1 = Z_2 = Z_3 = Z_4 = 0$.

Notice that

$$Z_1 + Z_2 + Z_3 + Z_4 = MU - NU + NT - PT + PV - QV + QS - MS \tag{6}$$

and

$$\begin{aligned} Z_1 - Z_2 + Z_3 - Z_4 &= MN - NP + PQ - QM + 2(WS + WT - WU - WV) \\ &= 2(ST - UV), \end{aligned} \tag{7}$$

the last equality is due to the fact that $MNPQ$ is circumscribed.

Since we want $Z_i = 0$ for every $1 \leq i \leq 4$, we need to have the right hand terms from (6) and (7) each equal to 0. In other words, **necessary** conditions for finding the desired grid dissection are

$$MU - NU + PV - QV = PT - NT + MS - QS \qquad \text{and} \qquad UV = ST. \tag{8}$$

There is a two-fold advantage we gain by reducing the number of equations from four to two: first, the system is significantly simpler and second, we avoid using point W - the common vertex of all four small quadrilaterals which is also the point with the most complicated coordinates.

System (8) has two equations and two unknowns - m and q - and it is small enough for MAPLE to handle. Still, after eliminating variable q, the resultant is a polynomial of degree 10 of m with polynomial functions of A, B, C and D as coefficients.

This polynomial can be factored and the value of m we are interested in is a root of a quadratic. Although m does not have a rational expression depending on A, B, C and D it can still be written in terms of $\sqrt{\sin a}$, $\sqrt{\cos a}$, $\ldots$, $\sqrt{\sin d}$, $\sqrt{\cos d}$.

Explicit Formulation of Theorem 8. Let $MNPQ$ be a circumscribed quadrilateral as described in figure 12. Denote

$$\begin{array}{llll} s_1=\sqrt{\sin a}, & s_2=\sqrt{\sin b}, & s_3=\sqrt{\sin c}, & s_4=\sqrt{\sin d} \\ c_1=\sqrt{\cos a}, & c_2=\sqrt{\cos b}, & c_3=\sqrt{\cos c}, & c_4=\sqrt{\cos d}. \end{array}$$

Define points $U\in MN$, $T\in NP$, $V\in TQ$ and $S\in QM$ such that

$$\frac{MU}{MN}=m, \quad \frac{NT}{NP}=n, \quad \frac{PV}{PQ}=p, \quad \frac{QS}{QM}=q$$

where

$$m=\frac{s_2s_4c_2^2(s_4s_1(s_1^2c_2^2+s_2^2c_1^2)+s_2s_3)}{(s_1^2c_2^2+s_2^2c_1^2)(s_1s_2s_4+s_3c_1^2)(s_1s_2s_3+s_4c_2^2)}, \tag{9}$$

$$n=\frac{s_3s_1c_3^2(s_1s_2(s_2^2c_3^2+s_3^2c_2^2)+s_3s_4)}{(s_2^2c_3^2+s_3^2c_2^2)(s_2s_3s_1+s_4c_2^2)(s_2s_3s_4+s_1c_3^2)}, \tag{10}$$

$$p=\frac{s_4s_2c_4^2(s_2s_3(s_3^2c_4^2+s_4^2c_3^2)+s_4s_1)}{(s_3^2c_4^2+s_4^2c_3^2)(s_3s_4s_2+s_1c_3^2)(s_3s_4s_1+s_2c_4^2)}, \tag{11}$$

$$q=\frac{s_1s_3c_1^2(s_3s_4(s_4^2c_1^2+s_1^2c_4^2)+s_1s_2)}{(s_4^2c_1^2+s_1^2c_4^2)(s_4s_1s_3+s_2c_4^2)(s_4s_1s_2+s_3c_1^2)}. \tag{12}$$

Denote $W=ST\cap UV$. Then, quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ are all circumscribed (*i.e.*, $Z_1=Z_2=Z_3=Z_4=0$).
Verifying these assertions was done in MAPLE. Recall that m and q were obtained as solutions of the system $Z_1+Z_2+Z_3+Z_4=0$, $Z_1-Z_2+Z_3-Z_4=0$. At this point it is not clear why for these choices of m, n, p and q we actually have $Z_i=0$, for all $1\le i\le 4$.

Using the expressions of m, n, p and q given above, we can write the coordinates of all points that appear in figure 12 in terms of s_i and c_i where $1\le i\le 4$. We can then calculate the lengths of all the twelve segments which appear as sides of the smaller quadrilaterals.

For instance we obtain:

$$MU=\frac{(s_1^3s_4c_2^2+s_1s_2^2s_4c_1^2+s_2s_3)s_2s_4}{c_1^2(s_1s_2s_3+c_2^2s_4)(s_1s_2s_4+c_1^2s_3)},$$

$$NU=\frac{(s_2^3s_3c_1^2+s_1^2s_2s_3c_2^2+s_1s_4)s_1s_3}{c_2^2(s_1s_2s_3+c_2^2s_4)(s_1s_2s_4+c_1^2s_3)}.$$

and similar relations can be written for NT, PV, QS and PT, QV, MS by circular permutations of the expressions for MU and NU, respectively.

In the same way it can be verified that

$$\begin{aligned} &UV=ST \\ &=\frac{(s_1^2c_4^2+s_4^2c_1^2)(s_1^2s_4^2+s_3^2s_2^2+2s_1s_2s_3s_4(s_1^2c_2^2+s_2^2c_1^2))(s_3^2c_1^2+s_1^2c_3^2+2s_1s_2s_3s_4)}{(s_1s_2s_3+c_2^2s_4)(s_2s_3s_4+c_3^2s_1)(s_3s_4s_1+c_4^2s_2)(s_1s_2s_4+c_1^2s_3)} \end{aligned}$$

and

$$UW = \frac{\lambda_r \cdot UV}{\lambda_r + \mu_r}, \quad VW = \frac{\mu_r \cdot UV}{\lambda_r + \mu_r}, \quad SW = \frac{\lambda_s \cdot ST}{\lambda_s + \mu_s}, \quad TW = \frac{\mu_s \cdot ST}{\lambda_s + \mu_s},$$

where

$$\begin{aligned}
\lambda_r &= (s_1^3 s_2 c_4^2 + s_1 s_2 s_4^2 c_1^2 + s_3 s_4)(s_2 s_3 s_4 + s_1 c_3^2)(s_3 s_4 s_1 + s_2 c_4^2)\\
\mu_r &= (s_3^3 s_4 c_2^2 + s_3 s_4 s_2^2 c_3^2 + s_1 s_2)(s_4 s_1 s_2 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)\\
\lambda_s &= (s_4^3 s_1 c_3^2 + s_4 s_1 s_3^2 c_4^2 + s_2 s_3)(s_2 s_3 s_4 + s_1 c_3^2)(s_1 s_2 s_3 + s_4 c_2^2)\\
\mu_s &= (s_2^3 s_3 c_1^2 + s_2 s_3 s_1^2 c_2^2 + s_4 s_1)(s_4 s_1 s_2 + s_3 c_1^2)(s_3 s_4 s_1 + s_2 c_4^2).
\end{aligned}$$

Still, verifying that $Z_i = 0$ is not as simple as it may seem. The reason is that the quantities s_i and c_i are not independent. For instance we have $s_i^4 + c_i^4 = 1$, for all $1 \le i \le 4$. Also, since $a+b+c+d = 180°$ we have $\sin(a+b) = \sin(c+d)$ which translates to $s_1^2\,c_2^2 + s_2^2\,c_1^2 = s_3^2\,c_4^2 + s_4^2\,c_3^2$. Similarly, $\cos(a+b) = -\cos(c+d)$ which means $c_1^2\,c_2^2 - s_1^2\,s_2^2 = s_3^2\,s_4^2 - c_3^2\,c_4^2$. There are $4+3+3 = 10$ such side relations which have to be used to prove that two expressions which look different are in fact equal. MAPLE cannot do this directly.

For example, it is not at all obvious that the expressions of m, n, p and q defined above represent numbers from the interval $(0, 1)$. Since each expression is obtained via circular permutations from the preceding one it is enough to look at m.

Clearly, since $s_i > 0$ and $c_i > 0$ for all $1 \le i \le 4$ we have that $m > 0$. On the other hand, using the side relations we mentioned above we get that

$$1 - m = \frac{s_3 s_1 c_1^2(s_2 s_3(s_1^2 c_2^2 + s_2^2 c_1^2) + s_1 s_4)}{(s_1^2 c_2^2 + s_2^2 c_1^2)(s_1 s_2 s_4 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)}.$$

Obviously, $1 - m > 0$ and therefore $0 < m < 1$.

As previously eluded the construction works in the case when $MNPQ$ is a trapezoid as well. In this case if $MN \parallel PQ$ then $UV \parallel MN$ too. In conclusion, it is quite tricky to check that the values of m, n, p and q given by equalities (9) - (12) imply that $Z_1 = Z_2 = Z_3 = Z_4 = 0$. The MAPLE file containing the complete verification of theorem 8 is about 15 pages long. On request, we would be happy to provide a copy. □

4. Conclusions and Directions of Future Research

In this paper we mainly investigated what types of geometric properties can be preserved when dissecting a convex quadrilateral. The original contributions are contained in section 3 in which we dealt exclusively with grid dissections. There are many very interesting questions which are left unanswered.

1. The results from Theorems 6 and 7 suggest that if a cyclic quadrilateral $ABCD$ has an m-by-n grid dissection into cyclic quadrilaterals with $m \cdot n$ a large odd integer, then $ABCD$ has to be “close” to a rectangle. It would be desirable to quantify this relationship.

2. Conjecture 1 implies that orthodiagonal quadrilaterals are "bad" when it comes to class preserving dissections. On the other hand, theorem 8 proves that circumscribed quadrilaterals are very well behaved in this respect. Why does this happen? After all, the characterization Theorem 2 b) and 2 c) suggest that these two properties are not radically different.

More precisely, let us define an α-quadrilateral to be a convex quadrilateral $ABCD$ with $AB^\alpha + CD^\alpha = BC^\alpha + AD^\alpha$, where α is a real number. Notice that for $\alpha = 1$ we get the circumscribed quadrilaterals and for $\alpha = 2$ the orthodiagonal ones. In particular, a kite is an α-quadrilateral for all values of α. The natural question is:

Problem. For which values of α does every α-quadrilateral have a 2-by-2 grid dissection into α-quadrilaterals?

3. Theorem 8 provided a constructive method for finding a grid dissection of any circumscribed quadrilateral into smaller circumscribed quadrilaterals. Can this construction be extended to a 4-by-4 class preserving grid dissection? Notice that extending the opposite sides of each one of the four small cyclic quadrilaterals which appear in Figure 12 we obtain the same pair of points, E and F. It is therefore tempting to verify whether iterating the procedure used for $MNPQ$ for each of these smaller quads would lead to a 4-by-4 grid dissection of $MNPQ$ into 16 cyclic quadrilaterals. Maybe even a 2^n-by-2^n grid dissection is possible. If true, it is desirable to first find a simpler way of proving Theorem 8.

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Dan Ismailescu: Department of Mathematics, Hofstra University, Hempstead, New York 11549, USA

E-mail address: matdpi@hofstra.edu

Adam Vojdany: Paul D. Schreiber High School, 101 Campus Dr, Port Washington, New York 11050, USA

E-mail address: avojdany1@yahoo.com

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On the Construction of Regular Polygons and Generalized Napoleon Vertices

Dimitris Vartziotis and Joachim Wipper

Abstract. An algebraic foundation for the derivation of geometric construction schemes transforming arbitrary polygons with n vertices into k-regular n-gons is given. It is based on circulant polygon transformations and the associated eigenpolygon decompositions leading to the definition of generalized Napoleon vertices. Geometric construction schemes are derived exemplarily for different choices of n and k.

1. Introduction

Because of its geometric appeal, there is a long, ongoing tradition in discovering geometric constructions of regular polygons, not only in a direct way, but also by transforming a given polygon with the same number of vertices [2, 6, 9, 10]. In the case of the latter, well known results are, for example, Napoleon's theorem constructing an equilateral triangle by taking the centroids of equilateral triangles erected on each side of an arbitrary initial triangle [5], or the results of Petr, Douglas, and Neumann constructing k-regular n-gons by $n-2$ iteratively applied transformation steps based on taking the apices of similar triangles [8, 3, 7]. Results like these have been obtained, for example, by geometric creativity, target-oriented constructions or by analyzing specific configurations using harmonic analysis.

In this paper the authors give an algebraic foundation which can be used in order to systematically derive geometric construction schemes for k-regular n-gons. Such a scheme is hinted in Figure 1 depicting the construction of a 1-regular pentagon (left) and a 2-regular pentagon (right) starting from the same initial polygon marked yellow. New vertex positions are obtained by adding scaled parallels and perpendiculars of polygon sides and diagonals. This is indicated by intermediate construction vertices whereas auxiliary construction lines have been omitted for the sake of clarity.

The algebraic foundation is derived by analyzing circulant polygon transformations and the associated Fourier basis leading to the definition of eigenpolygons. By choosing the associated eigenvalues with respect to the desired symmetric configuration and determining the related circulant matrix, this leads to an algebraic representation of the transformed vertices with respect to the initial vertices and the eigenvalues. Interpreting this algebraic representation geometrically yields the desired construction scheme.

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The authors would like to thank Bernd Scholz from TWT GmbH, Engineering Department, for pointing out a modified triangle transformation based on the results of [11] which attracted our interest on deriving geometric construction schemes from circulant polygon transformations.

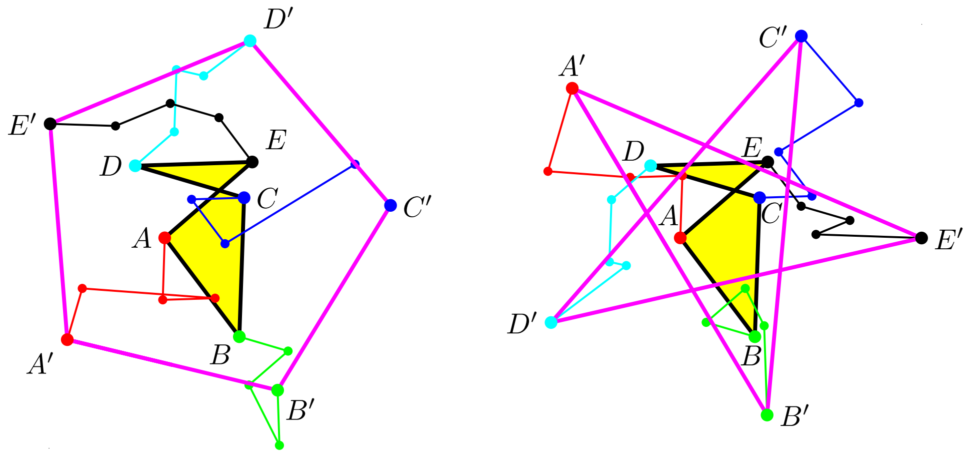

Figure 1. Construction of regular pentagons.

A special choice of parameters leads to a definition of generalized Napoleon vertices, which coincide with the vertices given by Napoleon's theorem in the case of $n = 3$. Geometric construction schemes based on such representations are derived for triangles, quadrilaterals, and pentagons.

2. Eigenpolygon decompositions

Let $z \in \mathbb{C}^n$ denote a polygon with n vertices z_k, $k \in \{0, \ldots, n-1\}$, in the complex plane using zero-based indexes. In order to obtain geometric constructions leading to regular polygons, linear transformations represented by complex circulant matrices $M \in \mathbb{C}^{n \times n}$ will be analyzed. That is, each row of M results from a cyclic shift of its preceding row, which reflects that new vertex positions are constructed in a similar fashion for all vertices.

The eigenvectors $f_k \in \mathbb{C}^n$, $k \in \{0, \ldots, n-1\}$, of circulant matrices are given by the columns of the Fourier matrix

$$F := \frac{1}{\sqrt{n}} \begin{pmatrix} r^{0 \cdot 0} & \ldots & r^{0 \cdot (n-1)} \\ \vdots & \ddots & \vdots \\ r^{(n-1) \cdot 0} & \ldots & r^{(n-1) \cdot (n-1)} \end{pmatrix},$$

where $r := \exp(2\pi \mathrm{i}/n)$ denotes the n-th complex root of unity [1]. Hence, the eigenvector $f_k = (1/\sqrt{n})(r^{0 \cdot k}, r^{1 \cdot k}, \ldots, r^{(n-1) \cdot k})^{\mathrm{t}}$ represents the k-th Fourier polygon obtained by successively connecting counterclockwise n times each k-th scaled root of unity starting by $r^0/\sqrt{n} = 1/\sqrt{n}$. This implies that f_k is a $\big(n/\gcd(n,k)\big)$-gon with vertex multiplicity $\gcd(n,k)$, where $\gcd(n,k)$ denotes the greatest common divisor of the two natural numbers n and k. In particular, f_0 degenerates to one vertex with multiplicity n, and f_1 as well as f_{n-1} are convex regular n-gons with opposite orientation. Due to its geometric configuration f_k is called *k-regular*, which will also be used in the case of similar polygons.

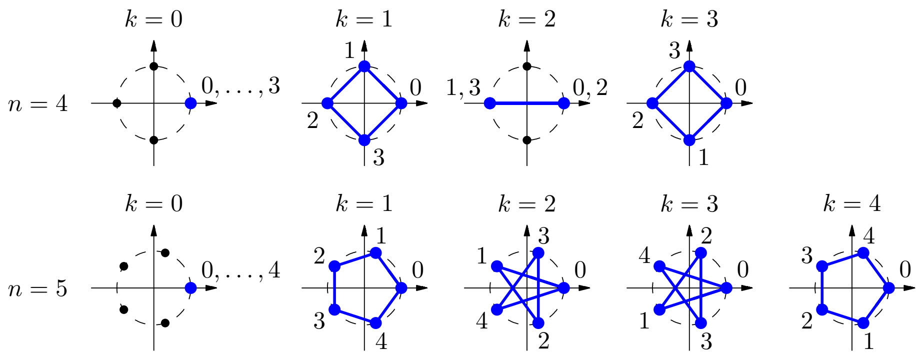

Figure 2. Fourier polygons f_k for $n \in \{4, 5\}$ and $k \in \{0, \dots, n-1\}$.

Examples of Fourier polygons are depicted in Figure 2. In this, black markers indicate the scaled roots of unity lying on a circle with radius $1/\sqrt{n}$, whereas blue markers denote the vertices of the associated Fourier polygons. Also given is the vertex index or, in the case of multiple vertices, a comma separated list of indexes. If n is a prime number, all Fourier polygons except for $k = 0$ are regular n-gons as is shown in the case of $n = 5$. Otherwise reduced Fourier polygons occur as is depicted for $n = 4$ and $k = 2$.

Since F is a unitary matrix, the diagonalization of M based on the eigenvalues $\eta_k \in \mathbb{C}$, $k \in \{0, \dots, n-1\}$, and the associated diagonal matrix $D = \operatorname{diag}(\eta_0, \dots, \eta_{n-1})$ is given by $M = FDF^*$, where F^* denotes the conjugate transpose of F. The coefficients c_k in the representation of $z = \sum_{k=0}^{n-1} c_k f_k$ in terms of the Fourier basis are the entries of the vector $c = F^* z$ and lead to the following definition.

Definition. The k-th *eigenpolygon* of a polygon $z \in \mathbb{C}^n$ is given by

$$e_k := c_k f_k = \frac{c_k}{\sqrt{n}} \left(r^{0 \cdot k}, r^{1 \cdot k}, \dots, r^{(n-1) \cdot k} \right)^{\mathrm{t}}, \tag{1}$$

where $c_k := (F^* z)_k$ and $k \in \{0, \dots, n-1\}$.

Since e_k is f_k times a complex coefficient c_k representing a scaling and rotation depending on z, the symmetric properties of the Fourier polygons f_k are preserved. In particular, the coefficient $c_0 = (F^* z)_0 = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} z_k$ implies that $e_0 = \frac{1}{n} \left(\sum_{k=0}^{n-1} z_k \right) (1, \dots, 1)^{\mathrm{t}}$ is n times the centroid of the initial polygon. This is also depicted in Figure 3 showing the eigenpolygon decomposition of two random polygons. In order to clarify the rotation and orientation of the eigenpolygons, the first three vertices are colored red, green, and blue.

Due to the representation of the transformed polygon

$$z' := Mz = M \left(\sum_{k=0}^{n-1} e_k \right) = \sum_{k=0}^{n-1} M e_k = \sum_{k=0}^{n-1} \eta_k e_k \tag{2}$$

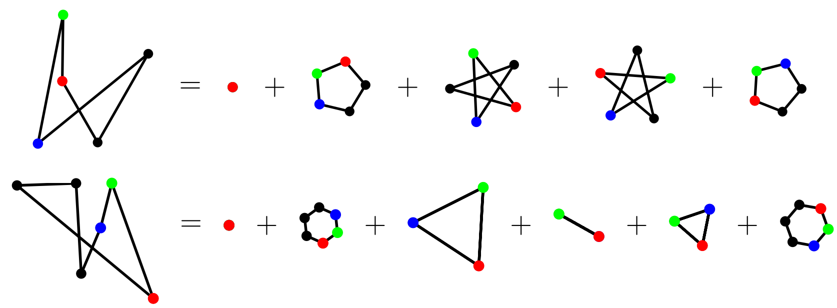

Figure 3. Eigenpolygon decomposition of a 5- and 6-gon.

applying the transformation M scales each eigenpolygon according to the associated eigenvalue $\eta_k \in \mathbb{C}$ of M. This is utilized by geometric construction schemes leading to scaled eigenpolygons. One is the Petr-Douglas-Neumann theorem [8, 3, 7] which is based on $n-2$ polygon transformations each consisting of taking the apices of similar isosceles triangles erected on the sides of the polygon. In each step a different apex angle taken from the set $\{k2\pi/n \,|\, k = 1, \dots, n-1\}$ is used. The characteristic angles are chosen in such a way that an eigenvalue in the decomposition (2) becomes zero in each case. Since all transformation steps preserve the centroid, $n-2$ steps successively eliminate the associated eigenpolygons until one scaled eigenpolygon with preserved centroid remains. In the case of $n = 3$ this leads to the familiar Napoleon's theorem [5] in which one transformation step suffices to obtain a regular triangle.

3. Construction of regular polygons

The eigenpolygon decomposition presented in the previous section can be used to prove that specific geometric transformations result in regular polygons. Beyond that, it can also be used to find new geometric construction schemes leading to predefined symmetric configurations. This is done by an appropriate choice of the eigenvalues η_k and by interpreting the resulting transformation matrix $M = FDF^*$ geometrically.

3.1. *General case.* In this subsection, a specific choice of eigenvalues will be analyzed in order to derive transformations, which lead to k-regular polygons and additionally preserve the centroid. The latter implies $\eta_0 = 1$ since e_0 already represents the centroid. By choosing $\eta_j = 0$ for all $j \in \{1, \dots, n-1\} \setminus \{k\}$ and $\eta_k \in \mathbb{C} \setminus \{0\}$, the transformation eliminates all eigenpolygons except the centroid e_0 and the designated eigenpolygon e_k which is scaled by the absolute value of η_k and rotated by the argument of η_k. This implies

$$\begin{aligned} M &= F \operatorname{diag}(1, 0, \dots, 0, \eta_k, 0, \dots, 0)\, F^* \\ &= F \operatorname{diag}(1, 0, \dots, 0)\, F^* + \eta_k F \operatorname{diag}(0, \dots, 0, 1, 0, \dots, 0)\, F^* . \end{aligned} \tag{3}$$

Hence, M is a linear combination of matrices of the type $E_k := F I_k F^*$, where I_k denotes a matrix with the only nonzero entry $(I_k)_{k,k} = 1$. Taking into account that $(F)_{\mu,\nu} = r^{\mu\nu}/\sqrt{n}$ and $(F^*)_{\mu,\nu} = r^{-\mu\nu}/\sqrt{n}$, the matrix $I_k F^*$ has nonzero elements only in its k-th row, where $(I_k F^*)_{k,\nu} = r^{-k\nu}/\sqrt{n}$. Therefore, the ν-th column of $E_k = F I_k F^*$ consists of the k-th column of F scaled by $r^{-k\nu}/\sqrt{n}$, thus resulting in $(E_k)_{\mu,\nu} = (F)_{\mu,k} r^{-k\nu}/\sqrt{n} = r^{\mu k} r^{-k\nu}/n = r^{k(\mu-\nu)}/n$. This yields the representation

$$(M)_{\mu,\nu} = (E_0)_{\mu,\nu} + \eta_k (E_k)_{\mu,\nu} = \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right) ,$$

since all entries of E_0 equal $1/n$. Hence, transforming an arbitrary polygon $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}}$ results in the polygon $z' = Mz$ with vertices

$$z'_\mu = (Mz)_\mu = \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right) z_\nu ,$$

where $\mu \in \{0, \ldots, n-1\}$. In the case of $\mu = \nu$ the weight of the associated summand is given by $\omega := (1 + \eta_k)/n$. Substituting this expression in the representation of z'_μ using $\eta_k = n\omega - 1$, hence $\omega \neq 1/n$, yields the decomposition

$$\begin{aligned} z'_\mu &= \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + (n\omega - 1) r^{k(\mu-\nu)}\right) z_\nu \\ &= \underbrace{\frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu}_{=:u_\mu} + \omega \underbrace{\sum_{\nu=0}^{n-1} r^{k(\mu-\nu)} z_\nu}_{=:v_\mu} = u_\mu + \omega v_\mu \end{aligned} \tag{4}$$

of z'_μ into a geometric location u_μ not depending on ω, and a complex number v_μ, which can be interpreted as vector scaled by the parameter ω. It should also be noticed that due to the substitution u_μ does not depend on z_μ, since the associated coefficient becomes zero.

A particular choice is $\omega = 0$, which leads to $z'_\mu = u_\mu$. As will be seen in the next section, in the case of $n = 3$ this results in the configuration given by Napoleon's theorem, hence motivating the following definition.

Definition. For $n \geq 3$ let $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}} \in \mathbb{C}^n$ denote an arbitrary polygon, and $k \in \{1, \ldots, n-1\}$. The vertices

$$u_\mu := \frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu , \qquad \mu \in \{0, \ldots, n-1\} ,$$

defining a k-regular n-gon are called *generalized Napoleon vertices*.

According to its construction, M acts like a filter on the polygon z removing all except the eigenpolygons e_0 and e_k. The transformation additionally weightens e_k by the eigenvalue $\eta_k \neq 0$. As a consequence, if e_k is not contained in the eigenpolygon decomposition of z, the resulting polygon $z' = Mz$ degenerates to the centroid e_0 of z.

The next step consists of giving a geometric interpretation of the algebraically derived entities u_0 and v_0 for specific choices of n, k, and ω resulting in geometric construction schemes to transform an arbitrary polygon into a k-regular polygon. Examples will be given in the next subsections.

3.2. *Transformation of triangles.* The general results obtained in the previous subsection will now be substantiated for the choice $n = 3$, $k = 1$. That is, a geometric construction is to be found, which transforms an arbitrary triangle into a counterclockwise oriented equilateral triangle with the same centroid. Due to the circulant structure, it suffices to derive a construction scheme for the first vertex of the polygon, which can be applied in a similar fashion to all other vertices.

In the case of $n = 3$ the root of unity is given by $r = \exp(\frac{2}{3}\pi \mathrm{i}) = \frac{1}{2}(-1+\mathrm{i}\sqrt{3})$. By using (4) in the case of $\mu = 0$, as well as the relations $r^{-1} = r^2 = \overline{r}$ and $r^{-2} = r$, this implies

$$\begin{aligned} u_0 &= \frac{1}{3}\sum_{\nu=0}^{2}(1-r^{-\nu})z_\nu = \frac{1}{3}\left[\left(\frac{3}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_1 + \left(\frac{3}{2}-\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_2\right] \\ &= \frac{1}{2}(z_1+z_2) - \mathrm{i}\,\frac{1}{3}\,\frac{\sqrt{3}}{2}(z_2-z_1) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \sum_{\nu=0}^{2} r^{-\nu} z_\nu = z_0 + \left(-\frac{1}{2}-\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_1 + \left(-\frac{1}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_2 \\ &= z_0 - \frac{1}{2}(z_1+z_2) + \mathrm{i}\,\frac{\sqrt{3}}{2}(z_2-z_1)\,. \end{aligned}$$

Thereby, the representations of u_0 and v_0 have been rearranged in order to give geometric interpretations as depicted in Figure 4.

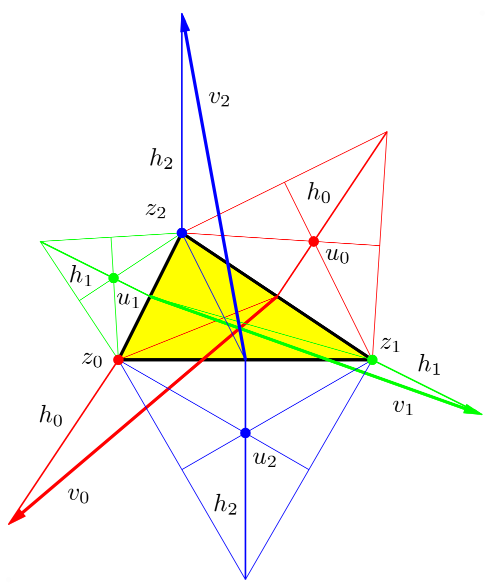

Figure 4. Napoleon vertices u_μ and directions v_μ in the case $n = 3$, $k = 1$.

Since multiplication by $-\mathrm{i}$ denotes a clockwise rotation by $\pi/2$, u_0 represents the centroid of a properly oriented equilateral triangle erected on the side z_1z_2. In a vectorial sense, v_0 represents the vector from the midpoint of the side z_1z_2 to z_0 added by the opposite directed height h_0 of the equilateral triangle erected on z_1z_2. Due to the circulant structure of M, the locations u_1, u_2 and the vectors v_1, v_2 can be constructed analogously. Using this geometric interpretation of the algebraically derived elements, the task is now to derive a construction scheme which combines the elements of the construction.

Algebraically, an obvious choice in the representation $z'_\mu = u_\mu + \omega v_\mu$ is $\omega = 0$, which leads to the familiar Napoleon configuration since in this case $z'_\mu = u_\mu$. Geometrically, an alternative construction is obtained by parallel translation of v_μ to u_μ. This is equivalent to the choice $\omega = 1$, hence $z'_0 = z_0 + \frac{\mathrm{i}}{\sqrt{3}}(z_2 - z_1)$. An according geometric construction scheme is depicted in Figure 5.

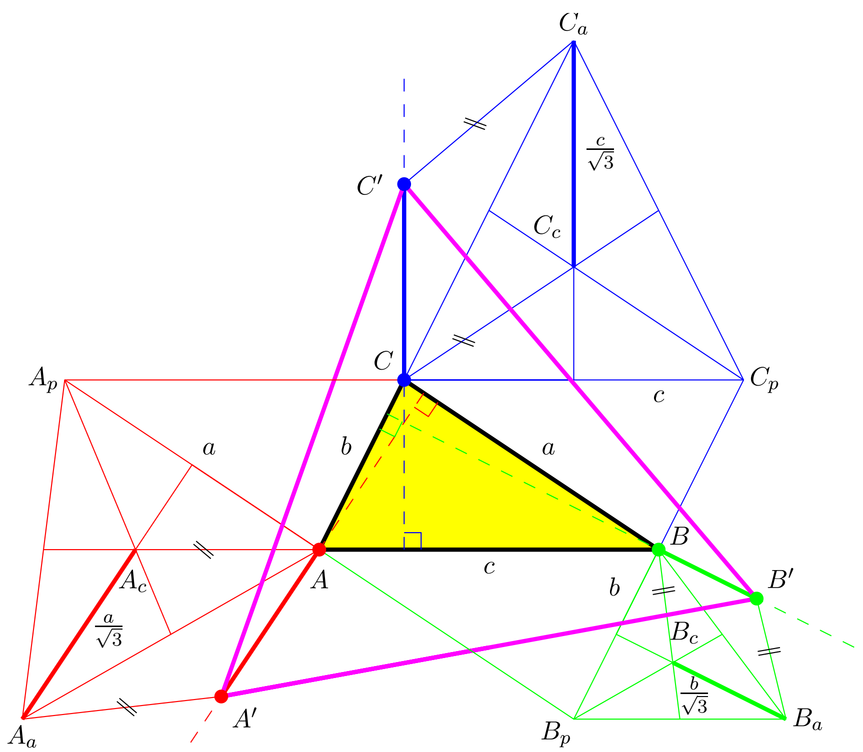

Figure 5. Construction of an equilateral triangle.

Thereby, the new position A' of A is derived as follows. First, the parallelogram $ABCA_p$ is constructed and an equilateral triangle is erected on AA_p. Since the distance from the centroid A_c to the apex A_a of this triangle is of the required length $a/\sqrt{3}$, where $a = |BC|$, one can transfer it by parallel translation to the vertical line on BC through A. The other vertices are constructed analogously as is also depicted in Figure 5.

According to the choice of parameters in the definition of M, the resulting triangle $A'B'C'$ is equilateral and oriented counterclockwise. A geometric proof is given by the fact that the triangle $A_pB_pC_p$ of the associated parallelogram vertices is similar to ABC with twice the side length. Due to their construction the new vertices A', B', and C' are the Napoleon vertices of $A_pB_pC_p$, hence $A'B'C'$ is equilateral. In particular, the midpoints of the sides of $A'B'C'$ yield the Napoleon triangle of ABC. Thus, A' can also be constructed by intersecting the line through A_p and A_c with the vertical line on BC through A.

3.3. *Transformation of quadrilaterals.* As a second example, the generalized Napoleon configuration in the case of $n = 4$, $k = 1$ is presented, that is $\omega = 0$ resulting in $z'_\mu = u_\mu$, $\mu \in \{0, \ldots, 3\}$. Using $r = \mathrm{i}$ and the representation (4) implies

$$\begin{aligned} u_0 &= \frac{1}{4}\sum_{\nu=0}^{3}(1 - r^{-\nu})z_\nu = \frac{1}{4}\Big((1+\mathrm{i})z_1 + (1+1)z_2 + (1-\mathrm{i})z_3\Big) \\ &= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{4}(z_3 - z_1) - \mathrm{i}\,\frac{1}{4}(z_3 - z_1)\,, \end{aligned}$$

which leads to the construction scheme depicted in Figure 6.

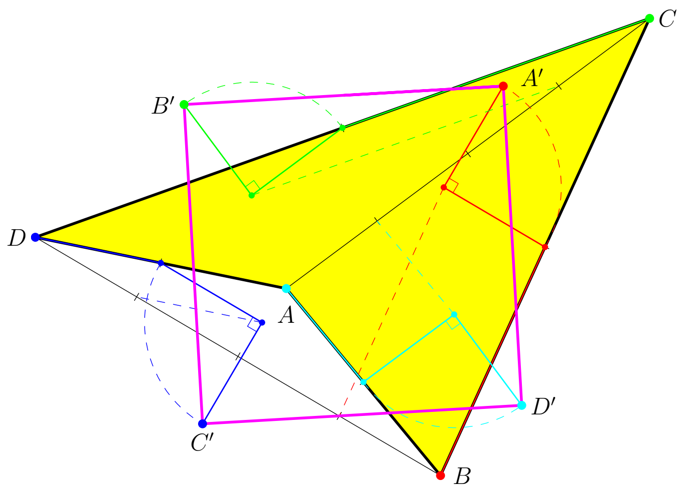

Figure 6. Construction of a regular quadrilateral.

As in the case of $n = 3$, the generalized Napoleon vertices can be constructed with the aid of scaled parallels and perpendiculars. Figure 6 depicts the intermediate vertices obtained by successively adding the summands given in the representation of u_μ from left to right. Parallels, as well as rotations by $\pi/2$ are marked by dashed lines. Diagonals, as well as subdivision markers are depicted by thin black lines.

3.4. *Transformation of pentagons.* In the case of $n = 5$, the root of unity is given by $r = (-1 + \sqrt{5})/4 + \mathrm{i}\sqrt{(5 + \sqrt{5})/8}$. The pentagon depicted on the left of Figure 1 has been transformed by using $k = 1$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = {} & z_0 + \frac{1}{\sqrt{5}}(z_1 - z_2) + \frac{1}{\sqrt{5}}(z_4 - z_3) \\ & - \mathrm{i}\,\frac{\sqrt{10+2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10-2\sqrt{5}}}{5}(z_2 - z_3)\,. \end{aligned}$$

Due to the choice $k = 1$, z' is a regular convex pentagon. The same initial polygon transformed by using $k = 2$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = {} & z_0 + \frac{1}{\sqrt{5}}(z_2 - z_1) + \frac{1}{\sqrt{5}}(z_3 - z_4) \\ & - \mathrm{i}\,\frac{\sqrt{10-2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10+2\sqrt{5}}}{5}(z_3 - z_2) \end{aligned}$$

is depicted on the right. Since $k = 2$ is not a divisor of $n = 5$, a star shaped nonconvex 2-regular polygon is constructed. Again, the representation also gives the intermediate constructed vertices based on scaled parallels and perpendiculars, which are marked by small markers. Thereby, auxiliary construction lines have been omitted in order to simplify the figure.

3.5. *Constructibility.* According to (4) the coefficients of the initial vertices z_μ in the representation of the new vertices z'_μ are given by $1 - r^{k(\mu-\nu)}$ and $\omega r^{k(\mu-\nu)}$ respectively. Using the polar form of the complex roots of unity, these involve the expressions $\cos(2\pi\xi/n)$ and $\sin(2\pi\xi/n)$, where $\xi \in \{0, \ldots, n-1\}$. Hence a compass and straightedge based construction scheme can only be derived if there exists a representation of these expressions and ω only using the constructible operations addition, subtraction, multiplication, division, complex conjugate, and square root.

Such representations are given exemplarily in the previous subsections for the cases $n \in \{3, 4, 5\}$. As is well known, Gauß proved in [4] that the regular polygon is constructible if n is a product of a power of two and any number of distinct Fermat prime numbers, that is numbers $F_m = 2^{(2^m)} + 1$ being prime. A proof of the necessity of this condition was given by Wantzel [13]. Thus, the first non constructible case using this scheme is given by $n = 7$. Nevertheless, there exists a neusis construction using a marked ruler to construct the associated regular heptagon.

4. Conclusion

A method of deriving construction schemes transforming arbitrary polygons into k-regular polygons has been presented. It is based on the theory of circulant matrices and the associated eigenpolygon decomposition. Following a converse approach, the polygon transformation matrix is defined by the choice of its eigenvalues representing the scaling and rotation parameters of the eigenpolygons. As has been shown for the special case of centroid preserving transformations leading to k-regular polygons, a general representation of the vertices of the new polygon can be derived in terms of the vertices of the initial polygon and an arbitrary transformation parameter ω. Furthermore, this leads to the definition of generalized Napoleon vertices, which are in the case of $n = 3$ identical to the vertices given by Napoleon's theorem.

In order to derive a new construction scheme, the number of vertices n and the regularity index k have to be chosen first. Since the remaining parameter ω has influence on the complexity of the geometric construction it should usually be chosen in order to minimize the number of construction steps. Finally giving a geometric interpretation of the algebraically derived representation of the new vertices is still a creative task. Examples for $n \in \{3, 4, 5\}$ demonstrate this procedure. Naturally, the problems in the construction of regular convex n-gons also apply in the presented scheme, since scaling factors of linear combinations of vertices have also to be constructible.

It is evident that construction schemes for arbitrary linear combinations of eigenpolygons leading to other symmetric configurations can be derived in a similar fashion. Furthermore, instead of setting specific eigenvalues to zero, causing the associated eigenpolygons to vanish, they could also be chosen in order to successively damp the associated eigenpolygons if the transformation is applied iteratively. This has been used by the authors to develop a new mesh smoothing scheme presented in [11, 12]. It is based on successively applying transformations to low quality mesh elements in order to regularize the polygonal element boundary iteratively. In this context transformations based on positive real valued eigenvalues are of particular interest, since they avoid the rotational effect known from other regularizing polygon transformations.

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Dimitris Vartziotis: NIKI Ltd. Digital Engineering, Research Center, 205 Ethnikis Antistasis Street, 45500 Katsika, Ioannina, Greece
E-mail address: dimitris.vartziotis@nikitec.gr

Dimitris Vartziotis and Joachim Wipper: TWT GmbH Science & Innovation, Research Department, Bernhäuser Straße 40–42, 73765 Neuhausen, Germany

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A Simple Barycentric Coordinates Formula

Nikolaos Dergiades

Abstract. We establish a simple formula for the barycentric coordinates with respect to a given triangle ABC of a point P specified by the oriented angles BPC, CPA and APB. Several applications are given.

We establish a simple formula for the homogeneous barycentric coordinates of a point with respect to a given triangle.

Theorem 1. *With reference to a given a triangle ABC, a point P specified by the oriented angles*

$$x = \measuredangle BPC, \quad y = \measuredangle CPA, \quad z = \measuredangle APB,$$

has homogeneous barycentric coordinates

$$\left(\frac{1}{\cot A - \cot x} : \frac{1}{\cot B - \cot y} : \frac{1}{\cot C - \cot z}\right). \tag{1}$$

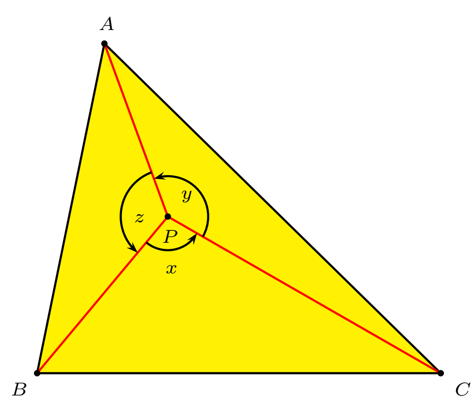

Figure 1.

Proof. Construct the circle through B, P, C, and let it intersect the line AP at A' (see Figure 2). Clearly, $\angle A'BC = \angle A'PC = \pi - \angle CPA = \pi - y$ and similarly, $\angle A'CB = \pi - z$. It follows from Conway's formula [5, §3.4.2] that in barycentric coordinates

$$A' = (-a^2 : S_C + S_{\pi-z} : S_B + S_{\pi-y}) = (-a^2 : S_C - S_z : S_B - S_y).$$

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Similarly, the lines BP intersects the circle CPA at a point B', and CP intersects the circle APB at C' whose coordinates can be easily written down. These be reorganized as

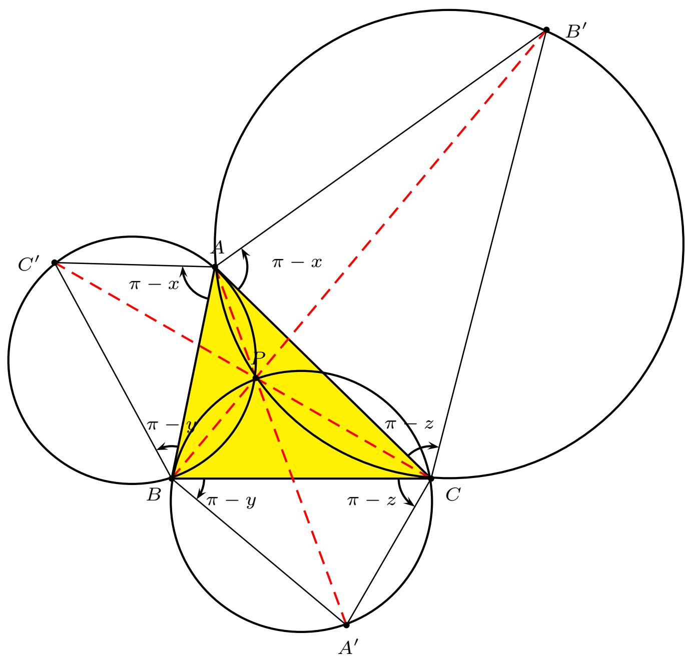

Figure 2.

$$A' = \left(-\frac{a^2}{(S_B - S_y)(S_C - S_z)} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right),$$
$$B' = \left(\frac{1}{S_A - S_x} : -\frac{b^2}{(S_C - S_z)(S_A - S_x)} : \frac{1}{S_C - S_z}\right),$$
$$C' = \left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : -\frac{c^2}{(S_A - S_x)(S_B - S_y)}\right).$$

According the version of Ceva's theorem given in [5, §3.2.1], the lines AA', BB', CC' intersect at a point, which is clearly P, whose coordinates are

$$\left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right).$$

Since by definition $S_\theta = S \cdot \cot\theta$, this formula is clearly equivalent to (1). □

Remark. This note is a revision of [1]. Antreas Hatzipolakis has subsequently given a traditional trigonometric proof [3].

The usefulness of formula (1) is that it is invariant when we substitute x, y, z by directed angles.

Corollary 2 (Schaal)**.** *If for three points A', B', C' the directed angles $x = (A'B,\ A'C)$, $y = (B'C,\ B'A)$ and $z = (C'A,\ C'B)$ satisfy $x + y + z \equiv 0 \bmod \pi$, then the circumcircles of triangles $A'BC$, $B'CA$, $C'AB$ are concurrent at P.*

Proof. Referring to Figure 2, if the circumcircles of triangles $A'BC$ and $B'CA$ intersect at P, then from concyclicity,

$$\begin{aligned}(PB,\ PC) &= \ (A'B,\ A'C) = x,\\ (PC,\ PA) &= \ (B'C,\ B'A) = y.\end{aligned}$$

It follows that

$$(PA,\ PB) = (PA,\ PC) + (PC,\ PB) = -y - x \equiv z = (C'A,\ C'B) \bmod \pi,$$

and C', A, B, P are concyclic. Now, it is obvious that the barycentrics of P are given by (1). □

For example, if the triangles $A'BC$, $B'CA$, $C'AB$ are equilateral on the exterior of triangle ABC, then $x = y = z = -\frac{\pi}{3}$, and $x + y + z \equiv 0 \bmod \pi$. By Corollary 2, we conclude that the circumcircles of these triangles are concurrent at

$$\begin{aligned}P &= \left(\frac{1}{\cot A - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot B - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot C - \cot\left(-\frac{\pi}{3}\right)}\right)\\ &= \left(\frac{1}{\cot A + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot B + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot C + \cot\left(-\frac{\pi}{3}\right)}\right).\end{aligned}$$

This is the first Fermat point, X_{13} of [4].

Corollary 3 (Hatzipolakis [2])**.** *Given a reference triangle ABC and two points P and Q, let R_a be the intersection of the reflections of the lines BP, CP in the lines BQ, CQ respectively* (see Figure 3). *Similarly define the points R_b and R_c. The circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at a point*

$$f(P, Q) = \left(\frac{1}{\cot A - \cot(2x' - x)} \ : \ \frac{1}{\cot B - \cot(2y' - y)} \ : \ \frac{1}{\cot C - \cot(2z' - z)}\right), \tag{2}$$

where

$$\begin{aligned}x &= (PB,\ PC), \qquad y = (PC,\ PA), \qquad z = (PA,\ PB);\\ x' &= (QB,\ QC), \qquad y' = (QC,\ QA), \qquad z' = (QA,\ QB).\end{aligned} \tag{3}$$

Proof. Let $x'' = (R_aB,\ R_aC)$. Note that

$$\begin{aligned}x'' &= \ (R_aB,\ QB) + (QB,\ QC) + (QC,\ R_aC)\\ &= \ (QB,\ QC) + (R_aB,\ QB) + (QC,\ R_aC)\\ &= \ (QB,\ QC) + (QB,\ PB) + (PC,\ QC)\\ &= \ (QB,\ QC) + (QB,\ QC) - (PB,\ PC)\\ &= \ 2x' - x.\end{aligned}$$

Similarly, $y'' = (R_bC,\ R_bA) = 2y' - y$ and $z'' = (R_cA,\ R_cB) = 2z' - z$. Hence,

$$x'' + y'' + z'' \equiv 2(x' + y' + z') - (x + y + z) \equiv 0 \bmod \pi.$$

By Corollary 2, the circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at the point $R = f(P, Q)$ given by (2). □

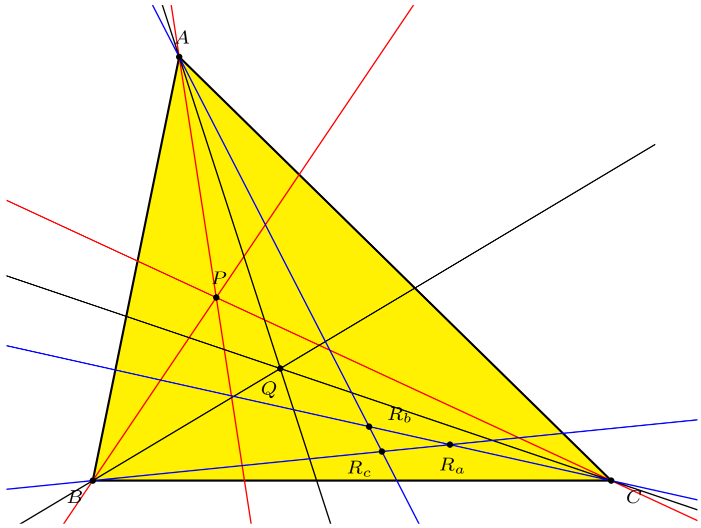

Figure 3.

Clearly, for the incenter I, $f(P, I) = P^*$, since $R_a = R_b = R_c = P^*$, the isogonal conjugate of P.

Corollary 4. *The mapping f preserves isogonal conjugation,* i.e.,

$$f^*(P, Q) = f(P^*, Q^*).$$

Proof. If the points P and Q are defined by the directed angles in (3), and $R = f(P, Q)$, $S = f(P^*,\ Q^*)$, then by Corollary 3, $(R^*B,\ R^*C) = A - (2x' - x)$ and

$$\begin{aligned}
(SB,\ SC) &\equiv\ 2(Q^*B,\ Q^*C) - (P^*B,\ P^*C) \\
&\equiv\ 2(A - x') - (A - x) \\
&\equiv\ A - (2x' - x) \\
&\equiv\ (R^*B,\ R^*C) \bmod \pi.
\end{aligned}$$

Similarly, $(SC, SA) \equiv (R^*C,\ R^*A)$ and $(SA,\ SB) \equiv (R^*A,\ R^*B) \bmod \pi$. Hence, $R^* = S$, or $f^*(P, Q) = f(P^*, Q^*)$. □

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Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

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Conic Homographies and Bitangent Pencils

Paris Pamfilos

Abstract. Conic homographies are homographies of the projective plane preserving a given conic. They are naturally associated with bitangent pencils of conics, which are pencils containing a double line. Here we study this connection and relate these pencils to various groups of homographies associated with a conic. A detailed analysis of the automorphisms of a given pencil specializes to the description of affinities preserving a conic. While the algebraic structure of the groups involved is simple, it seems that a geometric study of the various questions is lacking or has not been given much attention. In this respect the article reviews several well known results but also adds some new points of view and results, leading to a detailed description of the group of homographies preserving a bitangent pencil and, as a consequence, also the group of affinities preserving an affine conic.

1. Introduction

Deviating somewhat from the standard definition I call *bitangent* the pencils $\mathcal{P}$ of conics which are defined in the projective plane through equations of the form

$$\alpha c + \beta e^2 = 0.$$

Here $c(x, y, z) = 0$ and $e(x, y, z) = 0$ are the equations in *homogeneous coordinates* of a non-degenerate conic and a line and α, β are arbitrary, no simultaneously zero, real numbers. To be short I use the same symbol for the set and an equation representing it. Thus c denotes the set of points of a conic and $c = 0$ denotes an equation representing this set in some system of homogeneous coordinates. To denote bitangent pencils I use the letter $\mathcal{P}$ but also the more specific symbol (c, e). For any other member-conic c' of the pencil (c', e) represents the same pencil. I call line e and the pole E of e with respect to c respectively *invariant line* and *center* of the pencil. The intersection points $c \cap e$, if any, are called *fixed* or *base* points of the pencil. As is seen from the above equation, if such points exist, they lie on every member-conic of the pencil.

Traditionally the term *bitangent* is used only for pencils (c, e) for which line e either intersects c or is disjoint from it. This amounts to a second order (real or complex) contact between the members of the pencil, wherefore also the stem of the term. Pencils for which c and e^2 are tangent have a fourth order contact between their members and are classified under the name *superosculating* pencils

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([4, vol.II, p.188], [12, p.136]) or *penosculating* pencils ([9, p.268]). Here I take the liberty to incorporate this class of pencils into the bitangent ones, thus considering as a distinguished category the class of pencils which contain among their members a double line. This is done under the perspective of the tight relationship of conic homographies with bitangent pencils under this wider sense. An inspiring discussion in synthetic style on pencils of conics, which however, despite its wide extend, does not contain the relationship studied here, can be found in Steiner's lectures ([11, pp.224–430]).

Every homography[1] f of the plane preserving the conic c defines a bitangent pencil (c, e) to which conic c belongs as a member and to which f acts by preserving each and every member of the pencil. The pencil contains a double line e, which coincides with the *axis* of the homography. In this article I am mainly interested in the investigation of the geometric properties of four groups: $\mathcal{G}(c)$, $\mathcal{G}(c,e)$, $\mathcal{K}(c,e)$ and $\mathcal{A}(c)$, consisting respectively of homographies (i) preserving a conic c, (ii) preserving a pencil (c, e), (iii) permuting the members of a pencil, and (iv) preserving an affine conic. Last group is identical with a group of type $\mathcal{G}(c,e)$ in which line e is identified with the line at infinity. In Section 2 (*Conic homographies*) I review the well-known basic facts on homographies of conics stating them as propositions for easy reference. Their proofs can be found in the references given (especially [4, vol.II, Chapter 16], [12, Chapter VIII]). Section 3 (*Bitangent pencils*) is a short review on the classification of bitangent pencils. In Section 4 (*The isotropy at a point*) I examine the isotropy of actions of the groups referred above. In this, as well as in the subsequent sections, I supply the proofs of propositions for which I could not find a reference. Section 5 (*Automorphisms of pencils*) is dedicated to an analysis of the group $\mathcal{G}(c,e)$. Section 6 (*Bitangent flow*) comments on the vector-field point of view of a pencil and the characterization of its flow through a simple configuration on the invariant line. Section 7 (*The perspectivity group of a pencil*) contains a discussion on the group $\mathcal{K}(c,e)$ permuting the members of a pencil. Finally, Section 8 (*Conic affinities*) applies the results of the previous sections to the description of the group of affinities preserving an affine conic.

2. Conic homographies

Conic homographies are by definition restrictions on c of homographies of the plane that preserve a given conic c. One can define also such maps intrinsically, without considering their extension to the ambient plane. For this fix a point A on c and a line m and define the image $Y = f(X)$ of a point X by using its representation $f' = p \circ f \circ p^{-1}$ through the (stereographic) projection p of the conic onto line m centered at A.

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[1] I use this term coming from my native language (greek) as an alternative equivalent to *projectivity*.

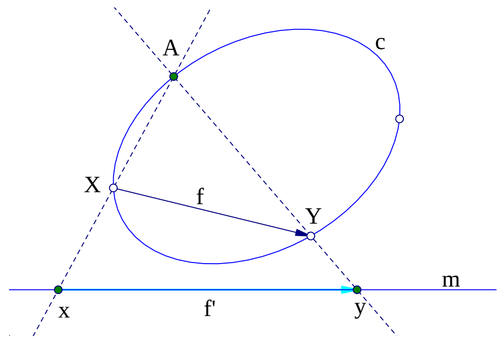

Figure 1. Conic homography

Homography f is the defined using a Moebius transformation ([10, p.40]) (see Figure 1)

$$y = f'(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

It can be shown ([4, vol.II, p.179]) that the two definitions are equivalent. Depending on the kind of the question one can prefer the first definition, through the restriction of a global homography, or the second through the projection. Later point of view implies the following ([10, p.47]).

Proposition 1. *A conic homography on the conic c is completely determined by giving three points A, B, C on the conic and their images A', B', C'. In particular, if a conic homography fixes three points on c it is the identity.*

The two ways to define conic homographies on a conic c reflect to the representation of their group $\mathcal{G}(c)$. In the first case, since every conic can be brought in appropriate homogeneous coordinates to the form ([2, p.209])

$$x^2 + y^2 - z^2 = 0$$

their group is represented through the group preserving this quadratic form which is $O(2,1)$. By describing homographies through Moebius transformations $\mathcal{G}(c)$ is represented with the group $PGL(2,\mathbb{R})$. The two representations are isomorphic but not *naturally* isomorphic ([4, vol.II, p.180]). An isomorphism between them can be established by fixing $A \in c$ and associating to each $f \in O(2,1)$ the corresponding induced in m transformation $f' \in PGL(2,\mathbb{R})$ ([15, p.235]), in the way this was defined above through the stereographic projection from A onto some line m (see Figure 1).

Next basic property of conic homographies is the existence of their *homography axis* ([4, vol.II, p.178]).

Proposition 2. *Given a conic homography f of the conic c, for every pair of points A, B on c, lines AB' and BA', with $A' = f(A), B' = f(B)$ intersect on a fixed line e, the homography axis of f. The fixed points of f, if any, are the intersection points of c and e.*

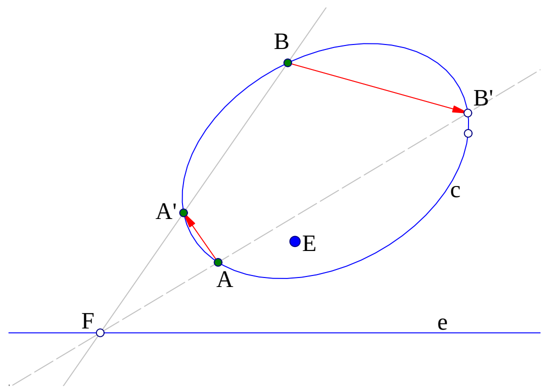

Figure 2. Homography axis e

Remark. This property implies (see Figure 2) an obvious geometric construction of the image B' of an arbitrary point B under the homography once we know the axis and a single point A and its image A' on the conic: Draw $A'B$ to find its intersection $F = A'B \cap e$ and from there draw line FA to find its intersection $B' = FA \cap c$.

Note that the existence of the axis is a consequence of the existence of at least a fixed point P for every homography f of the plane ([15, p.243]). If f preserves in addition a conic c, then it is easily shown that the polar e of P with respect to the conic must be invariant and coincides either with a tangent of the conic at a fixed point of f or coincides with the axis of f.

Next important property of conic homographies is the preservation of the whole bitangent family (c, e) generated by the conic c and the axis e of the homography. Here the viewpoint must be that of the restriction on c (and e) of a global homography of the plane.

Proposition 3. *Given a conic homography f of the conic c with homograpy axis e, the transformation f preserves every member $c' = \alpha c + \beta e^2$ of the pencil generated by c and the (double) line e. The pole E of the axis e with respect to c is a fixed point of the homography. It is also the pole of e with respect to every conic of the pencil. Line e is the axis of the conic homography induced by f on every member c' of the pencil.*

To prove the claims show first that line e is preserved by f (see Figure 3). For this take on c points A, B collinear with the pole E and consider their images $A' = f(A), B' = f(B)$. Since AB contains the pole of e, the pole Q of AB will be on line e. By Proposition 2 lines AB', BA' intersect at a point G of line e. It follows that the intersection point F of AA', BB' is also on e and that $A'B'$ passes through E. Hence the pole Q' of $A'B'$ will be on line e. Since homographies preserve polarity it must be $Q' = f(Q)$ and f preserves line e. From this follow easily all other claims of the proposition.

I call pencil $\mathcal{P} = (c, e)$ the *associated to f bitangent pencil*. I use also for E the name *center of the pencil* or/and *center of the conic homography f*.

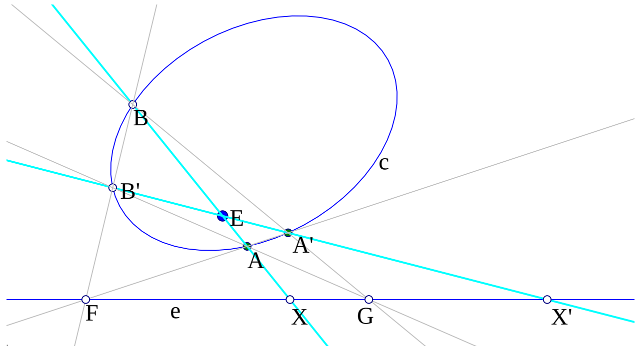

Figure 3. Invariance of axis e

Next deal with conic homographies is their distinction in *involutive* and non-involutive, i.e. homographies of period two and all others ([4, vol.II, p.179], [12, p.223]). Following proposition identifies involutive homographies with *harmonic homologies* (see Section 7) preserving a conic.

Proposition 4. *Every involutive conic homography f of the conic c fixes every point of its axis e. Inversely if it fixes its axis and $E \notin e$ it is involutive. Equivalently for each point $P \in c$ with $P' = f(P)$, line PP' passes through E the pole of the axis e of f. Point E is called in this case the center or Fregier point of the involution.*

Involutions are important because they can represent through their compositions every conic homography. The bitangent pencils (c, e) of interest, though, are those created by non-involutive conic homographies $f : c \to c$, and it will be seen that the automorphisms of such pencils consist of all homographies of the conic which commute with f. The following proposition clarifies the decomposition of every conic homography in two involutions ([4, vol.II, p.178], [12, p.224]).

Proposition 5. *Every conic homography f of a conic c can be represented as the product $f = I_2 \circ I_1$ of two involutions I_1, I_2. The centers of the involutions are necesserilly on the axis e of f. In addition the center of one of them may be any arbitrary point $P_1 \in e$ (not a fixed point of f), the center of the other $P_2 \in e$ is then uniquely determined.*

Following well known proposition signals also an important relation between a non-involutive conic homography and the associated to it bitangent pencil. I call the method suggested by this proposition the *tangential generation* of a non-involutive conic homography. It expresses for non-involutive homographies the counterpart of the property of involutive homographies to have all lines PP', with $(P \in c,\ P' = f(P))$, passing through a fixed point.

Proposition 6. *For every non-involutive conic homography f of a conic c and every point $P \in c$ and $P' = f(P)$ lines PP' envelope another conic c'. Conic c' is a member of the associated to f bitangent pencil. Inversely, given two member* *conics c, c' of a bitangent pencil the previous procedure defines a conic homography on c having its axis identical with the invariant line e of the bitangent pencil. Further the contact point Q' of line PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection point Q of PP' with the axis e of f.*

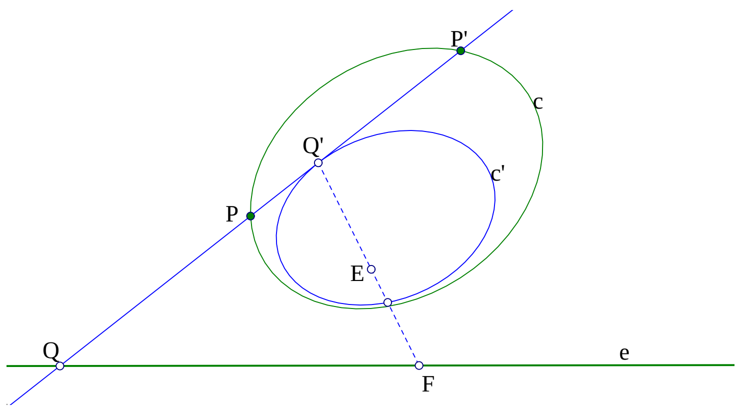

Figure 4. Tangential Generation

An elegant proof of these statements up to the last is implied by a proposition proved in [8, p.253], see also [4, vol.II, p.214] and [5, p.245]. Last statement follows from the fact that Q is the pole of line $Q'E$ (see Figure 4).

Propositions 5 and 6 allow a first description of the *automorphism group* $\mathcal{G}(c, e)$ of a given pencil (c, e) i.e. the group of homographies mapping every member-conic of the pencil onto itself. The group consists of homographies of two kinds. The first kind are the involutive homographies which are completely defined by giving their center on line e or their axis through E. The other homographies preserving the pencil are the non-involutive, which are compositions of pairs of involutions of the previous kind. Since we can put the center of one of the two involutions anywhere on e (except the intersection points of e and c), the homographies of this kind are parameterized by the location of their other center.

Before to look closer at these groups I digress for a short review of the classification of bitangent pencils and an associated naming convention for homographies.

3. Bitangent pencils

There are three cases of bitangent pencils in the real projective plane which are displayed in Figure 5. They are distinguished by the relative location of the invariant line and the conic generating the pencil.

Proposition 7. *Every bitangent pencil of conics is projectively equivalent to one generated by a fixed conic c and a fixed line e in one of the following three possible configurations.*

(I) *The line e non-intersecting the conic c (elliptic).*

(II) *The line e intersecting the conic c at two points (hyperbolic).*

(III) *The line e being tangent to the conic c (parabolic).*

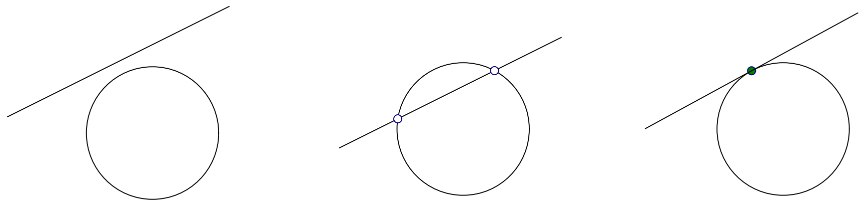

Figure 5. Bitangent pencils classification

The proof follows by reducing each case to a kind of *normal* form. For case (I) select a projective basis A, B, C making a *self-polar* triangle with respect to c. For this take A to be the pole of e with respect to c, take then B arbitrary on line e and define C to be the intersection of e and the polar p_B of B with respect to c. The triangle ABC thus defined is self-polar with respect to c and the equations of c and e take the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0, \quad x = 0.$$

In this we can assume that $\alpha > 0$, $\beta > 0$ and $\gamma < 0$. Applying then a simple projective transformation we reduce the equations in the form

$$x^2 + y^2 - z^2 = 0, \quad x = 0.$$

For case (II) one can define a projective basis A, B, C for which the equation of c and e take respectively the form

$$x^2 - yz = 0, \quad x = 0.$$

For this it suffices to take for A the intersection of the two tangents t_B, t_C to the conic at the intersections B, C of the line e with the conic c and the *unit* point of the basis on the conic. The projective equivalence of two such systems is obvious. Finally a system of type (III) can be reduced to one of type (II) by selecting again an appropriate projective base A, B, C. For this take B to be the contact point of the line and the conic. Take then A to be an arbitrary point on the conic and define C to be the intersection point of the tangents $t_A,\ t_B$. This reduces again the equations to the form ([4, vol.II, p.188])

$$xy - z^2 = 0, \quad x = 0.$$

The projective equivalence of two such *normal forms* is again obvious.

Remark. The disctinction of the three cases of bitangent pencils leads to a natural distinction of the non-involutive homographies in four general classes. The first class consists of homographies preserving a conic, such that the associated bitangent pencil is elliptic. It is natural to call these homographies *elliptic*. Analogously homographies preserving a conic and such that the associated bitangent pencil is hyperbolic or parabolic can be called respectively *hyperbolic* or *parabolic*. All other non-involutive homographies, not falling in one of these categories (i.e. not preserving a conic), could be called *loxodromic*. Simple arguments related to the set of fixed points of an homography show easily that the four classes are disjoint. In addition since, by Proposition 2, the fixed points of a homography f preserving a conic are its intersection points with the respective homography axis e, we see that the three classes of non-involutive homographies preserving a conic are characterized by the number of their fixed points on the conic ([12, p.101], [15, p 243]). This naming convention of the first three classes conforms also with the traditional naming of the corresponding kinds of real Moebius transformations induced on the invariant line of the associated pencil ([10, p.68]).

4. The isotropy at a point

Next proposition describes the structure of the isotropy group $\mathcal{G}_{AB}(c,e)$ for a hyperbolic pencil (c,e) at each one of the two intersection points $\{A,B\} = c \cap e$.

Proposition 8. *Every homography preserving both, a conic c, an intersecting the conic line e, and fixing one (A say) of the two intersection points A, B of c and e belongs to a group $\mathcal{G}_{AB}(c,e)$ of homographies, which is isomorphic to the multiplicative group $\mathbb{R}^*$ and can be parameterized by the points of the two disjoint arcs into which c is divided by A, B.*

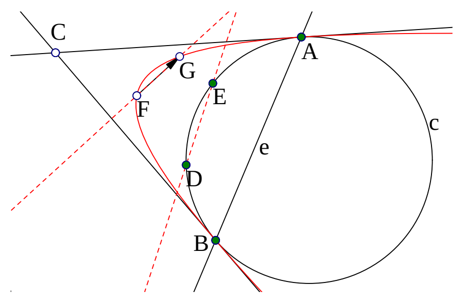

Figure 6. Isotropy of type IIb

Figure 6 illustrates the proof. Assume that homography f preserves both, the conic c, the line e, and also fixes A. Then it fixes also the other point B and also the pole C of line AB. Consequently f is uniquely determined by prescribing its value $f(D) = E \in c$ at a point $D \in c$. I denote this homography by f_{DE}. This map has a simple matrix representation in the projective basis $\{C, A, B, D\}$ in which conic c is represented by the equation $yz - x^2 = 0$ and line AB by $x = 0$, the unit point $D(1,1,1)$ being on the conic. In this basis and for $E \in c$ with coordinates (x,y,z) map f_{DE} is represented by non-zero multiples of the matrix

$$F_{DE} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

This representation shows that $\mathcal{G}_{AB}(c,e)$ is isomorphic to the multiplicative group $\mathbb{R}^*$ which has two connected components. The group $\mathcal{G}_{AB}(c,e)$ is the union of two cosets $\mathcal{G}_1, \mathcal{G}_2$ corresponding to the two arcs on c, defined by the two points

A, B. The arc containing point D corresponds to subgroup $\mathcal{G}_1$, coinciding with the connected component containing the identity. The other arc defined by AB corresponds to the other connected component $\mathcal{G}_2$ of the group. For points E on the same arc with D the corresponding homography f_{DE} preserves the two arcs defined by A, B, whereas for points E on the other arc than the one containing D the corresponding homography f interchanges the two arcs.

Obviously point D can be any point of c different from A and B. Selecting another place for D and varying E generates the same group of homographies. Clearly also there is a symmetry in the roles of A, B and the group can be identified with the group of homographies preserving conic c and fixing both points A and B.

Remark. Note that there is a unique involution I_0 contained in $\mathcal{G}_{AB}(c, e)$. It is the one having axis AB and center C, obtained for the position of E for which line DE passes through C, the corresponding matrix being then the $diagonal(-1, 1, 1)$.

Following proposition deals with the isotropy of pencils (c, e) at *normal* points of the conic c, i.e. points different from its intersection point(s) with the invariant line e.

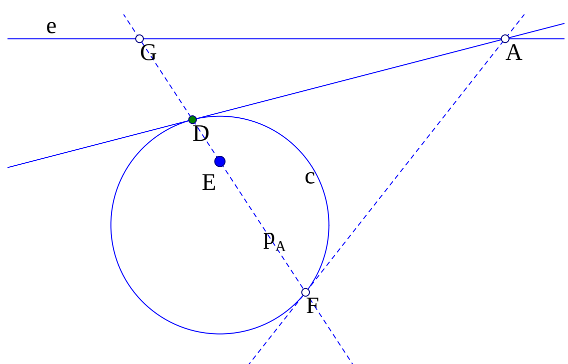

Figure 7. Isotropy at normal points

Proposition 9. *For every normal point D of the conic c the isotropy group $\mathcal{G}_D(c, e)$ is isomorphic to $\mathbb{Z}_2$. The different from the identity element of this group is the involution I_D with axis DE.*

For types (I) and (II) of pencils a proof is the following. Let the homography f preserve the conic c, the line e and fix point D. Then it preserves also the tangent t_D at D and consequently fixes also the intersection point A of this line with the axis e (see Figure 7). It is easily seen that the polar DF of A passes through the center E of the pencil and that f preserves DF. Thus the polar DF carries three points, which remain fixed under f. Since f has three fixed points on line DF it leaves the whole line fixed, hence it coincides with the involution with axis DF and center A.

For type (III) pencils the proof follows from the previous proposition. In fact, assuming $B = c \cap e$ and $A \in c, A \neq B$ an element f of the isotropy group $\mathcal{G}_A(c,e)$ fixes points A, B hence $f \in \mathcal{G}_{AB}$. But from all elements f of the last group only the involution I_B with axis AB preserves the members of the pencil (c,e). This is immediately seen by considering the decomposition of f in two involutions. Would f preserve the member-conics of the bitangent family (c,e) then, by Proposition 5, the centers of these involutions would be points of e but this is impossible for $f \in \mathcal{G}_{AB}$, since the involutions must in this case be centered on line AB.

A byproduct of the short investigation on the isotropy group $\mathcal{G}_{AB}$ of a hyperbolic pencil (c,e) is a couple of results concerning the orbits of $\mathcal{G}_{AB}$ on points of the plane, other than the fixed points A, B, C. To formulate it properly I adopt for triangle ABC the name of *invariant triangle*.

Proposition 10. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC the orbit $\mathcal{G}_{AB}F$ is the member conic c_F of the hyperbolic bitangent pencil (c,e) which passes through F.*

In fact, $\mathcal{G}_{AB}F \subset c_F$ since all $f \in \mathcal{G}_{AB}$ preserve the member-conics of the pencil (see Figure 6). By the continuity of the action the two sets must then be identical. The second result that comes as byproduct is the one suggested by Figure 8. In its formulation as well the formulation of next proposition I use the maps introduced in the course of the proof of Proposition 8.

Proposition 11. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, the intersection point H of lines DE and FG, where $G = f_{DE}(F)$, as E varies on the conic c, describes a conic passing through points A, B, C, D and F.*

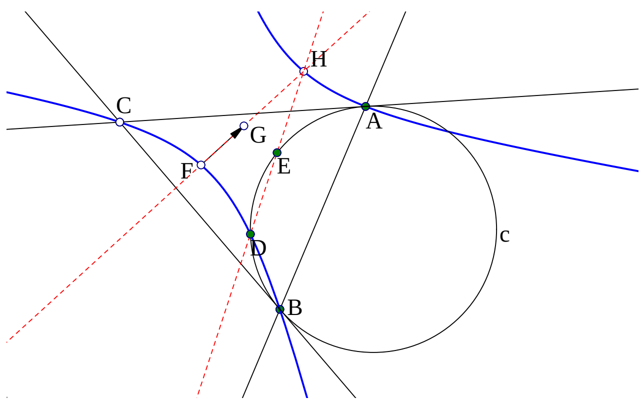

Figure 8. A triangle conic

To prove this consider the projective basis and the matrix representation of f_{DE} given above. It is easy to describe in this basis the map sending line DE to FG. Indeed let $E(x,y,z)$ be a point on the conic. Line DE has coefficients $(y-z, z-x, x-y)$. Thus, assuming F has coordinates (α, β, γ), its image will be described by the coordinates $(\alpha x, \beta y, \gamma z)$. The coefficients of the line FG will be then $(\beta\gamma(y-z), \gamma\alpha(z-x), \alpha\beta(x-y))$. Thus the correspondence of line FG to line DE will be described in terms of their coefficients by the projective transformation

$$(y-z, z-x, x-y) \mapsto (\,\beta\gamma(y-z),\ \gamma\alpha(z-x),\ \alpha\beta(x-y)\,).$$

The proposition is proved then by applying the *Chasles-Steiner* theorem, according to which the intersections of homologous lines of two pencils related by a homography describe a conic ([3, p.73], [4, vol.II, p.173]). According to this theorem the conic passes through the vertices of the pencils D, F. It is also easily seen that the conic passes through points A, B and C.

Proposition 12. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, lines EG with $G = f_{DE}(F)$ as E varies on c envelope a conic which belongs to the bitangent pencil $(c, e = AB)$.*

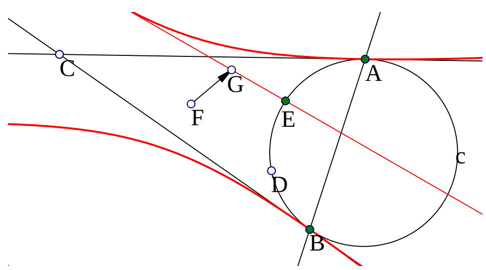

Figure 9. Bitangent member as envelope

The proof can be based on the dual of the argument of *Chasles-Steiner* ([3, p.89]), according to which the lines joining homologous points of a homographic relation between two ranges of points envelope a conic. Here lines EG (see Figure 9) join points (x, y, z) on the conic c with points $(\alpha x, \beta y, \gamma z)$ on the conic c_F, hence their coefficients are given by

$$(\,(\gamma-\beta)yz,\ (\alpha-\gamma)zx,\ (\beta-\alpha)xy\,).$$

Taking the traces of these lines on $x = 0$ and $y = 0$ we find that the corresponding coordinates $(0, y', z')$ and $(x'', 0, z'')$ satisfy an equation of the form $\tau'\tau'' = \kappa$, where $\tau' = y'/z',\ \tau'' = x''/z''$ and κ is a constant. Thus lines EG join points on $x = 0$ and $y = 0$ related by a homographic relation hence they envelope a conic. It is also easily seen that this conic passes through A, B and has there tangents CA, CB hence it belongs to the bitangent family.

Continuing the examination of possible isotropies, after the short digression on the three last propositions, I examine the isotropy group $\mathcal{G}_A(c, e)$ of a parabolic pencil (c, e), for which the axis e is tangent to the conic c at a point A. An element $f \in \mathcal{G}_A(c, e)$ may have A as its unique fixed point or may have an additional fixed point $B \neq A$.

An element $f \in \mathcal{G}_A(c, e)$ having A as a unique fixed point cannot leave invariant another line through A, since this would create a second fixed point on c. Also there is no other fixed point on the tangent e since this would also create another fixed point on c.

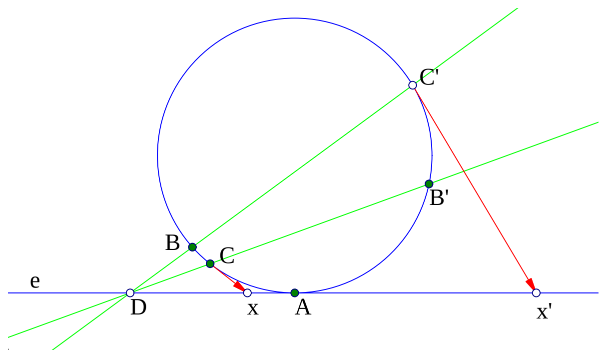

Figure 10. Parabolic isotropy

Proposition 13. *The group $\mathcal{G}_A^0$ including the identity and all homographies f, which preserve a conic c and have A as a unique fixed point, is isomorphic to the additive group $\mathbb{R}$. Every non-indetity homography in this group induces in the tangent e at A a parabolic transformation, which in line coordinates with origin at A is described by a function of the kind $x' = ax/(bx + a)$ or equivalently, by setting $d = b/a$, through the relation*

$$\frac{1}{x'} - \frac{1}{x} = d.$$

This function uniquely describes the conic homography from which it is induced in line e. All elements of this group are non-involutive.

In fact consider the induced Moebius transformation on line e with respect to coordinates with origin at A (see Figure 10). Since A is a fixed point this transformation will have the form $x' = ax/(bx + c)$. Since this is the only root of the equation $x(bx + c) = ax \Leftrightarrow bx^2 + (c - a)x = 0$, it must be $c = a$. Since for every point B other than A the tangents $t_B, t_{B'}$ where $B' = f(B)$ intersect line e at corresponding points $C, C' = f(C)$ the definition of f from its action on line e is complete and unique. The statement on the isomorphism results from the above representation of the transformation. The value $d = 0$ corresponds to the identity transformation. Every other value $d \in \mathbb{R}$ defines a unique parabolic transformation and the product of two such transformations corresponds to the sum $d + d'$ of these constants.

The group $\mathcal{G}_A$ of all homographies preserving a conic c and fixing a point A contains obviously the group $\mathcal{G}_A^0$. The other elements of this group will fix an additional point B on the conic. Consequently the group will be represented as a union $\mathcal{G}_A = \mathcal{G}_A^0 \cup_{B \neq A} \mathcal{G}_{AB}$. For another point C different from A and B the corresponding group $\mathcal{G}_{AC}$ is conjugate to $\mathcal{G}_{AB}$, by an element of the group $\mathcal{G}^0_A$. In fact, by the previous discussion there is a unique element $f \in \mathcal{G}^0_A$ mapping B to C. Then $Ad_f(\mathcal{G}_{AB}) = \mathcal{G}_{AC}$ i.e. every element $f_C \in \mathcal{G}_{AC}$ is represented as $f_C = f \circ f_B \circ f^{-1}$ with $f_B \in \mathcal{G}_{AB}$. These remarks lead to the following proposition.

Proposition 14. *The isotropy group $\mathcal{G}_A$ of conic homographies fixing a point A of the conic c is the semi-direct product of its subgroups $\mathcal{G}^0_A$ of all homographies f which preserve c and have the unique fixed point A on c and the subgroup $\mathcal{G}_{AB}$ of conic homographies which fix simultaneously A and another point $B \in c$ different from A.*

To prove this apply the criterion ([1, p.285]) by which such a decomposition of the group is a consequence of the following two properties: (i) Every element g of the group $\mathcal{G}_A$ is expressible in a unique way as a product $g = g_B \circ g_A$ with $g_A \in \mathcal{G}^0_A,\ g_B \in \mathcal{G}_{AB}$ and (ii) Group $\mathcal{G}^0_A$ is a normal subgroup of $\mathcal{G}_A$. Starting from property (ii) assume that $f \in \mathcal{G}_A$ has the form $f = g_B \circ g_A \circ g_B^{-1}$. Should f fix a point $C \in c$ different from A then it would be $g_A(g_B^{-1}(C)) = g_B^{-1}(C)$ i.e. $g_B^{-1}(C)$ would be a fixed point of g_A, hence $g_B^{-1}(C) = A$ which is impossible. To prove (i) show first that every element in $\mathcal{G}_A$ is expressible as a product $g = g_B \circ g_A$. This is clear if $g \in \mathcal{G}^0_A$ or $g \in \mathcal{G}_{AB}$. Assume then that g in addition to A fixes also the point $C \in c$ different from A. Then as remarked above g can be written in the form $g = g_A \circ g_B \circ g_A^{-1}$, hence $g = g_B \circ (g_B^{-1} \circ g_A \circ g_B \circ g_A^{-1})$ and the parenthesis is an element of $\mathcal{G}^0_A$. That such a representation is also unique follows trivially, since the equation $g_A \circ g_B = g'_A \circ g'_B$ would imply $g_A^{-1} \circ g'_A = g'_B \circ g_B^{-1}$ implying $g_A = g'_A$ and $g_B = g'_B$, since the two subgroups $\mathcal{G}^0_A$ and $\mathcal{G}_{AB}$ have in common only the identity element.

5. Automorphisms of pencils

In this section I examine the automorphism group $\mathcal{G}(c, e)$ of a pencil (c, e) and in particular the non-involutive automorphisms. Every such automorphism is a conic homography f of conic c preserving also the line e. Hence it induces on line e a homography which can be represented by a Moebius transformation

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Inversely, knowing the induced homography on line e from a non-involutive homography one can reconstruct the homography on every other member-conic c of the pencil. Figure 11 illustrates the construction of the image point $B' = f(B)$ by drawing the tangent t_B of c at B and finding its intersection C with e. The image $f(B)$ is found by taking the image point $C' = f(C)$ on e and drawing from there the tangents to c and selecting the appropriate contact point B' or B'' of the tangents from C'. The definition of the homography on c is unambiguous only for pencils of type (III). For the other two kinds of pencils one can construct two homographies f and f^*, which are related by the involution I_0 with center E and

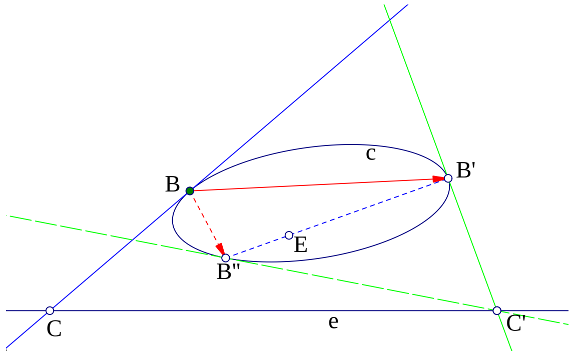

Figure 11. Using line e

axis e. The relation is $f^* = f \circ I_0 = I_0 \circ f$ (last equality is shown in Proposition 16).

Using this method one can easily answer the question of periodic conic homographies.

Proposition 15. *Only the elliptic bitangent pencils have homographies periodic of period $n > 2$. Inversely, if a conic homography is periodic, then it is elliptic.*

In fact, in the case of elliptic pencils, selecting the homography on e to be of the kind

$$x' = \frac{cos(\phi)x - sin(\phi)}{sin(\phi)x + cos(\phi)}, \quad \phi = \frac{2\pi}{n},$$

we define by the procedure described above an n-periodic homography preserving the pencil. For the cases of hyperbolic and parabolic pencils it is impossible to define a periodic homography with period $n > 2$. This because, for such pencils, every homography preserving them has to fix at least one point. If it fixes exactly one, then it is a parabolic homography, hence by Proposition 13 can not be periodic. If it fixes two points, then as we have seen in Proposition 8, the homography can be represented by a real *diagonal* matrix and this can not be periodic for $n > 2$. The inverse is shown by considering the associated bitangent pencil and applying the same arguments.

Since a general homography preserving a conic c can be written as the composition of two involutions, it is of interest to know the structure of the set of involutions preserving a given bitangent pencil. For non-parabolic pencils there is a particular involution I_0, namely the one having for axis the invariant line e of the family and for center E the pole of this line with respect to c.

If I is an arbitrary, other than I_0, involution preserving the bitangent family (c, e) then, since e is invariant by I, either its center Q is on line e or its axis coincides with e. Last case can be easily excluded by showing that the composition $f = I_0 \circ I$ is then an elation with axis e and drawing from this a contradiction. Consequently the axis $e_I = EF$ (see Figure 12) of the involution must pass through the pole E of e with respect to c. It follows that I commutes with I_0. A consequence of this is

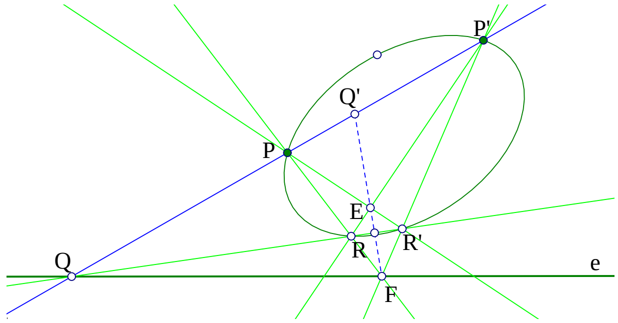

Figure 12. Involutive automorphisms

that $I' = I_0 \circ I$ is another involution the axis of which is line EQ and its center is F. Since by Proposition 5 every homography f preserving the bitangent pencil is a product of two involutions with centers on the axis e it follows that I_0 commutes with f. We arrive thus at the following.

Proposition 16. *The group $\mathcal{G}(c, e)$ of all homographies preserving a non-parabolic bitangent pencil is a subgroup of the group of homographies of the plane preserving line e, fixing the center E of the pencil and commuting with involution I_0.*

For the rest of the section I omit the reference to (c, e) and write simply $\mathcal{G}$ instead of $\mathcal{G}(c, e)$. Involution I_0 is a singularium and should be excluded from the set of all other involutions. It can be represented in infinite many ways as a product of involutions. In fact for any other involutive automorphism of the pencil I the involution $I' = I \circ I_0 = I_0 \circ I$ represents it as a product $I_0 = I \circ I'$. Counting it to the non-involutive automorphisms, it is easy to see that we can separate the group $\mathcal{G}$ into two disjoint sets. The set of non-involutive automorphisms $\mathcal{G}' \subset \mathcal{G}$ containing the identity and I_0 as particular elements, and the set $\mathcal{G}'' \subset \mathcal{G}$ of all other involutive automorphisms.

Proposition 17. *For non-parabolic pencils two involutions I, I' commute, if and only if their product is I_0. Further if the product of two involutions is an involution, then this involution is I_0. For parabolic pencils $I \circ I'$ is never commutative.*

For the first claim notice that $I' \circ I = I_0$ implies $I' = I_0 \circ I = I \circ I_0$. Last because every element of $\mathcal{G}$ commutes with I_0. Last equation implies $I \circ I' = I' \circ I$. Inversely, if last equation is valid it is readily seen that the two involutions have common fixed points on e and fix E hence their composition is $I' \circ I = I_0$. Next claim is a consequence of the previous, since $I' \circ I$ being involution implies $(I' \circ I) \circ (I' \circ I) = 1 \Rightarrow I' \circ I = I \circ I'$. Last claim is a consequence of the fact that $I \circ I'$ and $I' \circ I$ are inverse to each other and non-involutive, according to Proposition 13.

Proposition 18. *The automorphism group $\mathcal{G}$ of a pencil (c, e) is the union of two cosets $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$. $\mathcal{G}'$ consists of the non-involutive automorphisms (and I_0 for non-parabolic pencils) and builds a subgroup of $\mathcal{G}$. $\mathcal{G}''$ consists of all involutive* *automorphisms of the pencil (which are different from I_0 for non-parabolic pencils) and builds a coset of $\mathcal{G}'$ in $\mathcal{G}$. Further it is $\mathcal{G}''\mathcal{G}'' \subset \mathcal{G}'$ and $\mathcal{G}'\mathcal{G}'' \subset \mathcal{G}''$.*

In fact, given an involutive $I \in \mathcal{G}''$ and a non-involutive $f \in \mathcal{G}'$, we can, according to Proposition 5, represent f as a product $f = I \circ I'$ using involution I and another involution I' completely determined by f. Then $I \circ f = I' \in \mathcal{G}''$. This shows $\mathcal{G}''\mathcal{G}' \subset \mathcal{G}''$. The inclusion $\mathcal{G}'\mathcal{G}' \subset \mathcal{G}'$ proving $\mathcal{G}'$ a subgroup of $\mathcal{G}$ is seen similarly. The other statements are equally trivial.

Regarding commutativity, we can easily see that the (co)set of involutions contains non-commuting elements in general ($I' \circ I$ is the inverse of $I \circ I'$), whereas the subgroup $\mathcal{G}'$ is always commutative. More precisely the following is true.

Proposition 19. *The subgroup $\mathcal{G}' \subset \mathcal{G}$ of non-involutive automorphisms of the bitangent pencil (c, e) is commutative.*

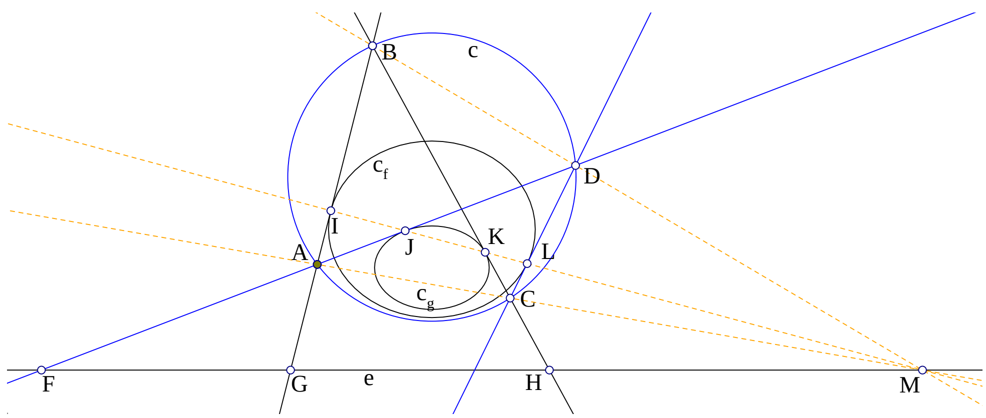

Figure 13. Commutativity for type I

The proof can be given on the basis of Figure 13, illustrating the case of elliptic pencils, the arguments though being valid also for the other types of pencils. In this figure the two products $f \circ g$ and $g \circ f$ of two non-involutive automorphisms of the pencil $f \in \mathcal{G}'$ and $g \in \mathcal{G}'$ are represented using the *tangential generation* of Proposition 6. For $A \in c$ point $B = f(A)$ has line AB tangent at I to a conic c_f of the pencil. Analogously $C = g(B)$ defines line BC tangent at K to a second conic c_g of the pencil. Let $D = g(A)$ and consequently AD be tangent at point J to c_g. It must be shown that $f(D) = C$ or equivalently that line DC is tangent at a point L to c_f. For this note first that lines $\{BD, IJ\}$ intersect at a point M on e. This happens because of the harmonic ratios $(A, B, G, I) = -1$ and $(A, D, F, J) = -1$. Similarly lines AC, IK intersect at a point M' of e. This follows again by the harmonic ratios $(B, A, I, G) = (B, C, K, H) = -1$. Hence $M' = M$ and consequently lines AC, BD intersect at M, hence according to Proposition 2, $C = f(D)$.

For hyperbolic pencils the result is also a consequence of the representation of these homographies through diagonal matrices, as in Proposition 8. For parabolic pencils the proof follows also directly from Proposition 13.

Note that for pencils (c, e) of type (II) for which c and e intersect at two points $\{A, B\}$, the involutions I_A, I_B with axes respectively BE, AE, do not belong to $\mathcal{G}$ but define through their composition $I_A \circ I_B = I_0$. This is noticed in Proposition 5 which represents every automorphism as the product of two involutions. It is though a case to be excluded in the following proposition, which results from Proposition 5 and the previous discussion.

Proposition 20. *If an automorphism $f \in \mathcal{G}$ of a pencil (c, e) is representable as a product of two involutions $f = I_2 \circ I_1$, then with the exception of $I_0 = I_A \circ I_B$ in the case of an hyperbolic pencil, in all other cases I_1 and I_2 are elements of $\mathcal{G}$.*

Regarding the transitivity of $\mathcal{G}(c, e)$ on the conics of the pencil, the following result can be easily proved.

Proposition 21. *(i) For elliptic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on the points of the conic c.*
(ii) For hyperbolic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A, B\}$, where $\{A, B\} = c \cap e$. All elements of $\mathcal{G}''$ interchange (A, B), whereas all elements of $\mathcal{G}'$ fix them.
(iii) For parabolic pencils each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A\}$ where $A = c \cap e$ and all of them fix point A.

6. Bitangent flow

Last proposition shows that every non-involutive conic homography f of a conic c is an element of a one-dimensional Lie group ([6, p.210], [13, p.82]) $\mathcal{G}$ acting on the projective plane . The invariant conic c is then a union of orbits of the action of this group. Group $\mathcal{G}$ is a subgroup of the Lie group $PGL(3, \mathbb{R})$ of all projectivities of the plane and contains a one-parameter group ([13, p.102]) of this group, which can be easily identified with the connected component of the subgroup $\mathcal{G}'$ containing the identity. Through the one-parameter group one can define a vector field on the plane, the integral curves of which are contained in the conics of the bitangent pencil associated to the non-involutive homography. Thus the bitangent pencil represents the flow of a vector field on the projective plane ([6, p.139], [14, p.292]). The fixed points correspond to the singularities of this vector field.

This point of view rises the problem of the determination of the simplest possible data needed in order to define such a flow on the plane. The answer (Proposition 26) to this problem lies in a certain involution on e related to the coset $\mathcal{G}''$ of the involutive automorphisms of the bitangent pencil. I start with non-parabolic pencils, characterized by the existence of the particular involution I_0.

Proposition 22. *For every non-parabolic pencil the correspondence $\mathcal{J} : Q \mapsto F$ between the centers of the involutions I and $I \circ I_0$ defines an involutive homography on line e. The fixed points of $\mathcal{J}$ coincide with the intersection points $\{A, B\} = c \cap e$.*

In fact considering the pencil E^* of lines through E it is easy to see that the correspondence $\mathcal{J} : F \mapsto Q$ (see Figure 14) is projective and has period two. The identification of the fixed points of $\mathcal{J}$ with $\{A, B\} = c \cap e$ is equally trivial.

Proposition 23. *The automorphism group $\mathcal{G}(c, e)$ of a non-parabolic pencil is uniquely determined by the triple*

$$(e, E, \mathcal{J})$$

consisting of a line e a point $E \notin e$ and an involutive homography on line e.

In fact $\mathcal{J}$ completely determines the involutive automorphisms I_Q of the pencil, since for each point Q on e point $F = \mathcal{J}(Q)$ defines the axis FE of the involution I_Q. The involutive automorphisms in turn, through their compositions, determine also the non involutive elements of the group.

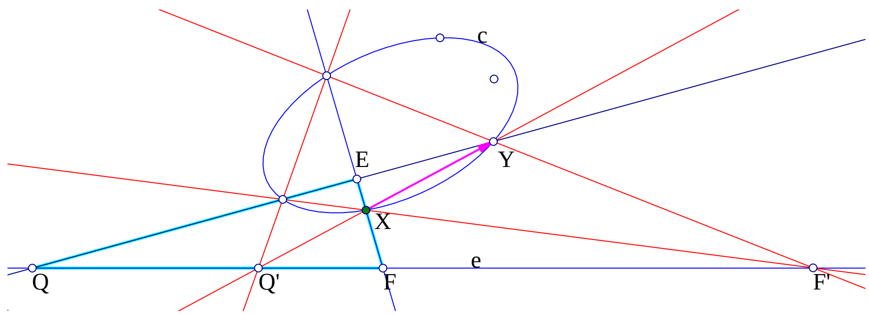

Figure 14. Quadrilateral in case I

Remark. For elliptic pencils involution $\mathcal{J}$ induces on every member-conic c of the pencil a correspondence of points $X \mapsto Y$ through its intersection with lines (EF, EQ) (see Figure 14). This defines an automorphism of the pencil of order 4 and through it infinite many convex quadrangles, each of which completely determines the pencil. Inversely, by the results of this section it will follow that for each convex quadrangle there is a well defined bitangent pencil having a member c circumscribed and a member c' inscribed in the quadrangle. Conic c is characterized by having its tangents at opposite vertices intersect on line e. Conic c' contacts the sides of the quadrangle at their intersections with lines $\{EI, EJ\}$ (see Figure 15). Note that for cyclic quadrilaterals in the euclidean plane the corresponding conic c does not coincide in general with their circumcircle. It is instead identical with the image of the circumcircle of the square under the unique projective map sending the vertices of the square to those of the given quadrilateral (e is the image under this map of the line at infinity).

Knowing the group $\mathcal{G}$ of its automorphisms, one would expect a complete reconstruction of the whole pencil, through the orbits $\mathcal{G}X$ of points X of the plane under the action of this group. Before to proceed to the proof of this property I modify slightly the point of view in order to encompass also parabolic pencils. For this consider the map $\mathcal{I} : e \mapsto E^*$ induced in the pencil E^* of lines emanating from E, the pole of the invariant line of the pencil. This map associates to every point $Q \in e$ the axis EF of the involution centered at Q. Obviously for non-parabolic pencils $\mathcal{I}$ determines $\mathcal{J}$ and vice versa. The first map though can be defined also

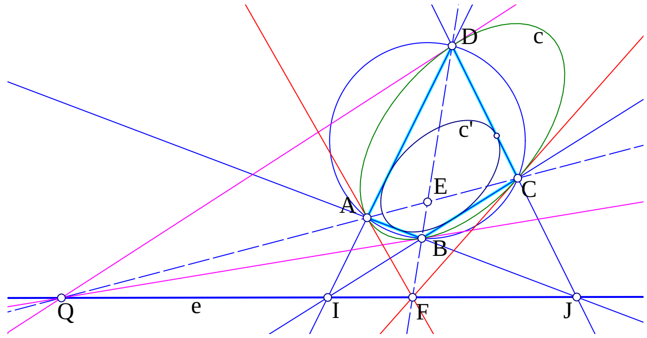

Figure 15. Circumcircle and circumconic

for parabolic pencils, since also in this case, for each point $Q \in e$ there is a unique line FQ representing the axis of the unique involutive automorphism of the pencil centered at Q. Following general fact is on the basis of the generation of the pencil through orbits.

Proposition 24. *Given a line e and a point E consider a projective map $\mathcal{I} : e \mapsto E^*$ of the line onto the pencil E^* of lines through E. Let e' denote the complement in e of the set $e'' = \{Q \in e : Q \in \mathcal{I}(Q)$. For every $Q \in e'$ denote the involution with center Q and axis $\mathcal{I}(Q)$. Then for every point $X \notin e$ of the plane the set $\{I_Q(X) : Q \in e'\} \cup e''$ is a conic.*

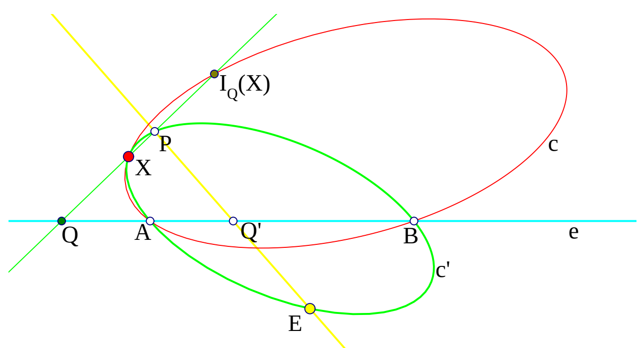

Figure 16. Orbits of involutions

In fact, by the Chasles-Steiner construction method of conics ([3, p.73]), lines XQ and $\mathcal{I}(Q)$ intersect at a point P describing a conic c', which passes through X and E. Every point $Q \in e''$ i.e. satisfying $Q \in \mathcal{I}(Q)$ coincides with a point of the intersection $c' \cap e$ and vice versa. Thus e'' has at most two points ($\{A, B\}$ in Figure 16).

The locus $\{I_Q(X) : Q \in e\}$ coincides then with the image c of the conic c', under the perspectivity p_X with center at X, axis the line e and homology coefficient $k = 1/2$.

Proposition 25. *The conics generated by the previous method belong to a bitangent pencil with axis e and center E if and only if they are invariant by all involutions I_Q for $Q \in e'$. The points in e'' are the fixed points of the pencil.*

The necessity of the condition is a consequence of Proposition 21. To prove the sufficiency assume that c is invariant under all $\{I_Q : Q \in e'\}$. Then for every $Q \in e'$ line $\mathcal{I}(Q)$ is the polar of Q with respect to c. Consequently line e is the polar of E with respect to c and, if $E \notin e$, the involution I_0 with axis e and center E leaves invariant c. Since the center of each involution from the pair $(I_Q,\, I_0)$ is on the axis of the other the two involutions commute and $I_Q \circ I_0$ defines an involution with center at the intersection $Q' = e \cap \mathcal{I}(Q)$ and axis the polar of this point with respect to c, which, by the previous arguments, coincides with $\mathcal{I}(Q')$. This implies that the map induced in line e by $\mathcal{J}' : Q \mapsto Q' = \mathcal{I}(Q) \cap e$ is an involution. Consider now the pencil (c, e). It is trivial to show that its member-conics coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$ and $\mathcal{J}'$ is identical with the involution $\mathcal{J}$ of the pencil. This completes the proof of the proposition for the case $E \notin e$.

The proof for the case $E \in e$ is analogous with minor modifications. In this case the assumption of the invariance of c under I_Q implies that line $\mathcal{I}(Q)$ is the polar of Q with respect to c. From this follows that c is tangent to e at E and $\mathcal{I}(E) = e$. Thus e'' contains the single element E. Then it is again trivial to show that the conics of the pencil (c, e) coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$.

The arguments in the previous proof show that non-parabolic pencils are completely determined by the involution $\mathcal{J}$ on line e, whereas parabolic pencils are completely defined by a projective map $\mathcal{I} : e \to E^*$ with the property $\mathcal{I}(E) = e$. Following proposition formulates these facts.

Proposition 26. *(i) Non-parabolic pencils correspond bijectively to triples $(e, E, \mathcal{J})$ consisting of a line e, a point $E \notin e$ and an involution $\mathcal{J} : e \to e$. The fixed points of the pencil coincide with the fixed points of $\mathcal{J}$.*
(ii) Parabolic pencils correspond bijectively to triples $(e, E, \mathcal{I})$ consisting of a line e, a point $E \in e$ and a projective map $\mathcal{I} : e \to E^*$ onto the pencil E^* of lines through E, such that $\mathcal{I}(E) = e$.

7. The perspectivity group of a pencil

Perspectivities are homographies of the plane fixing a line e, called the *axis* and leaving invariant every line through a point E, called the *center* of the perspectivity. If $E \in e$ then the perspectivity is called an *elation*, otherwise it is called *homology*. Tightly related to the group $\mathcal{G}$ of automorphisms of the pencil (c, e) is the group $\mathcal{K}$ of perspectivities, with center E the center of the pencil and axis the axis e of the pencil. As will be seen, this group acts on the pencil (c, e) by permuting its members. For non-parabolic pencils the perspectivities of this group are *homologies*, and for parabolic pencils the perspectivities are *elations*. The basic facts about perspectivities are summarized by the following three propositions ([12, p.72], [15, p.228], [7, p.247]).

Proposition 27. *Given a line e and three collinear points E, X, X', there is a unique perspectivity f with axis e and center E and $f(X) = X'$.*

Proposition 28. *For any perspectivity f with axis e and center E and two points (X, Y) with $(X' = f(X),\ Y' = f(Y))$, lines XY and $X'Y'$ intersect on e. For homologies the cross ratio $(X, X', E, X_e) = \kappa$, where $X_e = XX' \cap e$, is a constant κ called homology coefficient. Involutive homographies are homologies with $\kappa = -1$ and are called harmonic homologies.*

Proposition 29. *The set of homologies having in common the axis e and the center E builds a commutative group $\mathcal{K}$ which is isomorphic to the multiplicative group of real numbers.*

That the composition $h = g \circ f$ of two homologies with the previous characteristics is a homology follows directly from their definition. The homology coefficients multiply homomorphically $\kappa_h = \kappa_g \kappa_f$, this being a consequence of Proposition 27 and the well-known identity for cross ratios of five points (X, Y, Z, H) on a line d ([2, p.174])

$$(X, Y, E, H)(Y, Z, E, H)(Z, X, E, H) = 1,$$

where $H = d \cap e$. This implies also the commutativity.

Whereas the previous isomorphism is canonical, the following one, easily proved by using coordinates is not canonical. The usual way to realize it is to send e to infinity and have the elations conjugate to translations parallel to the direction determined by E ([4, vol.II, p.191]). The representation of the elation as a composition, given below follows directly from the definitions.

Proposition 30. *The set of all elations having in common the axis e and the center E builds a commutative group $\mathcal{K}$ which is isomorphic to the additive group of real numbers. Every elation f can be represented as a composition of two harmonic homologies $f = I_B \circ I_A$, which share with f the axis e and have their centers $\{A, B\}$ collinear with E. In this representation the center $A \notin e$ can be arbitrary, the other center B being then determined by f and lying on line AE.*

Returning to the pencil (c, e), the group $\mathcal{G}$ of its automorphisms and the corresponding group $\mathcal{K}$ of perspectivities, which are homologies in the non-parabolic case and elations in the parabolic, combine in the way shown by the following propositions.

Proposition 31. *For every bitangent pencil (c, e) the elements of $\mathcal{K}(c, e)$ commute with those of $\mathcal{G}(c, e)$.*

The proposition is easily proved first for involutive automorphisms of the pencil, characterized by having their centers Q on the perspectivity axis e and their axis q passing through the perspectivity center E. Figure 17 suggests the proof of the commutativity of such an involution f_Q with a homology f_E with center at E and axis the line e. Point $Y = f_E(X)$ satisfies the cross-ratio condition of the perspectivity $(X, Y, E, H) = \kappa$, where κ is the homology coefficient of the perspectivity.

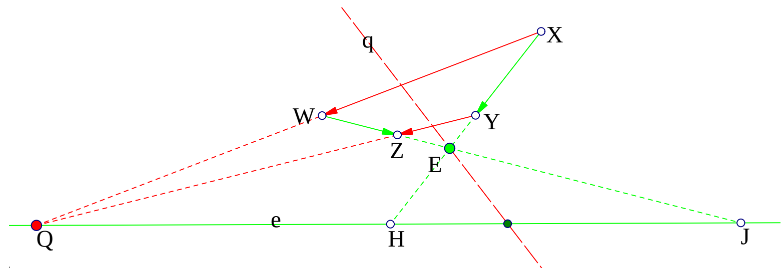

Figure 17. Homology commuting with involution

Then taking $Z = f_Q(Y)$ and the intersection W of line ZE with XQ it is readily seen that $f_Q(f_E(X)) = f_E(f_Q(X))$. Thus perspectivity f_E commutes with all involutive automorphisms of the pencil.

In the case of parabolic pencils, if EF is the axis of the involutive automorphism f_Q, $Q \in e$ of the pencil, according to Proposition 30, one can represent the elation f_E as a composition $I_B \circ I_A$ of two involutions with centers lying on EF and axis the invariant line e. Each of these involutions commutes then with f_Q, hence their composition will commute with f_Q too. Since the involutive automorphisms generate all automorphisms of the pencil it follows that f_E commutes with every automorphism of the pencil.

Proposition 32. *For non-hyperbolic pencils and every two member-conics (c, c') of the pencil there is a perspectivity with center at E and axis the line e, which maps c to c'. For hyperbolic pencils this is true if c and c' belong to the same connected component of the plane defined by lines (EA, EB), where $\{A, B\} = c \cap e$ are the base points of the pencil and E the center of the pencil.*

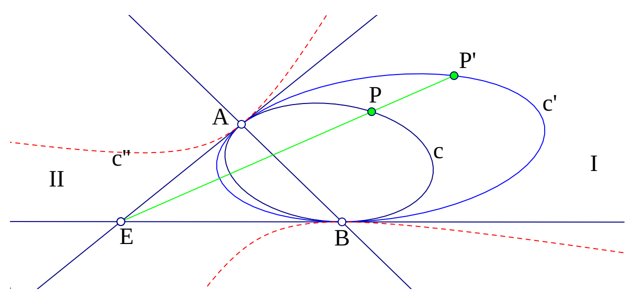

Figure 18. Perspectivity permuting member-conics

To prove the claim consider a line through E intersecting two conics of the pencil at points $P \in c$, $P' \in c'$ (see Figure 18). By Proposition 27 there is a perspectivity f mapping P to P'. By the previous proposition f commutes with all $g \in \mathcal{G}$ which can be used to map P to any other (than the base points of the pencil) point Q of c and point P' to $Q' \in c' \cap EQ$. This implies that $f(c) = c'$. The restriction for hyperbolic pencils is obviously necessary, since perspectivities leave invariant the lines through their center E.

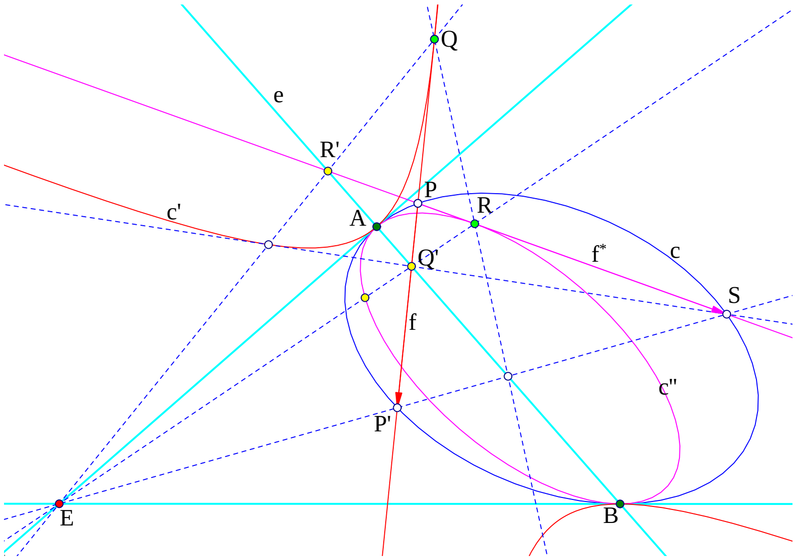

Figure 19. Conjugate member-conic

Proposition 33. *Let f be a non-involutive automorphism of the pencil (c, e) and c' be the member-conic determined by its tangential generation with respect to c(proposition-6). Then, for non-hyperbolic pencils there is a perspectivity $p_f \in \mathcal{K}$ mapping c to c'. This is true also for hyperbolic pencils provided f preserves the components of c cut out by e. Further p_f is independent of c.*

In fact, given $f \in \mathcal{G}$, according to *proposition-6*, there is a conic c' of the bitangent pencil such that lines PP', $P' = f(P)$ are tangent to c'. The exceptional case for pencils of type (II) occurs when f interchanges the two components cut out from c by the axis e. In this case conics c and c' are on different connected components of the plane defined by lines $\{EA, EB\}$. This is due to the fact (ibid) that the contact point Q of PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection $Q' = PP' \cap e$. Figure 19 illustrates this case and shows that for such automorphisms the resulting automorphism $f^* = f \circ I_0 = I_0 \circ f$, mapping P to $S = f^*(P) = I_0(f(P)) = I_0(P')$, defines through its corresponding tangential generation a kind of *conjugate* conic c'' to c' with respect to c.

To come back to the proof, first claim follows from the previous proposition. Last claim means that if the pencil is represented through another member-conic d by the pair (d, e), and the tangential generation of f is determined by a conic d', then the corresponding p'_f mapping d to d' is identical to p_f. The property is indeed a trivial consequence of the commutativity between the members of the groups $\mathcal{G}$ and $\mathcal{K}$. To see this consider a point $P \in c$ and its image $P' = f(P) \in c$. Consider also the perspectivity $g \in \mathcal{K}$ sending c to d and let $Q = g(P), Q' = g(P')$. By the commutativity of f, g it is $f(Q) = f(g(P)) = g(f(P)) = g(P') = Q'$. Thus the envelope c' of lines PP' maps via g to the envelope d' of lines QQ'. Hence if $p_f(c) = c'$ and $p'_f(d) = d'$ then $d' = g(c') = g(p_f(c))$ implying $p'_f(g(c)) = g(p_f(c))$ and from this $p'_f = g \circ p_f \circ g^{-1} = p_f$ since g and p_f commute.

Remark. Given a bitangent pencil (c, e) the correspondence of p_f to f considered above is univalent only for parabolic pencils. Otherwise it is bivalent, since both p_f and $p_f \circ I_0 = I_0 \circ p_f$ do the same job. Even in the univalent case the correspondence is not a homomorphism, since it is trivially seen that f and $g = f^{-1}$ have $p_f = p_g$. This situation is reflected also in simple configurations as, for example, in the case of the bitangent pencil (c, e) of concentric circles with common center E, the invariant line e being the line at infinity.

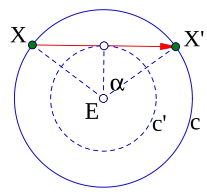

Figure 20. A case of $\mathcal{G}' \ni f \mapsto p_f \in \mathcal{K}$

In this case the rotation R_α by angle $\alpha \in (0, \pi)$ at E (see Figure 20), which is an element of the corresponding $\mathcal{G}'$, maps to the element $H_{cos(\frac{\alpha}{2})}$ of $\mathcal{K}$, which is the homothety with center E and ratio $cos(\frac{\alpha}{2})$.

8. Conic affinities

By identifying the invariant line e of a bitangent pencil (c, e) with the line at infinity all the results of the previous sections translate to properties of affine maps preserving affine conics ([2, p.184], [4, vol.II, p.146]). The automorphism group $\mathcal{G}$ of the pencil (c, e) becomes the group $\mathcal{A}$ of affinities preserving the conic c. Elliptic pencils correspond to *ellipses*, hyperbolic pencils correspond to *hyperbolas* and parabolic pencils to *parabolas*. The center E of the pencils becomes the *center* of the conic, for ellipses and hyperbolas, called collectively *central* conics. For these kinds of conics involution I_0 becomes the *symmetry* or *half turn* at the center of the conic. Every involution I other that I_0, becomes an *affine reflection* ([7, p.203]) with respect to the corresponding axis of the involution, which coincides with a *diameter* d of the conic. The center of the affine reflection I is a point at infinity defining the *conjugate* direction of lines XX' ($X' = f(X)$) of the reflection. This direction coincides with the one of the conjugate diameter to d. For an affine reflection I with diameter d the reflection $I \circ I_0$ is the reflection with respect to the *conjugate* diameter d' of the conic. Products of two affine reflexions are called *equiaffinities* ([7, p.208]) or *affine rotations*. For central conics the group $\mathcal{K}$ of perspectivities becomes the group of homotheties centered at E.

For parabolas the center of the pencil E is the contact point of the curve with the line at infinity. All affine reflections have in this case their axes passing through E

i.e. they are parallel to the direction defined by this point at infinity, which is also the contact point of the conic with e. The group $\mathcal{K}$ of elations in this case becomes the group of translations parallel to the direction defined by E.

In order to stress the differences between the three kinds of affine conics I translate the results of the previous paragraphs for each one separately.

By introducing an euclidean metric into the plane ([4, vol.I, p.200]) and taking for c the unit circle $c : x^2 + y^2 = 1$, the group of affinities of an ellipse becomes equal to the group of isometries of the circle. The subgroup $\mathcal{G}'$ equals then the group of rotations about the center of the circle and the coset $\mathcal{G}''$ equals the coset of reflections on diameters of the circle. I_0 is the symmetry at the center of the circle and the map $I \mapsto I \circ I_0$ sends the reflection on a diameter d to the reflection on the orthogonal diameter d' of the circle. An affine rotation is identified with an euclidean rotation and in particular a periodic affinity is identified with a periodic rotation. This and similar simple arguments lead to the following well-known results.

Proposition 34. (1) *The group $\mathcal{G}$ of affinities preserving the ellipse c is isomorphic to the rotation group of the plane.*
(2) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(3) *For every $n > 2$ there is a unique cyclic group of n elements $\{f, f^2, ..., f^n = 1\} \subset \mathcal{G}'$ with f periodic of period n.*
(4) *For every affine rotation f of an ellipse c the corresponding axis e is the line at infinity and the center E is the center of the conic.*
(5) *The pencil (c, e) consists of the conics which are homothetic to c with respect to its center.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the ellipse. To each affine rotation f of the conic corresponds a real number $r_f \in [0, 1]$ which is the homothety ratio of the element $p_f \in \mathcal{K} : c' = p_f(c)$, where c' realizes through its tangents the tangential generation of f.*

In the case of hyperbolas the groups differ slightly from the corresponding ones for ellipses in the connectedness of the cosets $\mathcal{G}'$, $\mathcal{G}''$ which now have two components. The existence of two components has a clear geometric meaning. The components result from the two disjoint parts into which is divided the axis e through its intersection points A, B with the conic c. Involutions I_P which have their center P in one of these parts have their axis non-intersecting the conic. These involutions are characterized in the affine plane by diameters non-intersecting the hyperbola. They represent affine reflections which have no fixed points on the hyperbola. The other connected component of the coset of affine reflections is characterized by the property of the corresponding diameters to intersect the hyperbola, thus defining two fixed points of the corresponding reflection.

Group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$ corresponding to $\mathcal{G}_{AB}$ of Proposition 8. This group is the disjoint union of the subgroup $\mathcal{G}_0'$ of affine *hyperbolic* rotations that preserve the components of the hyperbola and its coset $\mathcal{G}'_1 = I_0\mathcal{G}'_0$ of affine *hyperbolic crossed* rotations that interchange the two components of the hyperbola ([7, p.206]). By identifying c with the hyperbola $xy = 1$ one can describe the elements of $\mathcal{G}'_0$ through the affine maps $\{(x, y) \mapsto (\mu x, \frac{1}{\mu}y),\ \mu > 0\}$. The other component $\mathcal{G}'_1$ is then identified with the set of maps $\{(x, y) \mapsto (-\mu x, -\frac{1}{\mu}y),\ \mu > 0\}$. Following proposition summarizes the results.

Proposition 35. (1) *The group of affinities $\mathcal{G}$ of a given affine hyperbola c consists of affine reflections and affine rotations which are compositions of two reflections.*
(2) *The affine rotations build a commutative subgroup $\mathcal{G}' \subset \mathcal{G}$ and the affine reflections build the unique coset $\mathcal{G}'' \subset \mathcal{G}$ of this group. $\mathcal{G}'$ and $\mathcal{G}''$ are each homeomorphic to the pointed real line $\mathbb{R}^*$ and group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$.*
(3) *Group $\mathcal{G}' = \mathcal{G}'_0 \cup \mathcal{G}'_1$ has two components corresponding to rotations that preserve the components of the hyperbola and the others $\mathcal{G}'_1 = I_0\mathcal{G}'_0$, called crossed rotations, that interchange the two components. There are no periodic affinities preserving the hyperbola for a period $n > 2$.*
(4) *The coset of affine reflections of the hyperbola is the union $\mathcal{G}'' = \mathcal{G}''_0 \cup \mathcal{G}''_1$ of two components. Reflections $I \in \mathcal{G}''_0$ preserve hyperbola's components and have fixed points on them, whereas reflections $I \in \mathcal{G}''_1 = I_0\mathcal{G}''_0$ interchange the two components and have no fixed points.*
(5) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the hyperbola.*
(7) *For each non-involutive affinity f of an hyperbola c preserving the components the tangential generation of f defines another hyperbola c' homothetic to c with respect to E. If f permutes the components of c then $f^* = f \circ I_0$ defines through tangential generation c' homothetic c. The homotety ratios for the two cases are correspondingly $r_f > 1$ and $r_{f^*} \in (0, 1)$.*

The case of affine parabolas demonstrates significant differences from ellipses and hyperbolas. Since the affinities preserving a parabola fix its point at infinity E, their group is isomorphic to the group $\mathcal{G}_E$ discussed in Proposition 14. This group contains the subgroup $\mathcal{G}^0_E$ of so-called *parabolic rotations*, which are products of two *parabolic reflections*. These are the only affine reflections preserving the parabola and their set is a coset of $\mathcal{G}^0_E$ in $\mathcal{G}_E$. The most important addition in the case of parabolas are the isotropy groups $\mathcal{G}_{EB}$ fixing the point at infinity E of the parabola and an additional point B of it. Figure 21 illustrates an example of such a group $\mathcal{G}_{EB}$ in which the parabola is described in an affine frame by the equation $y = x^2$ and point B is the origin of coordinates. In this example the group $\mathcal{G}_{EB}$ is represented by affine transformations of the form

$$(x, y) \mapsto (rx, r^2y),\ r \in \mathbb{R}^*.$$

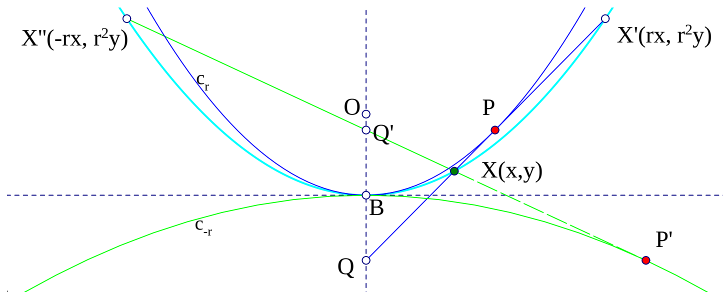

Figure 21. The group $\mathcal{G}_{EB}$

The figure displays also two other parabolas c_r, c_{-r}. These are the conics realizing the tangential generations (Proposition 6) for the corresponding affinities $(x,y) \mapsto (rx, r^2y)$ and $(x,y) \mapsto (-rx, r^2y)$ for $r > 0$. Note that, in addition to the unique affine reflection and the unique affine rotation sending a point X to another point X' and existing for affine conics of all kinds, there are for parabolas infinite more affinities doing the same job. In fact, in this case, by Proposition 14, for every two points (X, X') different from B there is precisely one element $f \in \mathcal{G}_{EB}$ mapping X to X'. This affinity preserves the parabola, fixes B and is neither an affine reflection nor an affine rotation.

Figure 22 shows the decomposition of the previous affinity into two involutions $f_r = I \circ I'$ with centers Q_1, Q_2 lying on the axis ($x = 0$) of the parabola. These are not affine since they do not preserve the line at infinity. They have though their axis parallel to the tangent d at B and their intersections with the axis R_i are symmetric to Q_i with respect to B. Thus, both of them map the line at infinity onto the tangent d at B, so that their composition leaves the line at infinity invariant. Since for another point $C \in c$ the corresponding group $\mathcal{G}_{EC} = Ad_f(\mathcal{G}_{EB})$ is conjugate to $\mathcal{G}_{EB}$ by an affine rotation $f \in \mathcal{G}_E^0$ the previous analysis transfers to the isotropy at C.

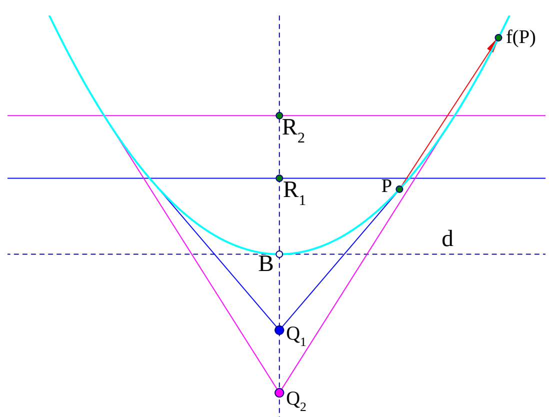

Figure 22. f_r as product of involutions

Another issue to be discussed when comparing the kinds of affine conics is that of area. Area in affine planes is defined up to a multiplicative constant ([4, vol.I, p.59]). To measure areas one fixes an affine frame, thus fixing simultaneously the *orientation* of the plane, and refers everything to this frame. Affinities preserving the area build a subgroup of the group of affinities of the plane, to which belong all affine rotations. Affine reflections reverse the sign of the area. Thus affine reflections and rotations, considered together build a group preserving the unsigned area. The analysis in the previous sections shows that affinities preserving an affine conic are automatically also unsigned-area-preserving in all cases with the exception of some types of affinities of parabolas. These affinities are the elements of the subgroups $\mathcal{G}_{EB}$ with the exception of the identity and the parabolic reflection I_B fixing B. Thus, while for all conics there are exactly two area preserving affinities mapping a point X to another point X' (an affine rotation and an affine reflection), for parabolas there is in addition a one-parameter infinity of area non-preserving affinities mapping X to X'.

Proposition 36. *In the following e denotes the line at infinity and E its unique common point with the parabola c.*
(1) *The group $\mathcal{G}$ of affinities preserving parabola c is the union $\mathcal{G} = \mathcal{G}_E^0 \cup_{B\in c} \mathcal{G}_{EB}$. This group is also the semidirect product of its subgroups $\mathcal{G}_E^0$ and $\mathcal{G}_{EB}$, the first containing all parabolic rotations and the second being the isotropy group at a point $B \in c$ of $\mathcal{G}$.*
(2) *$\mathcal{G}_E^0$ is the group of affine rotations, which are products of two affine reflections preserving the conic. This group is isomorphic to the additive group of real numbers. There are no periodic affinities preserving a parabola for a period $n > 2$.*
(3) *The set $\mathcal{G}''$ of all affine reflections preserving the parabola consists of affinities having their axis parallel to the axis of the parabola, which is the direction determined by its point at infinity E. This is a coset of the previous subgroup of $\mathcal{G}$ acting simply transitively on c.*
(4) *The group $\mathcal{G}_{EB}$ is isomorphic to the multiplicative group $\mathbb{R}^*$ and its elements, except the affine reflection $I_B \in \mathcal{G}_{EB}$, the axis of which passes through B, though they preserve c, are neither affine reflections nor affine rotations and do not preserve areas. This group acts simply transitively on $c - \{B\}$.*
(5) *For every pair of points $B \in c, C \in c$ there is a unique affine rotation $f \in \mathcal{G}_E^0$ such that $f(B) = C$. This element conjugates the corresponding isotropy groups: $Ad_f(\mathcal{G}_{EB}) = \mathcal{G}_{EC}$.*
(6) *Every coset of $\mathcal{G}_E^0$ intersects each subgroup $\mathcal{G}_{EB} \subset \mathcal{G}$ in exactly one element.*
(7) *For each affine rotation $f \in \mathcal{G}_E^0$ the tangential generation defines an element $p_f \in \mathcal{K}$. Last group coincides with the group of translations parallel to the axis of the parabola.*
(8) *For each element $f \in \mathcal{G}_{EB}$ the tangential generation defines a parabola which is a member of the bitangent pencil (c, EB). This pencil consists of all parabolas sharing with c the same axis and being tangent to c at B.*

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Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: pamfilos@math.uoc.gr

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Some Triangle Centers Associated with the Tritangent Circles

Nikolaos Dergiades and Juan Carlos Salazar

Abstract. We investigate two interesting special cases of the classical Apollonius problem, and then apply these to the tritangent circles of a triangle to find pair of perspective (or homothetic) triangles. Some new triangle centers are constructed.

1. An interesting construction

We begin with a simple construction of a special case of the classical Apollonius problem. Given two circles $O(r)$, $O'(r')$ and an external tangent $\mathcal{L}$, to construct a circle $O_1(r_1)$ tangent to the circles and the line, with point of tangency X between A and A', those of (O), (O') and $\mathcal{L}$ (see Figure 1). A simple calculation shows that $AX = 2\sqrt{r_1 r}$ and $XA' = 2\sqrt{r_1 r'}$, so that $AX : XA' = \sqrt{r} : \sqrt{r'}$. The radius of the circle is

$$r_1 = \frac{1}{4}\left(\frac{AA'}{\sqrt{r}+\sqrt{r'}}\right)^2.$$

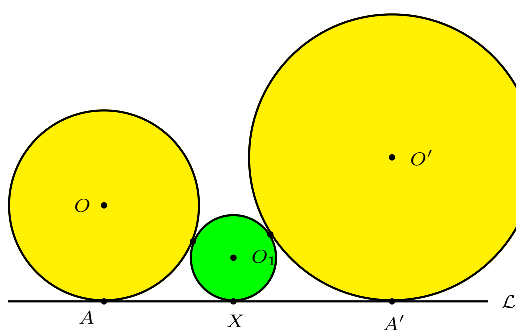

Figure 1

From this we design the following construction.

Construction 1. *On the line $\mathcal{L}$, choose two points P and Q be points on opposite sides of A such that $PA = r$ and $AQ = r'$. Construct the circle with diameter PQ to intersect the line OA at F such that O and F are on opposite sides of $\mathcal{L}$. The intersection of $O'F$ with $\mathcal{L}$ is the point X satisfying $AX : XA' = \sqrt{r} : \sqrt{r'}$. Let M be the midpoint of AX. The perpendiculars to OM at M, and to $\mathcal{L}$ at X intersect at the center O_1 of the required circle* (see Figure 2).

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Many thanks to the editor for many additions and especially for Construction 1.

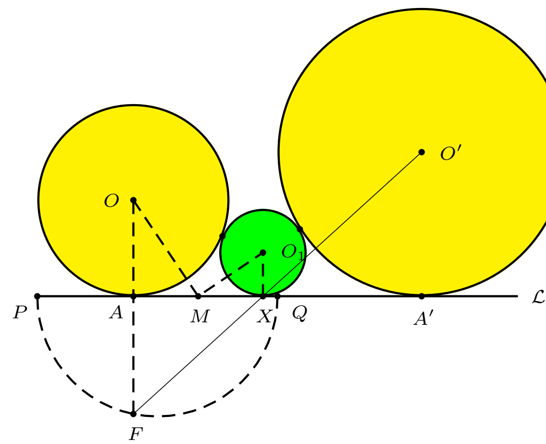

Figure 2

For a construction in the case when $\mathcal{L}$ is not necessarily tangent of (O) and (O'), see [1, Problem 471].

2. An application to the excircles of a triangle

We apply the above construction to the excircles of a triangle ABC. We adopt standard notations for a triangle, and work with homogeneous barycentric coordinates. The points of tangency of the excircles with the sidelines are as follows.

	BC	CA	AB
(I_a)	$A_a = (0 : s-b : s-c)$	$B_a = (-(s-b) : 0 : s)$	$C_a = (-(s-c) : s : 0)$
(I_b)	$A_b = (0 : -(s-a) : s)$	$B_b = (s-a : 0 : s-c)$	$C_b = (s : -(s-c) : 0)$
(I_c)	$A_c = (0 : s : -(s-a))$	$B_c = (s : 0 : -(s-c))$	$C_c = (s-a : s-b : 0)$

Consider the circle $O_1(X)$ tangent to the excircles $I_b(r_b)$ and $I_c(r_c)$, and to the line BC at a point X between A_c and A_b (see Figure 3).

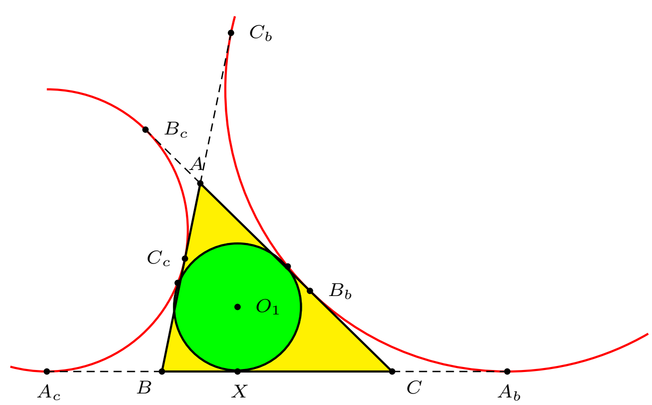

Figure 3

Lemma 2. *The point X has coordinates*

$$(0 : s\sqrt{s-c} - (s-a)\sqrt{s-b} : s\sqrt{s-b} - (s-a)\sqrt{s-c}).$$

Proof. If (O_1) is the circle tangent to (I_b), (I_c), and to BC at X between A_c and A_b, then $A_cX : XA_b = \sqrt{r_c} : \sqrt{r_b} = \sqrt{s-b} : \sqrt{s-c}$. Note that $A_cA_b = b + c$, so that

$$\begin{aligned} BX =& A_cX - A_cB \\ =& \frac{\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}} \cdot (b+c) - (s-a) \\ =& \frac{s\sqrt{s-b} - (s-a)\sqrt{s-c}}{\sqrt{s-b}+\sqrt{s-c}}. \end{aligned}$$

Similarly

$$XC = \frac{s\sqrt{s-c} - (s-a)\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}}.$$

It follows that the point X has coordinates given above. □

Similarly, there are circles $O_2(Y)$ and $O_3(Z)$ tangent to CA at Y and to AB at Z respectively, each also tangent to a pair of excircles. Their coordinates can be written down from those of X by cyclic permutations of a, b, c.

3. The triangle bounded by the polars of the vertices with respect to the excircles

Consider the triangle bounded by the polars of the vertices of ABC with respect to the corresponding excircles. The polar of A with respect to the excircle (I_a) is the line B_aC_a; similarly for the other two polars.

Lemma 3. *The polars of the vertices of ABC with respect to the corresponding excircles bound a triangle with vertices*

$$\begin{aligned} U =& (-a(b+c) : S_C : S_B), \\ V =& (S_C : -b(c+a) : S_A), \\ W =& (S_B : S_A : -c(a+b)). \end{aligned}$$

Proof. The polar of A with respect to the excircle (I_a) is the line B_aC_a, whose barycentric equation is

$$\begin{vmatrix} x & y & z \\ -(s-b) & 0 & s \\ -(s-c) & s & 0 \end{vmatrix} = 0,$$

or

$$sx + (s-c)y + (s-b)z = 0.$$

Similarly, the polars C_bA_b and A_cB_c have equations

$$\begin{aligned} (s-c)x + sy + (s-a)z =& 0, \\ (s-b)x + (s-a)y + sz =& 0. \end{aligned}$$

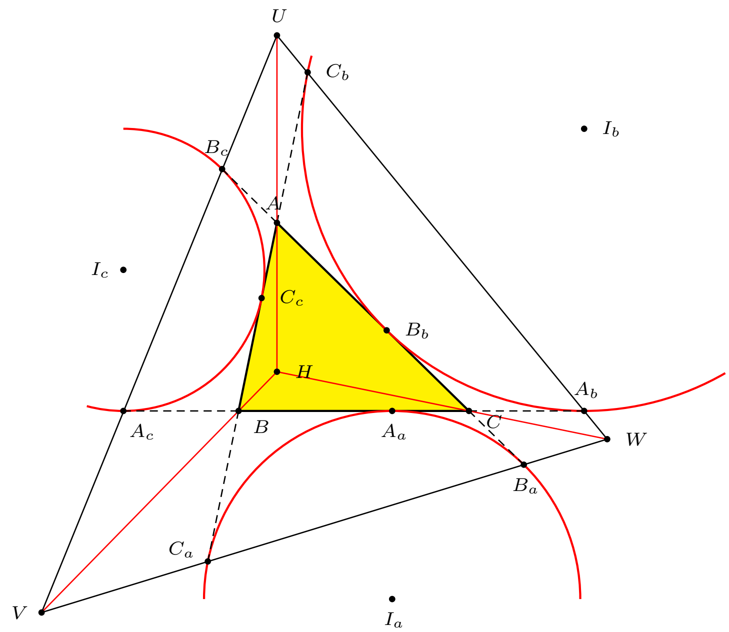

Figure 4

These intersect at the point

$$\begin{aligned} U =&(-a(2s-a) : ab - 2s(s-c) : ca - 2s(s-b)) \\ =&(-2a(b+c) : a^2+b^2-c^2 : c^2+a^2-b^2) \\ =&(-a(b+c) : S_C : S_B). \end{aligned}$$

The coordinates of V and W can be obtained from these by cyclic permutations of a, b, c. □

Corollary 4. *Triangles UVW and ABC are*
(a) *perspective at the orthocenter H,*
(b) *orthologic with centers H and I respectively.*

Proposition 5. *The triangle UVW has circumcenter H and circumradius $2R+r$.*

Proof. Since H, B, V are collinear, HV is perpendicular to CA. Similarly, HW is perpendicular to AB. Since VW makes equal angles with CA and AB, it makes equal angles with HV and HW. This means $HV = HW$. For the same reason, $HU = HV$, and H is the circumcenter of UVW.

Applying the law of sines to triangle AUB_c, we have we have

$$AU = AB_c \cdot \frac{\sin\frac{180^\circ - C}{2}}{\sin\frac{C}{2}} = (s-b)\cot\frac{C}{2} = r_a.$$

The circumradius of UVW is $HU = HA + AU = 2R\cos A + r_a = 2R + r$, as a routine calculation shows. □

Proposition 6. *The triangle UVW and the intouch triangle DEF are homothetic at the point*

$$J = \left(\frac{b+c}{b+c-a} : \frac{c+a}{c+a-b} : \frac{a+b}{a+b-c}\right). \tag{1}$$

The ratio of homothety is $-\frac{2R+r}{r}$.

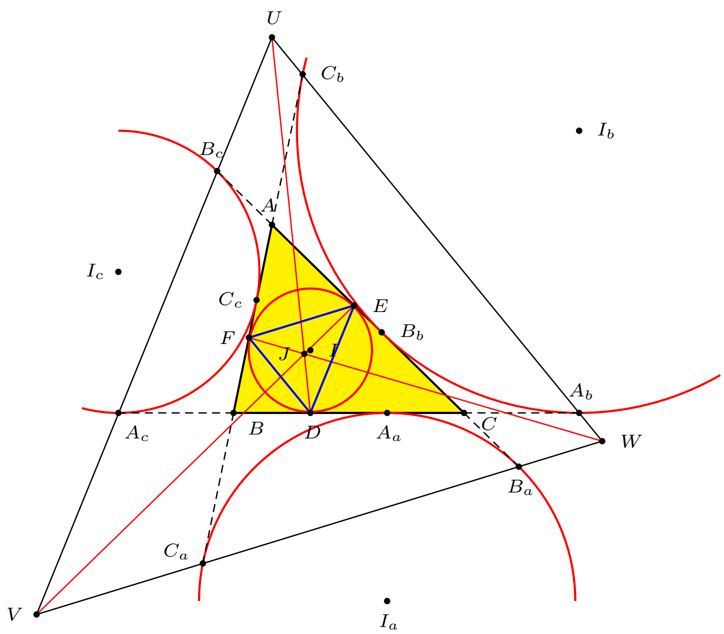

Figure 5

Proof. The homothety follows easily from the parallelism of VW and EF, and of WU, FD, and UV, DE. The homothetic center is the common point J of the lines DU, EV, and FW (see Figure 5). These lines have equations

$$\begin{aligned}
(b-c)(b+c-a)x + (b+c)(c+a-b)y - (b+c)(a+b-c)z =& 0,\\
-(c+a)(b+c-a)x + (c-a)(c+a-b)y + (c+a)(a+b-c)z =& 0,\\
(a+b)(b+c-a)x - (a+b)(c+a-b)y + (a-b)(a+b-c)z =& 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(b+c-a)x : (c+a-b)y : (a+b-c)z\\
=& \begin{vmatrix} c-a & c+a \\ -(a+b) & a-b \end{vmatrix} : \begin{vmatrix} c+a & -(c+a) \\ a+b & a-b \end{vmatrix} : \begin{vmatrix} -(c+a) & c-a \\ a+b & -(a+b) \end{vmatrix}\\
=& 2a(b+c) : 2a(c+a) : 2a(a+b)\\
=& b+c : c+a : a+b.
\end{aligned}$$

The coordinates of the homothetic center J are therefore as in (1) above.

Since the triangles UVF and DEF have circumcircles $H(2R+r)$ and $I(r)$, the ratio of homothety is $-\frac{2R+r}{r}$. The homothetic center J divides IH in the ratio $IJ : JH = r : 2R + r$. $\square$

Remark. The triangle center J appears as X_{226} in Kimberling's list [2].

4. Perspectivity of XYZ and UVW

Theorem 7. *Triangles XYZ and UVW are perspective at a point with coordinates*

$$\left(\frac{S_B}{\sqrt{s-c}}+\frac{S_C}{\sqrt{s-b}}-\frac{a(b+c)}{\sqrt{s-a}} : \frac{S_C}{\sqrt{s-a}}+\frac{S_A}{\sqrt{s-c}}-\frac{b(c+a)}{\sqrt{s-b}} : \frac{S_A}{\sqrt{s-b}}+\frac{S_B}{\sqrt{s-a}}-\frac{c(a+b)}{\sqrt{s-c}}\right).$$

Proof. With the coordinates of X and U from Lemmas 1 and 2, the line XU has equation

$$\begin{vmatrix} x & y & z \\ -a(b+c) & S_C & S_B \\ 0 & s\sqrt{s-c}-(s-a)\sqrt{s-b} & s\sqrt{s-b}-(s-a)\sqrt{s-c} \end{vmatrix} = 0.$$

Since the coefficient of x is

$$\begin{aligned} & (s(S_B+S_C)-aS_B)\sqrt{s-b}-(s(S_B+S_C)-aS_C)\sqrt{s-c} \\ =& a((as-S_B)\sqrt{s-b}-(as-S_C)\sqrt{s-c}) \\ =& a(b+c)((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c}). \end{aligned}$$

From this, we easily simplify the above equation as

$$\begin{aligned} & ((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c})x \\ & +(s\sqrt{s-b}-(s-a)\sqrt{s-c})y+((s-a)\sqrt{s-b}-s\sqrt{s-c})z=0. \end{aligned}$$

With $u=\sqrt{s-a}$, $v=\sqrt{s-b}$, and $w=\sqrt{s-c}$, we rewrite this as

$$-vw(v-w)x+(v(u^2+v^2+w^2)-u^2w)y+(u^2v-w(u^2+v^2+w^2))z=0. \tag{2}$$

Similarly the equations of the lines VY, WZ are

$$(v^2w-u(u^2+v^2+w^2))x-wu(w-u)y+(w(u^2+v^2+w^2)-uv^2)z=0, \tag{3}$$

$$(u(u^2+v^2+w^2)-vw^2)x+(w^2u-v(u^2+v^2+w^2))y-uv(u-v)z=0. \tag{4}$$

It is clear that the sum of the coefficients of x (respectively y and z) in (2), (3) and (4) is zero. The system of equations therefore has a nontrivial solution. Solving them, we obtain the coordinates of the common point of the lines XU, YV, ZW as

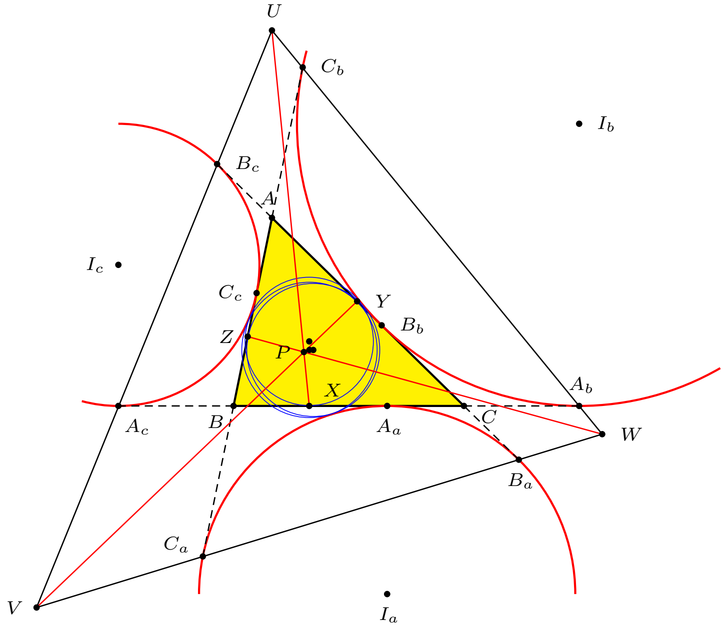

Figure 6

$$
\begin{aligned}
&x : y : z \\
=&uv(v^2(u^2+v^2+w^2)-w^2u^2)+wu(w^2(u^2+v^2+w^2)-u^2v^2) \\
&-vw(v^2+w^2)(2u^2+v^2+w^2) \\
:&vw(w^2(u^2+v^2+w^2)-u^2v^2)+uv(u^2((u^2+v^2+w^2)-v^2w^2) \\
&-wu(w^2+u^2)(u^2+2v^2+w^2) \\
:&wu(u^2(u^2+v^2+w^2)-v^2w^2)+vw(v^2((u^2+v^2+w^2)-w^2u^2) \\
&-uv(u^2+v^2)(u^2+v^2+2w^2)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{(s-b)s-(s-c)(s-a)}{w}+\frac{(s-c)s-(s-a)(s-b)}{v}-\frac{a(b+c)}{u} \\
:&\frac{(s-c)s-(s-a)(s-b)}{u}+\frac{(s-a)s-(s-b)(s-c)}{w}-\frac{b(c+a)}{v} \\
:&\frac{(s-a)s-(s-b)(s-c)}{v}+\frac{(s-b)s-(s-c)(s-a)}{u}-\frac{c(a+b)}{w} \\
=&\frac{S_B}{w}+\frac{S_C}{v}-\frac{a(b+c)}{u}:\frac{S_C}{u}+\frac{S_A}{w}-\frac{b(c+a)}{v}:\frac{S_A}{v}+\frac{S_B}{u}-\frac{c(a+b)}{w}.
\end{aligned}
$$

□

The triangle center constructed in Theorem 3 above does not appear in [2].

5. Another construction

Given three circles $O_i(r_i)$, $i = 1, 2, 3$, on one side of a line $\mathcal{L}$, tangent to the line, we construct a circle $O(r)$, tangent to each of these three circles externally.

For $i = 1, 2, 3$, let the circle $O_i(r_i)$ touch $\mathcal{L}$ at S_i and $O(r)$ at T_i. If the line S_1T_1 meets the circle (O) again at T, then the tangent to (O) at T is a line $\mathcal{L}'$ parallel to $\mathcal{L}$. Hence, T, T_2, S_2 are collinear; so are T, T_3, S_3. Since the line T_2T_3 is antiparallel to $\mathcal{L}'$ with respect to the lines TT_2 and TT_3, it is also antiparallel to $\mathcal{L}$ with respect to the lines TS_2 and TS_3, and the points T_2, T_3, S_3, S_2 are concyclic. From $TT_2 \cdot TS_2 = TT_3 \cdot TS_3$, we conclude that the point T lies on the radical axis of the circles $O_2(r_2)$ and $O_3(r_3)$, which is the perpendicular from the midpoint of S_2S_3 to the line O_2O_3. For the same reason, it also lies on the radical axis of the circles $O_3(r_3)$ and $O_1(r_1)$, which is the perpendicular from the midpoint of S_1S_3 to the line O_1O_3. Hence T is the radical center of the three given circles $O_i(r_i)$, $i = 1, 2, 3$, and the circle $T_1T_2T_3$ is the image of the line $\mathcal{L}$ under the inversion with center T and power $TT_1 \cdot TS_1$. From this, the required circle (O) can be constructed as follows.

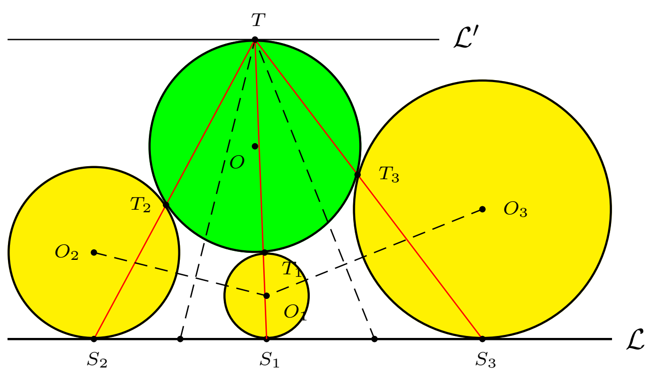

Figure 7

Construction 8. *Construct the perpendicular from the midpoint of S_1S_2 to O_1O_2, and from the midpoint of S_1S_3 to O_1O_3. Let T be the intersection of these two perpendiculars. For $i = 1, 2, 3$, let T_i be the intersection of the line TS_i with the circle (O_i). The required circle (O) is the one through T_1, T_2, T_3* (see Figure 7).

6. Circles tangent to the incircle and two excircles

We apply Construction 2 to obtain the circle tangent to the incircle (I) and the excircles (I_b) and (I_c). Let the incircle (I) touch the sides BC, CA, AB at D, E, F respectively.

Proposition 9. *The radical center of (I), (I_b), (I_c) is the point*

$$J_a = (b + c : c - a : b - a).$$

This is also the midpoint of the segment DU.

Proof. The radical axis of (I) and (I_b) is the line joining the midpoints of the segments DA_b and FC_b. These midpoints have coordinates $(0 : a-c : a+c)$ and $(c+a : c-a : 0)$. This line has equation

$$-(c-a)x + (c+a)y + (c-a)z = 0.$$

Similarly, the radical axis of (I) and (I_c) is the line

$$(a-b)x - (a-b)y + (a+b)z = 0.$$

The radical center J_a of the three circles is the intersection of these two radical axes. Its coordinates are as given above.

By Construction 2, J_a is the intersection of the lines through the midpoints of A_bD and A_cD perpendicular to II_b and II_c respectively. As such, it is the midpoint of DU. □

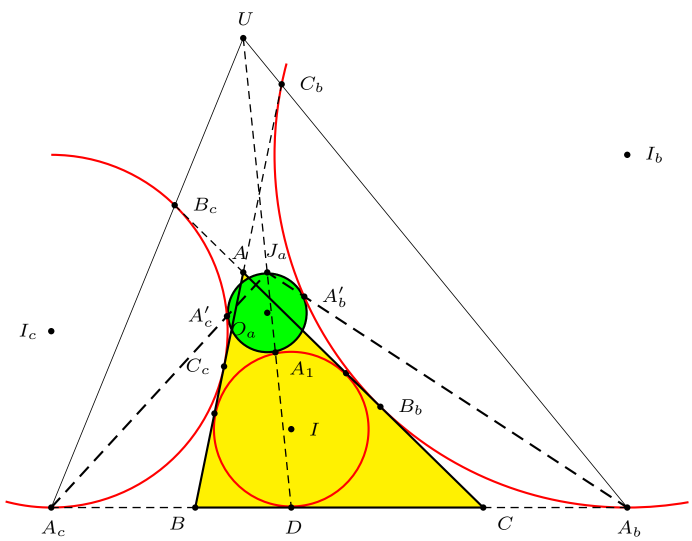

Figure 8.

The lines J_aD, J_aA_b and J_aA_c intersect the circles (I), (I_b) and (I_c) respectively again at

$$\begin{aligned}
A_1 =& ((b+c)^2(s-b)(s-c) : c^2(s-a)(s-c) : b^2(s-a)(s-b)),\\
A'_b =& ((b+c)^2 s(s-c) : -(ab-c(s-a))^2 : b^2 s(s-a)),\\
A'_c =& ((b+c)^2 s(s-b) : c^2 s(s-a) : -(ca-b(s-a))^2).
\end{aligned}$$

The circle through these points is the one tangent to (I), (I_b), and (I_c) (see Figure 8). It has radius $\frac{a}{b+c} \cdot \frac{(s-a)^2+r_a^2}{4r_a}$.

In the same way, we have a circle (O_b) tangent to (I), (I_c), (I_a) respectively at B_1, B'_c, B'_a, and passing through the radical center J_b of these three circles, and another circle (O_c) tangent to (I), (I_a), (I_b) respectively at C_1, C'_a, C'_b, passing through the radical center J_c of the circles. J_b and J_c are respectively the midpoints of the segments EV and FW. The coordinates of J_b, J_c, B_1, C_1 are as follows.

$$\begin{aligned}
J_b =&(c-b : c+a : a-b),\\
J_c =&(b-c : a-c : a+b);\\
B_1 =&(a^2(s-b)(s-c) : (c+a)^2(s-c)(s-a) : c^2(s-a)(s-b)),\\
C_1 =&(a^2(s-b)(s-c) : b^2(s-c)(s-a) : (a+b)^2(s-a)(s-b)).
\end{aligned}$$

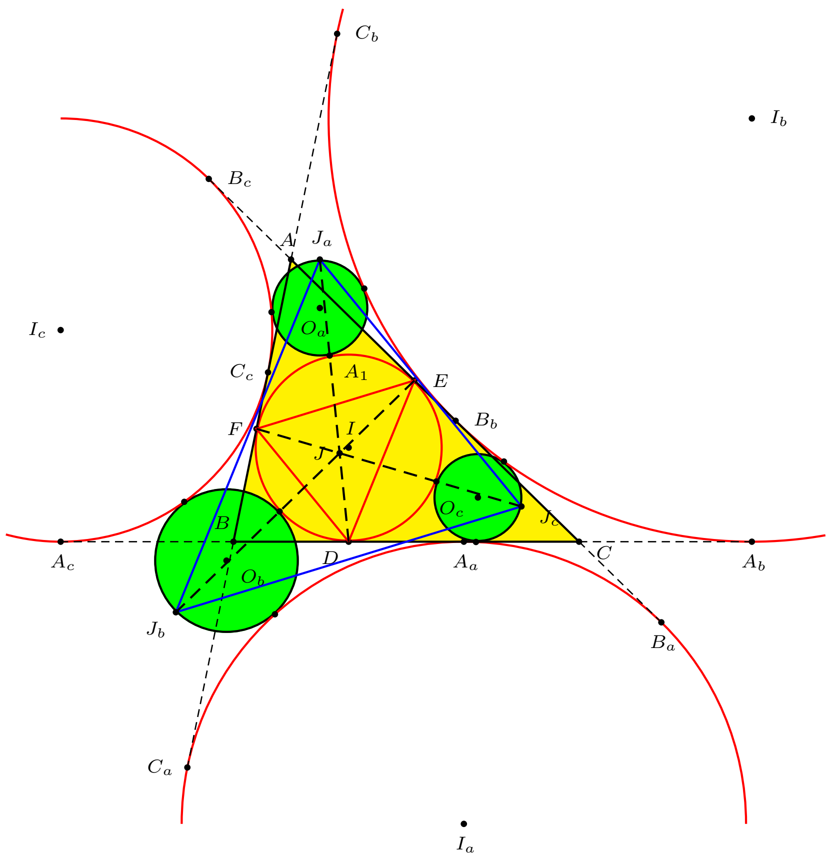

Figure 9.

Proposition 10. *The triangle $J_aJ_bJ_c$ is the image of the intouch triangle under the homothety* $\mathsf{h}(J, -\frac{R}{r})$.

Proof. Since UVW and DEF are homothetic at J, and J_a, J_b, J_c are the midpoints of DU, EV, FW respectively, it is clear that $J_aJ_bJ_c$ and DEF are also homothetic at the same J. Note that $J_bJ_c = \frac{1}{2}(VW - EF)$. The circumradius of $J_aJ_bJ_c$ is $\frac{1}{2}((2R+r)-r) = R$. The ratio of homothety of $J_aJ_bJ_c$ and DEF is $\frac{-R}{r}$. $\square$

Corollary 11. *J is the radical center of the circles (O_a), (O_b), (O_c).*

Proof. Note that $JJ_a \cdot JA_1 = \frac{R}{r} \cdot DJ \cdot JA_1$. This is $\frac{R}{r}$ times the power of J with respect to the incircle. The same is true for $JJ_b \cdot JB_1$ and $JJ_c \cdot JC_1$. This shows that J is the radical center of the circles (O_a), (O_b), (O_c). □

Since the incircle (I) is the inner Apollonius circle and the circumcircle (O_i), $i = 1, 2, 3$, it follows that $J_aJ_bJ_c$ is the outer Apollonius circle to the same three circles (see Figure 10). The center O' of the circle $J_aJ_bJ_c$ is the midpoint between the circumcenters of DEF and UVW, namely, the midpoint of IH. It is the triangle center X_{946} in [2].

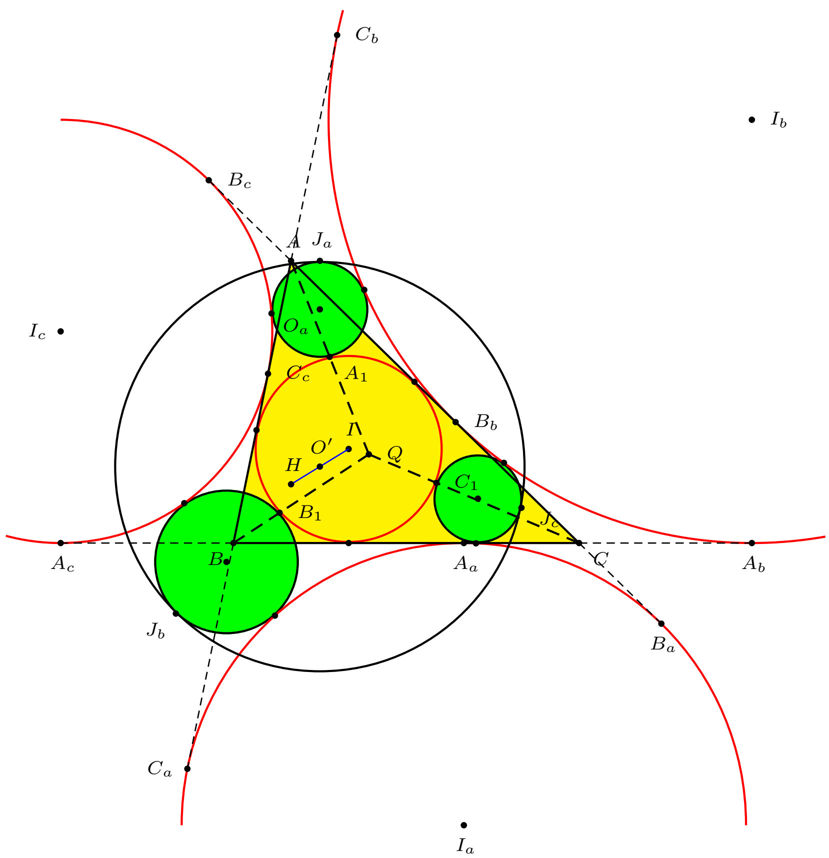

Figure 10.

Proposition 12. *$A_1B_1C_1$ is perspective with ABC at the point*

$$Q = \left(\frac{1}{a^2(s-a)} : \frac{1}{b^2(s-b)} : \frac{1}{c^2(s-c)}\right),$$

which is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle.

This is clear from the coordinates of A_1, B_1, C_1. The perspector Q is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle. It is not in the current listing in [2].

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Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Juan Carlos Salazar: Deceased

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Ten Concurrent Euler Lines

Nikolai Ivanov Beluhov

Dedicated to Svetlozar Doichev

Abstract. Let F_1 and F_2 denote the Fermat-Toricelli points of a given triangle ABC. We prove that the Euler lines of the 10 triangles with vertices chosen from A, B, C, F_1, F_2 (three at a time) are concurrent at the centroid of triangle ABC.

Given a (positively oriented) triangle ABC, construct externally on its sides three equilateral triangles BCT_a, CAT_b, and ABT_c with centers N_a, N_b and N_c respectively (see Figure 1). As is well known, triangle $N_aN_bN_c$ is equilateral. We call this the first Napoleon triangle of ABC.

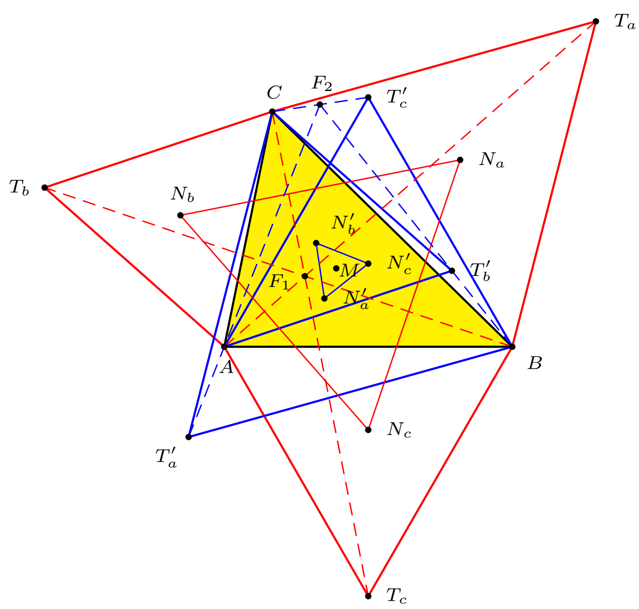

Figure 1.

The same construction performed internally gives equilateral triangles BCT'_a, CAT'_b and ABT'_c with centers N'_a, N'_b, and N'_c respectively, leading to the second Napoleon triangle $N'_aN'_bN'_c$. The centers of both Napoleon triangles coincide with the centroid M of triangle ABC.

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The lines AT_a, BT_b and CT_c make equal pairwise angles, and meet together with the circumcircles of triangles BCT_a, CAT_b, and ABT_c at the first Fermat-Toricelli point F_1. Denoting by $\angle XYZ$ the oriented angle $\angle(YX, YZ)$, we have $\angle AF_1B = \angle BF_1C = \angle CF_1A = 120°$. Analogously, the second Fermat-Toricelli point satisfies $\angle AF_2B = \angle BF_2C = \angle CF_2A = 60°$.

Clearly, the sides of the Napoleon triangles are the perpendicular bisectors of the segments joining their respective Fermat-Toricelli points with the vertices of triangle ABC (as these segments are the common chords of the circumcircles of the the equilateral triangles ABT_c, BCT_a, etc).

We prove the following interesting theorem.

Theorem *The Euler lines of the ten triangles with vertices from the set* $\{A,\ B,\ C,\ F_1,\ F_2\}$ *are concurrent at the centroid* M *of triangle* ABC.

Proof. We divide the ten triangles in three types:
(I): Triangle ABC by itself, for which the claim is trivial.
(II): The six triangles each with two vertices from the set $\{A, B, C\}$ and the remaining vertex one of the points F_1, F_2.
(III) The three triangles each with vertices F_1, F_2, and one from $\{A, B, C\}$.

For type (II), it is enough to consider triangle ABF_1. Let M_c be its centroid and M_C be the midpoint of the segment AB. Notice also that N_c is the circumcenter of triangle ABF_1 (see Figure 2).

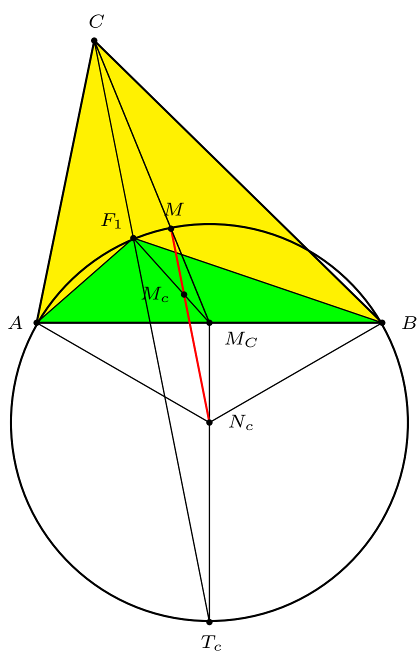

Figure 2.

Now, the points C, F_1 and T_c are collinear, and the points M, M_c and N_c divide the segments M_CC, M_CF_1 and M_CT_c in the same ratio $1 : 2$. Therefore, they are collinear, and the Euler line of triangle ABF contains M.

For type (III), it is enough to consider triangle CF_1F_2. Let M_c and O_c be its centroid and circumcenter. Let also M_C and M_F be the midpoints of AB and F_1F_2. Notice that O_c is the intersection of N_aN_b and $N'_aN'_b$ as perpendicular bisectors of F_1C and F_2C. Let also P be the intersection of N_bN_c and $N'_cN'_a$, and F' be the reflection of F_1 in M_C (see Figure 3).

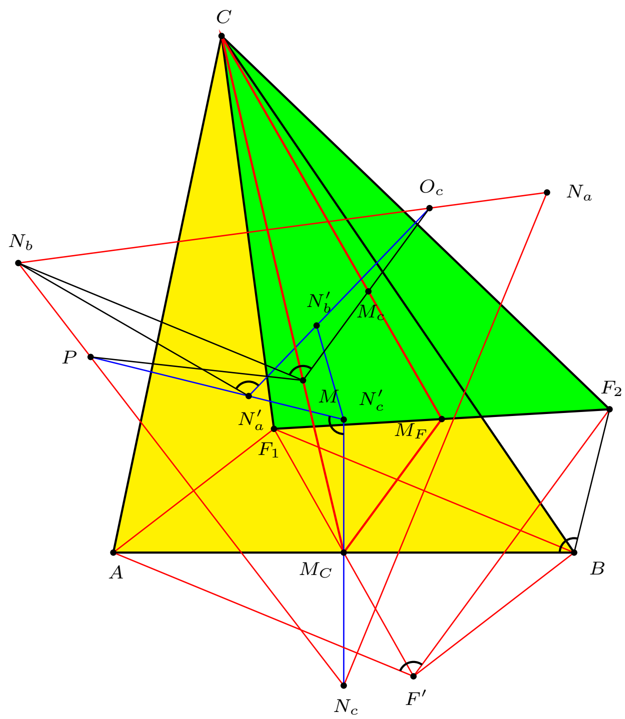

Figure 3.

The rotation of center M and angle 120° maps the lines N_aN_b and $N'_aN'_b$ into N_bN_c and $N'_cN'_a$ respectively. Therefore, it maps O_c to P, and $\angle O_cMP = 120^\circ$. Since $\angle O_cN'_aP = 120^\circ$, the four points O_c, M, N'_a, P are concyclic. The circle containing them also contains N_b since $\angle PN_bO_c = 60^\circ$. Therefore, $\angle O_cMN_b = \angle O_cN'_aN_b$.

The same rotation maps angle $O_cN'_aN_b$ onto angle PN'_cN_c, yielding $\angle O_cN'_aN_b = \angle PN'_cN_c$. Since $PN'_c \perp BF_2$ and $N_cN'_c \perp BA$, $\angle PN'_cN_c = \angle F_2BA$.

Since $\angle BF'A = \angle AF_1B = 120^\circ = 180^\circ - \angle AF_2B$, the quadrilateral AF_2BF' is also cyclic and $\angle F_2BA = \angle F_2F'A$. Thus, $\angle F_2F'A = \angle O_cMN_b$.

Now, $AF' \| F_1B \perp N_aN_c$ and $N_bM \perp N_aN_c$ yield $AF' \| N_bM$. This, together with $\angle F_2F'A = \angle O_cMN_b$, yields $F'F_2 \| MO_c$.

Notice now that the points M_c and M divide the segments CM_F and CM_C in ratio $1:2$, therefore $M_cM \| M_CM_F$. The same argument, applied to the segments F_1F_2 and F_1F' with ratio $1:1$, yields $M_CM_F \| F'F_2$.

In conclusion, we obtain $M_cM \| F'F_2 \| MO_c$. The collinearity of the points M_c, M and O_c follows. $\square$

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Nikolai Ivanov Beluhov: "Bulgarsko opalchenie" str. 5, Stara Zagora 6000, Bulgaria
E-mail address: `nbehulov@abv.bg`

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On the Possibility of Trigonometric Proofs of the Pythagorean Theorem

Jason Zimba

Abstract. The identity $\cos^2 x + \sin^2 x = 1$ can be derived independently of the Pythagorean theorem, despite common beliefs to the contrary.

1. Introduction

In a remarkable 1940 treatise entitled *The Pythagorean Proposition*, Elisha Scott Loomis (1852–1940) presented literally hundreds of distinct proofs of the Pythagorean theorem. Loomis provided both "algebraic proofs" that make use of similar triangles, as well as "geometric proofs" that make use of area reasoning. Notably, none of the proofs in Loomis's book were of a style one would be tempted to call "trigonometric". Indeed, toward the end of his book ([1, p.244]) Loomis asserted that all such proofs are circular:

> There are no trigonometric proofs [of the Pythagorean theorem], because all of the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$ etc.

Along the same lines but more recently, in the discussion page behind Wikipedia's Pythagorean theorem entry, one may read that a purported proof was once deleted from the entry because it "...depend[ed] on the veracity of the identity $\sin^2 x + \cos^2 x = 1$, which is the Pythagorean theorem ..." ([5]).

Another highly ranked Internet resource for the Pythagorean theorem is Cut-The-Knot.org, which lists dozens of interesting proofs ([2]). The site has a page devoted to fallacious proofs of the Pythagorean theorem. On this page it is again asserted that the identity $\cos^2 x + \sin^2 x = 1$ cannot be used to prove the Pythagorean theorem, because the identity "is based on the Pythagorean theorem, to start with" ([3]).

All of these quotations seem to reflect an implicit belief that the relation $\cos^2 x + \sin^2 x = 1$ cannot be derived independently of the Pythagorean theorem. For the record, this belief is false. We show in this article how to derive this identity independently of the Pythagorean theorem.

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The author would like to thank an anonymous referee for a number of suggestions, both general and specific, which greatly improved the manuscript during the editorial process.

2. Sine and cosine of acute angles

We begin by defining the sine and cosine functions for positive acute angles, independently of the Pythagorean theorem, as ratios of sides of similar right triangles. Given $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, let $\mathcal{R}_\alpha$ be the set of all right triangles containing an angle of measure α, and let $\mathbf{T}$ be one such triangle. Because the angle measures in $\mathbf{T}$ add up to π (see Euclid's *Elements*, I.32), [1] $\mathbf{T}$ must have angle measures $\frac{\pi}{2}$, $\frac{\pi}{2} - \alpha$ and α. The side opposite to the right angle is the longest side (see *Elements* I.19), called the hypotenuse of the right triangle; we denote its length by $H_{\mathbf{T}}$.

First consider the case $\alpha \neq \frac{\pi}{4}$. The three angle measures of $\mathbf{T}$ are distinct, so that the three side lengths are also distinct (see *Elements*, I.19). Let $A_{\mathbf{T}}$ denote the length of the side of $\mathbf{T}$ adjacent to the angle of measure α, and $O_{\mathbf{T}}$ the length of the opposite side. If $\mathbf{T}$ and $\mathbf{S}$ are any two triangles in $\mathcal{R}_\alpha$, then because $\mathbf{T}$ and $\mathbf{S}$ have angles of equal measures, corresponding side ratios in $\mathbf{S}$ and $\mathbf{T}$ are equal:

$$\frac{A_{\mathbf{T}}}{H_{\mathbf{T}}} = \frac{A_{\mathbf{S}}}{H_{\mathbf{S}}} \quad \text{and} \quad \frac{O_{\mathbf{T}}}{H_{\mathbf{T}}} = \frac{O_{\mathbf{S}}}{H_{\mathbf{S}}}$$

(see *Elements*, VI.4). Therefore, for $\alpha \neq \frac{\pi}{4}$ in the range $\left(0,\ \frac{\pi}{2}\right)$, we may define

$$\cos\alpha := \frac{A}{H} \quad \text{and} \quad \sin\alpha := \frac{O}{H},$$

where the ratios may be computed using any triangle in $\mathcal{R}_\alpha$. [2]

We next consider the case $\alpha = \frac{\pi}{4}$. Any right triangle containing an angle of measure $\frac{\pi}{4}$ must in fact have two angles of measure $\frac{\pi}{4}$ (see *Elements*, I.32), so its three angles have measures $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$. Such a triangle is isosceles (see *Elements*, I.6), and therefore has only two distinct side lengths, H and L, where $H > L$ is the length of the side opposite to the right angle and L is the common length shared by the two other sides (see *Elements*, I.19). Because any two right triangles containing angle $\alpha = \frac{\pi}{4}$ have the same three angle measures, any two such triangles are similar and have the same ratio $\frac{L}{H}$ (see *Elements*, VI.4). Now therefore define

$$\cos\frac{\pi}{4} := \frac{L}{H} \quad \text{and} \quad \sin\frac{\pi}{4} := \frac{L}{H},$$

where again the ratios may be computed using any triangle in $\mathcal{R}_{\pi/4}$.

The ratios $\frac{A}{H}$, $\frac{O}{H}$, and $\frac{L}{H}$ are all strictly positive, for the simple reason that a triangle always has sides of positive length (at least in the simple conception of a triangle that operates here). These ratios are also all strictly less than unity, because H is the longest side (*Elements*, I.19 again). Altogether then, we have defined the

[1] The Pythagorean theorem is proved in Book I of the *Elements* as Proposition I.47, and the theorem is proved again in Book VI using similarity arguments as Proposition VI.31. References to the Elements should not be taken to mean that we are adopting a classical perspective on geometry. The references are only meant to reassure the reader that the annotated claims do not rely on the Pythagorean theorem (by showing that they precede the Pythagorean theorem in Euclid's exposition).

[2] We shall henceforth assume that for any $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, there exists a right triangle containing an angle of measure α. The reader wishing to adopt a more cautious or classical viewpoint may replace the real interval $\left(0,\ \frac{\pi}{2}\right)$ everywhere throughout the paper by the set $\langle 0,\ \frac{\pi}{2}\rangle$ defined as the set of all $\alpha \in \left(0,\ \frac{\pi}{2}\right)$ for which there exists a right triangle containing an angle of measure α.

functions $\cos : \left(0,\ \frac{\pi}{2}\right) \to (0,1)$ and $\sin : \left(0,\ \frac{\pi}{2}\right) \to (0,1)$ independently of the Pythagorean theorem.

Because sine and cosine as defined above are independent of the Pythagorean theorem, any proof of the Pythagorean theorem may validly employ these functions. Indeed, Elements VI.8 very quickly leads to the Pythagorean theorem with the benefit of trigonometric notation. [3] However, our precise concern in this paper is to derive trigonometric identities, and to this we now turn.

3. Subtraction formulas

The sine and cosine functions defined above obey the following subtraction formulas, valid for all $\alpha,\ \beta \in \left(0,\ \frac{\pi}{2}\right)$ with $\alpha - \beta$ also in $\left(0,\ \frac{\pi}{2}\right)$:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \tag{1}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{2}$$

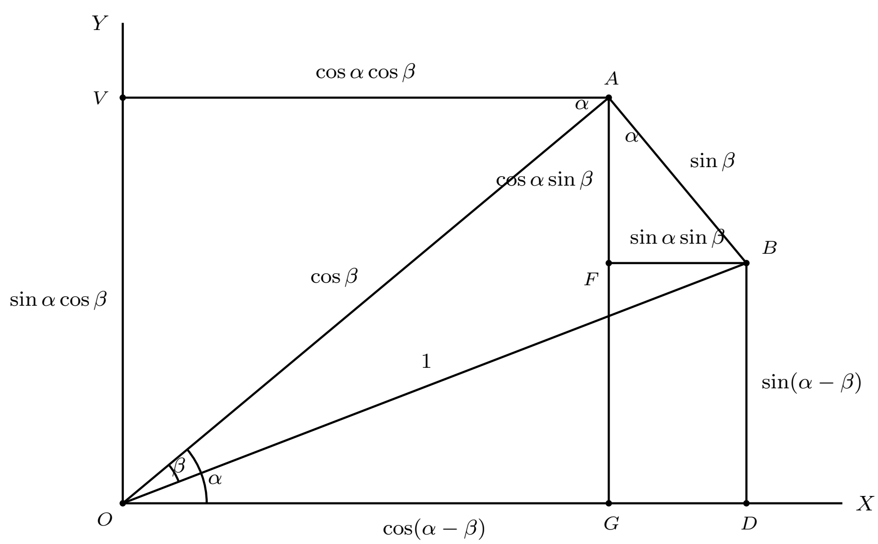

Figure 1.

The derivation of these formulas, as illustrated in Figure 1, is a textbook exercise. It is independent of the Pythagorean theorem, for although there are three hypotenuses OA, OB, and AB, their lengths are not calculated from the Pythagorean theorem, but rather from the sine and cosine we have just defined. Thus, assigning $OB = 1$, we have $OA = \cos\beta$ and $AB = \sin\beta$. The lengths of the horizontal and vertical segments are easily determined as indicated in Figure 1.

[3]See proof #6 in [2], specifically the observation attributed to R. M. Mentock.

4. The Pythagorean theorem from the subtraction formula

It is tempting to try to derive the identity $\cos^2 x + \sin^2 x = 1$ by setting $\alpha = \beta = x$ and $\cos 0 = 1$ in (1). [4] This would not be valid, however, because the domain of the cosine function does not include zero. But there is a way around this problem. Given any $x \in \left(0,\ \frac{\pi}{2}\right)$, let y be any number with $0 < y < x < \frac{\pi}{2}$. Then x, y, and $x - y$ are all in $\left(0,\ \frac{\pi}{2}\right)$. Therefore, applying (1) repeatedly, we have

$$\begin{aligned}
\cos y &= \cos(x - (x - y)) \\
&= \cos x \cos(x - y) + \sin x \sin(x - y) \\
&= \cos x(\cos x \cos y + \sin x \sin y) + \sin x(\sin x \cos y - \cos x \sin y) \\
&= (\cos^2 x + \sin^2 x) \cos y.
\end{aligned}$$

From this, $\cos^2 x + \sin^2 x = 1$.

5. Proving the Pythagorean theorem as a corollary

Because the foregoing proof is independent of the Pythagorean theorem, we may deduce the Pythagorean theorem as a corollary without risk of *petitio principii*. The identity $\cos^2 x + \sin^2 x = 1$ applied to a right triangle with legs a, b and hypotenuse c gives $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$.

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Jason Zimba: Bennington College, 1 College Drive, Bennington, Vermont 05201, USA
E-mail address: `jzimba@gmail.com`

[4] A similar maneuver was attempted in 1914 by J. Versluys, who took $\alpha + \beta = \frac{\pi}{2}$ and $\sin \frac{\pi}{2} = 1$ in (2). Versluys cited Schur as the source of this idea (see [4, p.94]). A sign of trouble with the approach of Versluys/Schur is that the diagram typically used to derive the addition formula cannot be drawn for the case $\alpha + \beta = \frac{\pi}{2}$.

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On the Construction of a Triangle from the Feet of Its Angle Bisectors

Alexey V. Ustinov

Abstract. We give simple examples of triangles not constructible by ruler and compass from the feet of its angle bisectors when the latter form a triangle with an angle of 60° or 120°.

Given a triangle ABC with sides a, b, c, we want to construct a triangle $A'B'C'$ such that that segments AA', BB' and CC' are its angle bisectors, internal or external. Restricted to internal bisectors, this is Problem 138 of Wernick's list [3] (see also [2]). Yiu [4] has given a conic solution of the problem. Implicit in this is the impossibility of a ruler-and-compass construction in general, though in the case of a right angled triangle, this is indeed possible ([4, §7]). The purpose of this note is to give simple examples of $A'B'C'$ not constructible from ABC by ruler-and-compass when the latter contains a 60° or 120° angle.

Following [4] we denote by $(x:y:z)$ the barycentric coordinates of the incenter of triangle $A'B'C'$ with respect to triangle ABC, when A, B, C are the feet of the internal angle bisectors, or an excenter when one of A, B, C is the foot of an internal bisector and the remaining two external. The vertices of triangle $A'B'C'$ have coordinates $(-x,y,z)$, $(x,-y,z)$, $(x,y,-z)$. These coordinates satisfy the following equations (see [4, §3]):

$$\begin{aligned}
-x(c^2y^2-b^2z^2)+yz((c^2+a^2-b^2)y-(a^2+b^2-c^2)z) &= 0,\\
-y(a^2z^2-c^2x^2)+xz((a^2+b^2-c^2)z-(b^2+c^2-a^2)x) &= 0, \qquad (1)\\
-z(b^2x^2-a^2y^2)+xy((b^2+c^2-a^2)x-(c^2+a^2-b^2)y) &= 0.
\end{aligned}$$

These three equations being dependent, it is enough to consider the last two. Elimination of z from these leads to a quartic equation in x and y. This fact already suggests the impossibility of a ruler-and-compass construction. However, this can be made precise if we put $c^2=a^2-ab+b^2$. In this case, angle C is 60° and we obtain, by writing $bx=t\cdot ay$, the following cubic equation in t:

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$$3(a-b)bt^3 - (a^2 - 4ab + b^2)t^2 + (a^2 - 4ab + b^2)t + 3a(a-b) = 0.$$

With $a = 8$, $b = 7$ (so that $c = \sqrt{57}$ and angle C is $60°$), this reduces to

$$7t^3 + 37t^2 - 37t + 8 = 0,$$

which is easily seen not to have rational roots. The roots of the cubic equation are not constructible by ruler and compass (see [1, Chapter 3]). Explicit solutions can be realized by taking $A = (7,0)$, $B = (4, 4\sqrt{3})$, $C = (0,0)$, with resulting $A'B'C'$ and the corresponding incenter (or excenter) exhibited in the table below.

t	$0.5492\cdots$	$0.3370\cdots$	$-6.1721\cdots$
A'	$(-0.3891, 6.8375)$	$(1.3112, 6.9711)$	$(5.8348, 0.7573)$
B'	$(1.4670, -25.7766)$	$(5.5301, 29.3999)$	$(7.6694, 0.9954)$
C'	$(8.5071, 7.0213)$	$(6.6557, 6.8857)$	$(6.3481, -0.9692)$
I	$(3.6999, 3.0537)$ incenter	$(3.7956, 3.9267)$ $B'-$ excenter	$(3.7956, 3.9267)$ $A'-$ excenter

On the other hand, if $c^2 = a^2 + ab + b^2$, the eliminant of z from (1) is also a cubic (in x and y) which, with the substitution $bx = t \cdot ay$, reduces to

$$3(a+b)bt^3 - (a^2 + 4ab + b^2)t^2 - (a^2 - 4ab + b^2)t + 3a(a+b) = 0.$$

With $a = 2$, $b = 1$ (so that $c = \sqrt{7}$ and angle C is $120°$), this reduces to

$$9t^3 - 13t^2 - 13t + 18 = 0,$$

with three irrational roots. Explicit solutions can be realized by taking $A = (1,0)$, $B = (-1, \sqrt{3})$, $C = (0,0)$, with resulting $A'B'C'$ and the corresponding excenter exhibited in the table below.

t	$1.0943\cdots$	$1.5382\cdots$	$-1.1881\cdots$
A'	$(0.6876, -0.3735)$	$(5.2374, -2.2253)$	$(0.0436, 0.0549)$
B'	$(-1.4112, 0.7665)$	$(1.2080, -0.5132)$	$(-0.0555, -0.0699)$
C'	$(0.1791, 0.2609)$	$(0.7473, 0.6234)$	$(0.1143, -0.0586)$
I	$(0.1791, 0.2609)$ $C'-$ excenter	$(0.3863, 0.3222)$ $A'-$ excenter	$(-0.1261, 0.0646)$ $C'-$ excenter

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Alexey V. Ustinov: Institute of Applied Mathematics, Dzerzhinskogo 54, Khabarovsk 680000, Russia

E-mail address: ustinov.alexey@gmail.com

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Pythagorean Triangles with Square of Perimeter Equal to an Integer Multiple of Area

John F. Goehl, Jr.

Abstract. We determine all primitive Pythagorean triangles with square on perimeter equal to an integer multiple of its area.

Complete solutions can be found for several special cases of the problem of solving $P^2 = nA$, where P is the perimeter and A is the area of an integer-sided triangle, and n is an integer. The general problem is considered in a recent paper [1]. We consider the case of right triangles. Let the sides be a, b, and c, where c is the hypotenuse. We require

$$n = \frac{2(a+b+c)^2}{ab}.$$

By the homogeneity of the problem, it is enough to consider primitive Pythagorean triangles. It is well known that there are positive integers p and q, relatively prime and of different parity, such that

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.$$

With these, $n = \frac{4p(p+q)}{q(p-q)} = \frac{4t(t+1)}{t-1}$, where $t = \frac{p}{q}$. Rewriting this as

$$4t^2 - (n-4)t + n = 0,$$

we obtain

$$t = \frac{(n-4) \pm d}{8},$$

where

$$d^2 = (n-4)^2 - 16n = (n-12)^2 - 128. \tag{1}$$

Since t is rational, d must be an integer (which we may assume positive). Equation (1) may be rewritten as

$$(n-12-d)(n-12+d) = 128 = 2^7.$$

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From this,

$$n - 12 - d = 2^k,$$
$$n - 12 + d = 2^{7-k},$$

for $k = 1, 2, 3$. We have

$$t = \frac{n-4+d}{8} = 2^{4-k} + 1 \quad \text{or} \quad t = \frac{n-4-d}{8} = \frac{2^k+8}{8}.$$

Since p and q are relatively prime integers of different parity, we exclude the cases when t is an odd integer. Thus, the primitive Pythagorean triangles solving $P^2 = nA$ are precisely those shown in the table below.

k	t	(p,q)	(a,b,c)	n	A	P
1	$\frac{5}{4}$	$(5,4)$	$(9,40,41)$	45	180	90
2	$\frac{3}{2}$	$(3,2)$	$(5,12,13)$	30	30	30
3	2	$(2,1)$	$(3,4,5)$	24	6	12

Among these three solutions, only in the case of $(3,4,5)$ can the square on the perimeter be tessellated by n copies of the triangle.

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John F. Goehl, Jr.: Department of Physical Sciences, Barry University, 11300 NE Second Avenue, Miami Shores, Florida 33161, USA

E-mail address: jgoehl@mail.barry.edu

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Trilinear Polars and Antiparallels

Shao-Cheng Liu

Abstract. We study the triangle bounded by the antiparallels to the sidelines of a given triangle ABC through the intercepts of the trilinear polar of a point P other than the centroid G. We show that this triangle is perspective with the reference triangle, and also study the condition of concurrency of the antiparallels. Finally, we also study the configuration of induced GP-lines and obtain an interesting conjugation of finite points other than G.

1. Perspector of a triangle bounded by antiparallels

We use the barycentric coordinates with respect to triangle ABC throughout. Let $P = (u : v : w)$ be a finite point in the plane of ABC, distinct from its centroid G. The trilinear polar of P is the line

$$\mathcal{L}: \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,$$

which intersects the sidelines BC, CA, AB respectively at

$$P_a = (0 : v : -w), \quad P_b = (-u : 0 : w), \quad P_c = (u : -v : 0).$$

The lines through P_a, P_b, P_c antiparallel to the respective sidelines of ABC are

$$\begin{array}{ll}
\mathcal{L}_a: & (b^2w - c^2v)x + (b^2 - c^2)wy + (b^2 - c^2)vz = 0,\\
\mathcal{L}_b: & (c^2 - a^2)wx + (c^2u - a^2w)y + (c^2 - a^2)uz = 0,\\
\mathcal{L}_c: & (a^2 - b^2)vx + (a^2 - b^2)uy + (a^2v - b^2u)z = 0.
\end{array}$$

They bound a triangle with vertices

$$\begin{aligned}
A' = \; & (-a^2(a^2(u^2 - vw) + b^2u(w - u) - c^2u(u - v))\\
& : (c^2 - a^2)(a^2v(w - u) + b^2u(v - w))\\
& : (a^2 - b^2)(c^2u(v - w) + a^2w(u - v))),\\
B' = \; & ((b^2 - c^2)(a^2v(w - u) + b^2u(v - w))\\
& : -b^2(b^2(v^2 - wu) + c^2v(u - v) - a^2v(v - w))\\
& : (a^2 - b^2)(b^2w(u - v) + c^2v(w - u))),\\
C' = \; & ((b^2 - c^2)(c^2u(v - w) + a^2w(u - v))\\
& : (c^2 - a^2)(b^2w(u - v) + c^2v(w - u))\\
& : -c^2(c^2(w^2 - uv) + a^2w(v - w) - b^2w(w - u))).
\end{aligned}$$

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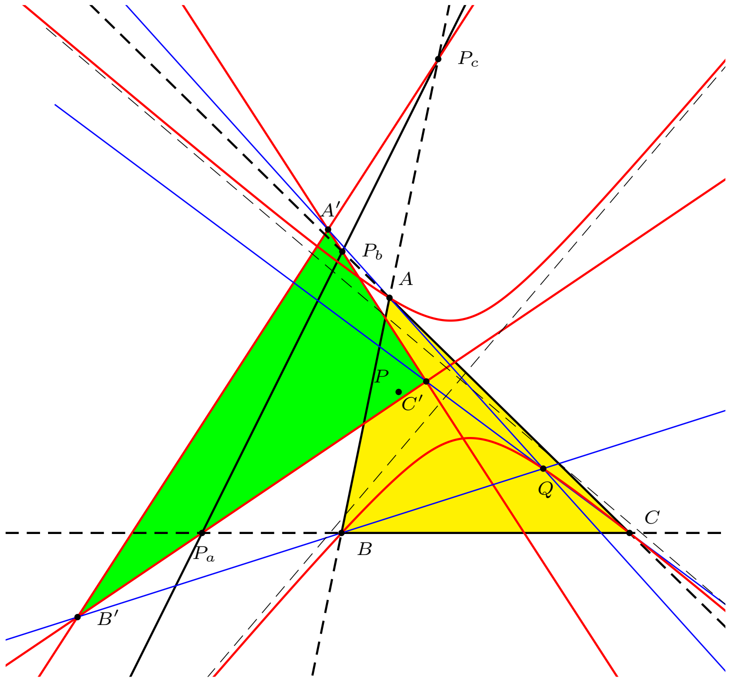

Figure 1. Perspector of triangle bounded by antiparallels

The lines AA', BB', CC' intersect at a point

$$Q = \left(\frac{b^2-c^2}{b^2w(u-v)+c^2v(w-u)} : \frac{c^2-a^2}{c^2u(v-w)+a^2w(u-v)} : \frac{a^2-b^2}{a^2v(w-u)+b^2u(v-w)}\right) \tag{1}$$

We show that Q is a point on the Jerabek hyperbola. The coordinates of Q in (1) can be rewritten as

$$Q = \left(\frac{a^2(b^2-c^2)}{\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2}} : \frac{b^2(c^2-a^2)}{\frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2}} : \frac{c^2(a^2-b^2)}{\frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2}}\right).$$

If we also write this in the form $\left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$, then

$$\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2} : \frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2} : \frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2} = (b^2-c^2)x : (c^2-a^2)y : (a^2-b^2)z.$$

$$\frac{\frac{1}{v}-\frac{1}{w}}{a^2} : \frac{\frac{1}{w}-\frac{1}{u}}{b^2} : \frac{\frac{1}{u}-\frac{1}{v}}{c^2} = -(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z : \cdots : \cdots$$
$$\frac{1}{v}-\frac{1}{w} : \frac{1}{w}-\frac{1}{u} : \frac{1}{u}-\frac{1}{v} = a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z) : \cdots : \cdots$$

Since $\left(\frac{1}{v}-\frac{1}{w}\right)+\left(\frac{1}{w}-\frac{1}{u}\right)+\left(\frac{1}{u}-\frac{1}{v}\right)=0$, we have

$$\begin{aligned} 0 = & \sum_{\text{cyclic}} a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z) \\ = & \sum_{\text{cyclic}} (-a^2(b^2-c^2)+b^2(b^2-c^2)+c^2(b^2-c^2))x \\ = & \sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x. \end{aligned}$$

This is the equation of the Euler line. It shows that the point Q lies on the Jerabek hyperbola. We summarize this in the following theorem, with a slight modification of (1).

Theorem 1. *Let $P=(u:v:w)$ be a point in the plane of triangle ABC, distinct from its centroid. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC at a point*

$$Q(P)=\left(\frac{b^2-c^2}{b^2\left(\frac{1}{u}-\frac{1}{v}\right)+c^2\left(\frac{1}{w}-\frac{1}{u}\right)} : \cdots : \cdots\right)$$

on the Jerabek hyperbola.

Here are some examples.

P	X_1	X_3	X_4	X_6	X_9	X_{23}	X_{24}	X_{69}
$Q(P)$	X_{65}	X_{64}	X_4	X_6	X_{1903}	X_{1177}	X_3	X_{66}
P	X_{468}	X_{847}	X_{193}	X_{93}	X_{284}	X_{943}	X_{1167}	X_{186}
$Q(P)$	X_{67}	X_{68}	X_{69}	X_{70}	X_{71}	X_{72}	X_{73}	X_{74}

Table 1. The perspector $Q(P)$

Note that for the orthocenter $X_4=H$ and $X_6=K$, we have $Q(H)=H$ and $Q(K)=K$. In fact, for $P=H$, the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ bound the orthic triangle. On the other hand, for $P=K$, these lines bound the tangential triangle, anticevian triangle of K. We prove that these are the only points satisfying $Q(P)=P$.

Proposition 2. *The perspector $Q(P)$ coincides with P if and only if P is the orthocenter or the symmedian point.*

Proof. The perspector R coincides with P if and only if the lines AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent, so are the triples BP, $\mathcal{L}_c$, $\mathcal{L}_a$ and CP, $\mathcal{L}_a$, $\mathcal{L}_b$. Now, AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if

$$\begin{vmatrix} 0 & w & -v \\ (c^2-a^2)w & (c^2u-a^2w) & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & (a^2v-b^2u) \end{vmatrix} = 0,$$

or

$$a^2(a^2-b^2)v^2w + a^2(c^2-a^2)vw^2 - b^2(c^2-a^2)w^2u - c^2(a^2-b^2)uv^2 = 0.$$

From the other two triples we obtain

$$-a^2(b^2-c^2)vw^2 + b^2(b^2-c^2)w^2u + b^2(a^2-b^2)wu^2 - c^2(a^2-b^2)u^2v = 0$$

and

$$-a^2(b^2-c^2)v^2w - b^2(c^2-a^2)wu^2 + c^2(c^2-a^2)u^2v + c^2(b^2-c^2)uv^2 = 0.$$

From the difference of the last two, we have, apart from a factor b^2-c^2,

$$u(b^2w^2-c^2v^2) + v(c^2u^2-a^2w^2) + w(a^2v^2-b^2u^2) = 0.$$

This shows that P lies on the Thomson cubic, the isogonal cubic with pivot the centroid G. The Thomson cubic is appears as K002 in Bernard Gibert's catalogue [2]. The same point, as a perspector, lies on the Jerabek hyperbola. Since the Thomson cubic is self-isogonal, its intersections with the Jerabek hyperbola are the isogonal conjugates of the intersections with the Euler line. From [2], P^* is either G, O or H. This means that P is K, H, or O. Table 1 eliminates the possiblility $P = O$, leaving H and K as the only points satisfying $Q(P) = P$. □

Proposition 3. *Let P be a point distinct from the centroid G, and Γ the circum-hyperbola containing G and P. If T traverses Γ, the antiparallels through the intercepts of the triliner polar of T bound a triangle perspective with ABC with the same perspector $Q(P)$ on the Jerabek hyperbola.*

Proof. The circum-hyperbola containing G and P is the isogonal transform of the line KP^*. If we write $P^* = (u : v : w)$, then a point T on Γ has coordinates $\left(\frac{a^2}{u+ta^2} : \frac{b^2}{v+tb^2} : \frac{c^2}{w+tc^2}\right)$ for some real number t. By Theorem 1, we have

$$\begin{aligned} Q(T) &= \left(\frac{b^2-c^2}{b^2\left(\frac{u+ta^2}{a^2} - \frac{v+tb^2}{b^2}\right) + c^2\left(\frac{w+tc^2}{c^2} - \frac{u+ta^2}{a^2}\right)} : \cdots : \cdots\right) \\ &= \left(\frac{b^2-c^2}{b^2\left(\frac{u}{a^2} - \frac{v}{b^2}\right) + c^2\left(\frac{w}{c^2} - \frac{u}{a^2}\right)} : \cdots : \cdots\right) \\ &= Q(P). \end{aligned}$$

□

2. Concurrency of antiparallels

Proposition 4. *The three lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if*

$$-2(a^2-b^2)(b^2-c^2)(c^2-a^2)uvw+\sum_{\text{cyclic}} b^2c^2u((c^2+a^2-b^2)v^2-(a^2+b^2-c^2)w^2)=0. \tag{2}$$

Proof. The three lines are concurrent if and only if

$$\begin{vmatrix} b^2w-c^2v & (b^2-c^2)w & (b^2-c^2)v \\ (c^2-a^2)w & c^2u-a^2w & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & a^2v-b^2u \end{vmatrix}=0.$$

□

For $P=X_{25}$ (the homothetic center of the orthic and tangential triangles), the trilinear polar is parallel to the Lemoine axis (the trilinear polar of K), and the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ concur at the symmedian point (see Figure 2).

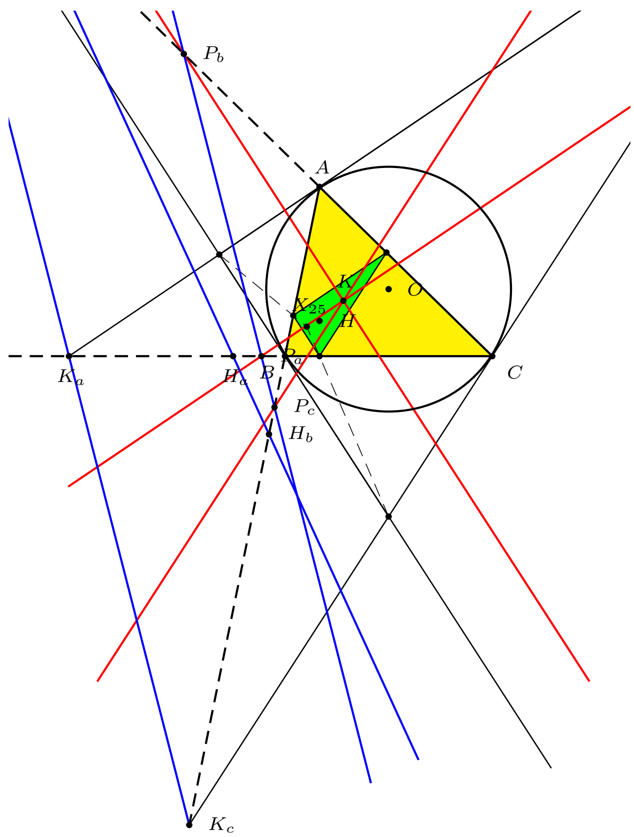

Figure 2. Antiparallels through the intercepts of X_{25}

The cubic defined by (2) can be parametrized as follows. If Q is the point $\left(\frac{a^2}{a^2(b^2+c^2-a^2)+t}:\cdots:\cdots\right)$ on the Jerabek hyperbola, then the antiparallels through the intercepts of the trilinear polar of

$$P_0(Q) = \left(\frac{a^2(b^2c^2+t)}{(b^2+c^2-a^2)(a^2(b^2+c^2-a^2)+t)} : \cdots : \cdots\right)$$

are concurrent at Q. On the other hand, given $P = (u : v : w)$, the antiparallels through the intercepts of the trilinear polars of

$$P_0 = \left(\frac{u(v-w)}{a^2(b^2+c^2-a^2)(b^2w(u-v)+c^2v(w-u))} : \cdots : \cdots\right)$$

are concurrent at $Q(P)$. Here are some examples.

P	X_1	X_3	X_4	X_6	X_{24}
Q	X_{65}	X_{64}	X_4	X_6	X_3
$P_0(Q)$	X_{278}	X_{1073}	X_{2052}	X_{25}	X_{1993}

Table 2. $P_0(Q)$ for Q on the Jerabek hyperbola

$$\begin{aligned}
P_0(X_{66}) =& \left(\frac{1}{a^2(b^4+c^4-a^4)} : \cdots : \cdots\right),\\
P_0(X_{69}) =& (b^2c^2(b^2+c^2-3a^2) : \cdots : \cdots),\\
P_0(X_{71}) =& (a^2(b+c-a)(a(bc+ca+ab)-(b^3+c^3)) : \cdots : \cdots),\\
P_0(X_{72}) =& (a^3-a^2(b+c)-a(b+c)^2+(b+c)(b^2+c^2) : \cdots : \cdots).
\end{aligned}$$

3. Triple of induced GP-lines

Let P be a point in the plane of triangle ABC, distinct from the centroid G, with trilinear polar intersecting BC, CA, AB respectively at P_a, P_b, P_c. Let the antiparallel to BC through P_a intersect CA and AB at B_a and C_a respectively; similarly define C_b, A_b, and A_c, B_c. These are the points

$$\begin{aligned}
B_a &= ((b^2-c^2)v : 0 : c^2v-b^2w), \quad & C_a &= ((b^2-c^2)w : c^2v-b^2w : 0);\\
A_b &= (0 : (c^2-a^2)u : a^2w-c^2u), \quad & C_b &= (a^2w-c^2u : (c^2-a^2)w : 0);\\
A_c &= (0 : b^2u-a^2v : (a^2-b^2)u), \quad & B_c &= (b^2u-a^2v : 0 : (a^2-b^2)v).
\end{aligned}$$

The triangles AB_aC_a, A_bBC_b, A_cB_cC are all similar to ABC. For every point T with reference to ABC, we can speak of the corresponding points in these triangles with the same homogeneous barycentric coordinates. Thus, the P-points in these triangles are

$$\begin{aligned}
P_A =& (b^2c^2(u+v+w)(v-w)-c^4v^2+b^4w^2 : b^2w(c^2v-b^2w) : c^2v(c^2v-b^2w)),\\
P_B =& (a^2w(a^2w-c^2u) : c^2a^2(u+v+w)(w-u)-a^4w^2+c^4u^2 : c^2u(a^2w-c^2u)),\\
P_C =& (a^2v(b^2u-a^2v) : b^2u(b^2u-a^2v) : a^2b^2(u+v+w)(u-v)-b^4u^2+a^4v^2).
\end{aligned}$$

On the other hand, the centroids of these triangles are the points

$$\begin{aligned}
G_A =& \ (2b^2c^2(v-w) - c^4v + b^4w : b^2(c^2v - b^2w) : c^2(c^2v - b^2w)),\\
G_B =& \ (a^2(a^2w - c^2u) : 2c^2a^2(w-u) - a^4w + c^4u : c^2(a^2w - c^2u)),\\
G_C =& \ (a^2(b^2u - a^2v) : b^2(b^2u - a^2v) : 2a^2b^2(u-v) - b^4u + a^4v).
\end{aligned}$$

We call G_AP_A, G_BP_B, G_CP_C the triple of GP-lines induced by antiparallels through the intercepts of the trilinear polar of P, or simply the triple of induced GP-lines.

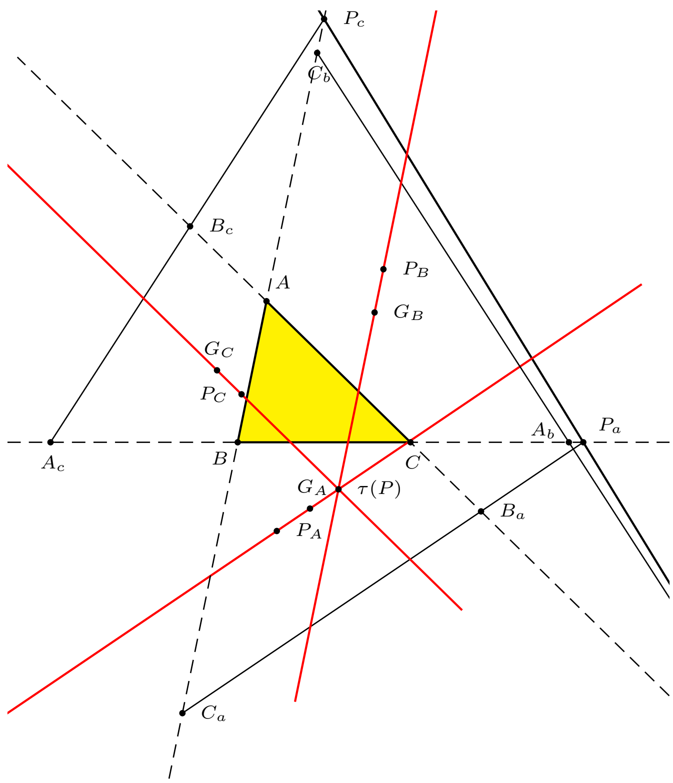

Figure 3. Triple of induced GP-lines

Theorem 5. *The triple of induced GP-lines are concurrent at*

$$\begin{aligned}
\tau(P) = (&-a^2(u^2 - v^2 + vw - w^2) + b^2u(u+v-2w) + c^2u(w+u-2v)\\
&: a^2v(u+v-2w) - b^2(v^2 - w^2 + wu - u^2) + c^2v(v+w-2u)\\
&: a^2w(w+u-2v) + b^2w(v+w-2u) - c^2(w^2 - u^2 + uv - v^2)).
\end{aligned}$$

Proof. The equations of the lines G_AP_A, G_BP_B, G_CP_C are

$$\begin{aligned}(c^2v-b^2w)x+(c^2(w+u-v)-b^2w)y-(b^2(u+v-w)-c^2v)z=&\ 0,\\ -(c^2(v+w-u)-a^2w)x+(a^2w-c^2u)y+(a^2(u+v-w)-c^2u)z=&\ 0,\\ (b^2(v+w-u)-a^2v)x-(a^2(w+u-v)-b^2u)y+(b^2u-a^2v)z=&\ 0.\end{aligned}$$

These three lines intersect at $\tau(P)$ given above. □

Remark. If T traverses the line GP, then $\tau(T)$ traverses the line $G\tau(P)$.

Note that the equations of induced GP-lines are invariant under the permutation $(x,y,z)\leftrightarrow(u,v,w)$, *i.e.*, these can be rewritten as

$$\begin{aligned}(c^2y-b^2z)u+(c^2(z+x-y)-b^2z)v-(b^2(x+y-z)-c^2y)w=&\ 0,\\ -(c^2(y+z-x)-a^2z)u+(a^2z-c^2x)v+(a^2(x+y-z)-c^2x)w=&\ 0,\\ (b^2(y+z-x)-a^2y)u-(a^2(z+x-y)-b^2x)v+(b^2x-a^2y)w=&\ 0.\end{aligned}$$

This means that the mapping τ is a conjugation of the finite points other than the centroid G.

Corollary 6. *The triple of induced GP-lines concur at Q if and only if the triple of induced GQ-lines concur at P.*

We conclude with a list of pairs of triangle centers conjugate under τ.

$X_1,\ X_{1054}$	$X_3,\ X_{110}$	$X_4,\ X_{125}$	$X_6,\ X_{111}$	X_{23}, X_{182}	X_{69}, X_{126}
X_{98}, X_{1316}	X_{100}, X_{1083}	$X_{184},\ X_{186}$	$X_{187},\ X_{353}$	$X_{352},\ X_{574}$	

Table 3. Pairs conjugate under τ

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Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: `liu471119@yahoo.com.tw`

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A Sequence of Triangles and Geometric Inequalities

Dan Marinescu, Mihai Monea, Mihai Opincariu, and Marian Stroe

Abstract. We construct a sequence of triangles from a given one, and deduce a number of famous geometric inequalities.

1. A geometric construction

Throughout this paper we use standard notations of triangle geometry. Given a triangle ABC with sidelengths a, b, c, let s, R, r, and Δ denote the semiperimeter, circumradius, inradius, and area respectively. We begin with a simple geometric construction. Let H be the orthocenter of triangle ABC. Construct a circle, center H, radius $R' = \sqrt{2Rr}$ to intersect the half lines HA, HB, HC at A', B', C' respectively (see Figure 1).

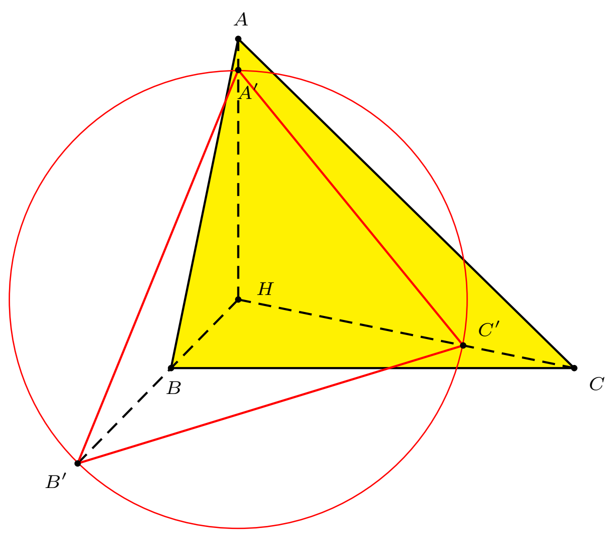

Figure 1.

If the triangle ABC has a right angle at A with altitude AD (D on the hypotenuse BC), we choose A' on the line AD such that A is between D and A'.

Lemma 1. *Triangle $A'B'C'$ has*
(a) *angle measures* $A' = \frac{\pi}{2} - \frac{A}{2}$, $B' = \frac{\pi}{2} - \frac{B}{2}$, $C' = \frac{\pi}{2} - \frac{C}{2}$,
(b) *sidelengths* $a' = \sqrt{a(b+c-a)}$, $b' = \sqrt{b(c+a-b)}$, $c' = \sqrt{c(a+b-c)}$, *and*
(c) *area* $\Delta' = \Delta$.

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Proof. (a) $\angle B'A'C' = \frac{1}{2}\angle B'HC' = \frac{1}{2}\angle BHC = \frac{\pi-A}{2}$; similarly for B' and C'.

(b) By the law of sines,

$$a' = 2R'\sin A' = 2\sqrt{2Rr}\cos\frac{A}{2} = 2\sqrt{2\cdot\frac{abc}{4\Delta}\cdot\frac{\Delta}{s}}\cdot\sqrt{\frac{s(s-a)}{bc}} = \sqrt{a(b+c-a)};$$

similarly for b' and c'.

(c) Triangle $A'B'C'$ has area

$$\begin{aligned}\Delta' =&\ \frac{1}{2}b'c'\sin A' = \frac{1}{2}b'c'\cos\frac{A}{2}\\ =&\ \frac{1}{2}\sqrt{b(c+a-b)}\cdot\sqrt{c(a+b-c)}\cdot\sqrt{\frac{s(s-a)}{bc}}\\ =&\ \sqrt{s(s-a)(s-b)(s-c)}\\ =&\ \Delta.\end{aligned}$$

□

Proposition 2. (a) $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2 - (b-c)^2 - (c-a)^2 - (a-b)^2$.
(b) $a'^2 + b'^2 + c'^2 \le a^2 + b^2 + c^2$.
(c) $a' + b' + c' \le a + b + c$.
(d) $\sin A' + \sin B' + \sin C' \ge \sin A + \sin B + \sin C$.
(e) $R' \le R$.
(f) $r' \ge r$.

In each case, equality holds if and only if ABC is equilateral.

Proof. (a) follows from Lemma 1(b); (b) follows from (a). For (c),

$$\begin{aligned}a'+b'+c' =&\ \sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}\\ \le&\ \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2}\\ =&\ a+b+c.\end{aligned}$$

For (d), we have

$$\begin{aligned}&\sin A + \sin B + \sin C\\ =&\frac{1}{2}\left(\sin B + \sin C + \sin C + \sin A + \sin A + \sin B\right)\\ =&\ \sin\frac{B+C}{2}\cos\frac{B-C}{2} + \sin\frac{C+A}{2}\cos\frac{C-A}{2} + \sin\frac{A+B}{2}\cos\frac{A-B}{2}\\ \le&\sin\frac{B+C}{2} + \sin\frac{C+A}{2} + \sin\frac{A+B}{2}\\ =&\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\\ =&\sin A' + \sin B' + \sin C'.\end{aligned}$$

(e) $R' = \frac{a'+b'+c'}{2(\sin A'+\sin B'+\sin C')} \le \frac{a+b+c}{2(\sin A+\sin B+\sin C)} = R$.

(f) $r' = \frac{\Delta'}{s'} \ge \frac{\Delta}{s} = r$. □

Remark. The inequality $R' \leq R$ certainly follows from Euler's inequality $R \geq 2r$. From the direct proof of (e), Euler's inequality also follows (see Theorem 6(b) below).

2. A sequence of triangles

Beginning with a triangle ABC, we repeatedly apply the construction in §1 to obtain a sequence of triangles $(A_nB_nC_n)_{n\in\mathbb{N}}$ with $A_0B_0C_0 \equiv ABC$, and angle measures and sidelengths defined recursively by

$$\begin{aligned} A_{n+1} &= \frac{\pi - A_n}{2}, \quad B_{n+1} = \frac{\pi - B_n}{2}, \quad C_{n+1} = \frac{\pi - C_n}{2}; \\ a_{n+1} &= \sqrt{a_n(b_n + c_n - a_n)}, \quad b_{n+1} = \sqrt{b_n(c_n + a_n - a_n)}, \\ c_{n+1} &= \sqrt{c_n(a_n + b_n - c_n)}. \end{aligned}$$

Denote by s_n, R_n, r_n, Δ_n the semiperimeter, circumradius, inradius, and area of triangle $A_nB_nC_n$. Note that $\Delta_n = \Delta$ for every n.

Lemma 3. *The sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$ are convergent and*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}.$$

Proof. It is enough to consider the sequence $(A_n)_{n\in\mathbb{N}}$. Rewrite the relation $A_{n+1} = \frac{\pi}{2} - \frac{A_n}{2}$ as

$$A_{n+1} - \frac{\pi}{3} = -\frac{1}{2}\left(A_n - \frac{\pi}{3}\right).$$

It follows that the sequence $\left(A_n - \frac{\pi}{3}\right)_{n\in\mathbb{N}}$ is a geometric sequence with common ratio $-\frac{1}{2}$. It converges to 0, giving $\lim_{n\to\infty} A_n = \frac{\pi}{3}$. □

Proposition 4. *The sequence $(R_n)_{n\in\mathbb{N}}$ is convergent and* $\lim_{n\to\infty} R_n = \frac{2}{3}\sqrt{\sqrt{3}\Delta}$.

Proof. Since $R_n = \frac{a_nb_nc_n}{4\triangle_n} = \frac{8R_n^3 \sin A_n \sin B_n \sin C_n}{4\triangle_n}$, we have

$$R_n^2 = \frac{\triangle}{2\sin A_n \sin B_n \sin C_n}.$$

The result follows from Lemma 3. □

Proposition 5. *The sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 2\sqrt{\frac{\triangle}{\sqrt{3}}}.$$

Proof. This follows from $a_n = 2R_n \sin A_n$, Lemma 3 and Proposition 4. □

From these basic results we obtain a number of interesting convergent sequences. In each case, the increasing or decreasing property is clear from Proposition 2.

	Sequence		Limit	Reference
(a)	Δ_n	constant	Δ	Lem.1(c)
(b)	$\sin A_n + \sin B_n + \sin C_n$	increasing	$\frac{3\sqrt{3}}{2}$	Prop.2(d), Lem.3
(c)	R_n	decreasing	$\frac{2}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(e), 4
(d)	s_n	decreasing	$\sqrt{3\sqrt{3}\Delta}$	Prop.2(c), 4
(e)	r_n	increasing	$\frac{1}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(f)
(f)	$\frac{R_n}{r_n}$	decreasing	2	
(g)	$a_n^2 + b_n^2 + c_n^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(b), 5
(h)	$a_n^2 + b_n^2 + c_n^2 - (b_n - c_n)^2$ $-(c_n - a_n)^2 - (a_n - b_n)^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(a, b), 5

3. Geometric inequalities

The increasing or decreasing properties of these sequences, along with their limits, lead easily to a number of famous geometric inequalities [1, 3].

Theorem 6. *The following inequalities hold for an arbitrary angle ABC.*

(a) $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

(b) [Euler's inequality] $R \geq 2r$.

(c) [Weitzenböck inequality] $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$.

(d) [Hadwiger-Finsler inequality] $a^2+b^2+c^2-(b-c)^2-(c-a)^2-(a-b)^2 \geq 4\sqrt{3}\Delta$.

In each case, equality holds if and only if the triangle is equilateral.

Remark. Weitzenböck's inequality is usually proved as a consequence of the Hadwiger - Finsler's inequality ([2, 4]). Our proof shows that they are logically equivalent.

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Dan Marinescu: Colegiul Naţional “Iancu de Hunedoara”, Hunedoara, Str.l Libertatii No.2 Bl. 9 Ap.14, 331032, Romania
E-mail address: marinescuds@yahoo.com

Mihai Monea: Colegiul Naţional “Decebal” Deva, Str. Decebal Bl. 8 Ap.10, 330021, Romania
E-mail address: mihaimonea@yahoo.com

Mihai Opincariu: Colegiul Naţional “Avram Iancu” Brad, Str. 1 Iunie No.13 Bl. 15 Ap.14, 335200, Romania
E-mail address: opincariumihai@yahoo.com

Marian Stroe: Coleguil Economic “Emanoil Gojdu” Hunedoara, Str. Viorele No. 4, Bl 10 Ap.10, 331093, Romania
E-mail address: maricu_stroe@yahoo.com

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Trilinear Polars of Brocardians

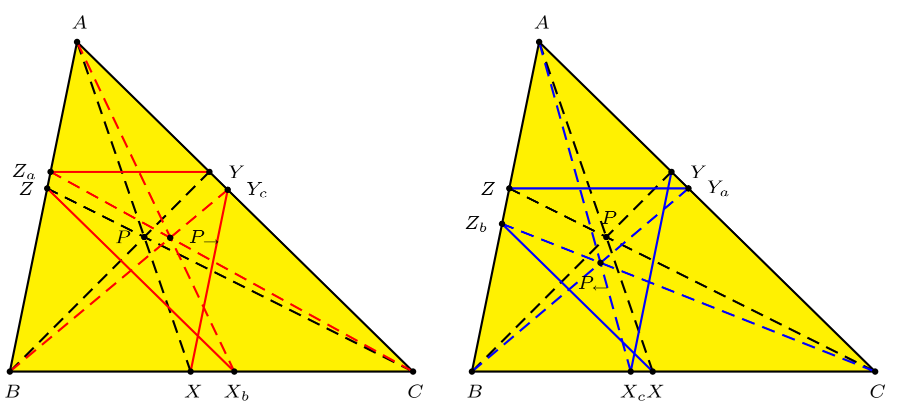
Francisco Javier García Capitán

Abstract. We study the trilinear polars of the Brocardians of a point, and investigate the condition for their orthogonality.

1. The Brocardians

Let P be a point not on any of the sidelines of triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$ and cevian triangle XYZ. Construct the parallels of $\begin{matrix}CA\\AB\\BC\end{matrix}$ through $\begin{matrix}Z\\X\\Y\end{matrix}$ to intersect $\begin{matrix}BC\\CA\\AB\end{matrix}$ at $\begin{matrix}X_b\\Y_c\\Z_a\end{matrix}$ (see Figure 1(a)). The triangle $X_bY_cZ_a$ is perspective with ABC at the point

$$P_{\rightarrow} := \left(\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right).$$



1(a) The Brocardian $P_{\rightarrow}$

1(b) The Brocardian $P_{\leftarrow}$

Likewise, the parallels of $\begin{matrix}AB\\BC\\CA\end{matrix}$ through $\begin{matrix}Y\\Z\\X\end{matrix}$ intersect $\begin{matrix}BC\\CA\\AB\end{matrix}$ at $\begin{matrix}X_c\\Y_a\\Z_b\end{matrix}$ such that triangle $X_cY_aZ_b$ is perspective with ABC at

$$P_{\leftarrow} := \left(\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right)$$

(see Figure 1(b)). The points $P_{\rightarrow}$ and $P_{\leftarrow}$ are called the Brocardians of P (see [2, §8.4]). For example, the Brocardians of the symmedian point are the Brocard points $\Omega = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$ and $\Omega' = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$.

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2. Noncollinearity of P and its Brocardians

The points P, $P_{\rightarrow}$ and $P_{\leftarrow}$ are never collinear since

$$\begin{vmatrix} u & v & w \\ \frac{1}{w} & \frac{1}{u} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{w} & \frac{1}{u} \end{vmatrix} = \frac{u^2+v^2+w^2-vw-wu-uv}{uvw} \neq 0.$$

It is well known that the Brocard points are equidistant from the symmedian point. It follows that the pedal of K on the line $\Omega\Omega'$ is the midpoint of the segment $\Omega\Omega'$, the triangle center $X_{39} = (a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2))$ in [1].

Now, for the Gergonne point $G_{\mathrm{e}} = \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$, the Brocardians are the points $G_{\mathrm{e}\rightarrow} = (a+b-c : b+c-a : c+a-b)$ and $G_{\mathrm{e}\leftarrow} = (c+a-b : a+b-c : b+c-a)$. The midpoint of $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$ is the incenter $I = (a : b : c)$. Indeed, I is the pedal of the Gergonne point on the line $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$

$$(b^2+c^2-a(b+c))x + (c^2+a^2-b(c+a))y + (a^2+b^2-c(a+b))z = 0.$$

3. Trilinear polars of the Brocardians

The trilinear polars of the Brocardians of P are the lines

$$\ell_{\rightarrow} \qquad wx + uy + vz = 0,$$

and

$$\ell_{\leftarrow} \qquad vx + wy + uz = 0.$$

These lines intersect at the point

$$Q = (u^2 - vw : v^2 - wu : w^2 - uv).$$

Since

$$(u^2-vw,\ v^2-wu,\ w^2-uv) = (u+v+w)(u,\ v,\ w) - (vw+wu+uv)(1,\ 1,\ 1),$$

the point Q divides the segment GP in the ratio

$$GQ : QP = (u+v+w)^2 : -3(vw+wu+uv).$$

The point Q is never an infinite point since

$$u^2+v^2+w^2-vw-wu-uv \neq 0.$$

It follows that the trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ are never parallel.

4. Orthogonality of trilinear polars of Brocardians

The trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ have infinite points $(u-v : v-w : w-u)$ and $(w-u : u-v : v-w)$ respectively. They are orthogonal if and only if

$$S_A(u-v)(w-u) + S_B(v-w)(u-v) + S_C(w-u)(v-w) = 0 \qquad (1)$$

(see [2, §4.5]). Now, (1) defines a conic with center $G = (1 : 1 : 1)$ (see [2, §10.7.2]). Since the conic contains G, it is necessarily degenerate. Solving for the infinite points of the conic, we obtain the condition that the conic consists of a pair of real lines if and only if

$$S_{AA} + S_{BB} + S_{CC} - 2S_{BC} - 2S_{CA} - 2S_{AB} \geq 0.$$

Equivalently,

$$5(a^4 + b^4 + c^4) - 6(b^2c^2 + c^2a^2 + a^2b^2) \geq 0. \tag{2}$$

Here is a characterization of triangles satisfying condition (2). Given two points B and C with $BC = a$, we set up a Cartesian coordinates system such that $B = \left(-\frac{a}{2},\ 0\right)$ and $C = \left(\frac{a}{2},\ 0\right)$. If $A = (x, y)$, then

$$\begin{aligned}\left(x - \frac{a}{2}\right)^2 + y^2 &= b^2,\\ \left(x + \frac{a}{2}\right)^2 + y^2 &= c^2.\end{aligned}$$

With these, condition (2) becomes

$$(4x^2 + 4y^2 - 8ay + 3a^2)(4x^2 + 4y^2 + 8ay + 3a^2) \geq 0.$$

This is the exterior of the two circles, centers $(0,\ \pm a)$, radii $\frac{a}{2}$. Here is a simple example. If we require $C = \frac{\pi}{2}$, then $S_C = 0$ and the degenerate conic (1) is the union of the two lines $v - w = 0$ and $S_A(z - x) + S_B(y - z) = 0$. These are the C-median and the line GK_c, K_c being the C-trace of the symmedian point K. Figure 2 illustrates the trilinear polars of the Brocardians of a point P on GK_c

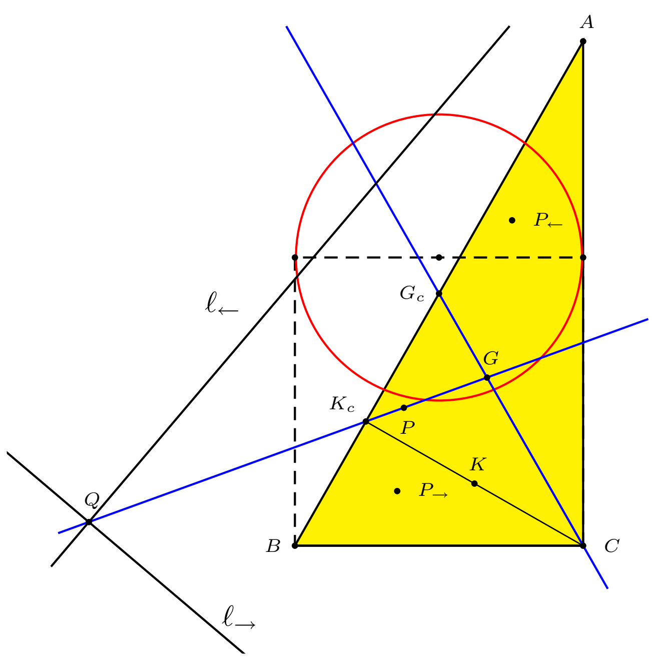

Figure 2.

On the other hand, for points A on the circumferences of the two circles, the triangle ABC has exactly one real line through the centroid G such that for every P on the line, the trilinear polars of the Brocardians intersect orthogonally (on the same line). It is enough to consider A on the circle $4(x^2+y^2)-8ay+3a^2=0$, with coordinates $\left(\frac{a}{2}\cos\theta,\ a+\frac{a}{2}\sin\theta\right)$. The center of triangle ABC is the point $G=\left(\frac{a}{6}\cos\theta,\ \frac{a}{6}(2+\sin\theta)\right)$. The line in question connects G to the fixed point $M=\left(0,\ \frac{a}{2}\right)$:

$$(1-\sin\theta)x+\cos\theta\left(y-\frac{a}{2}\right)=0.$$

The trilinear polars of the Brocardians of an arbitrary point P on this line are symmetric with respect to GM, and intersect orthogonally (see Figure 3).

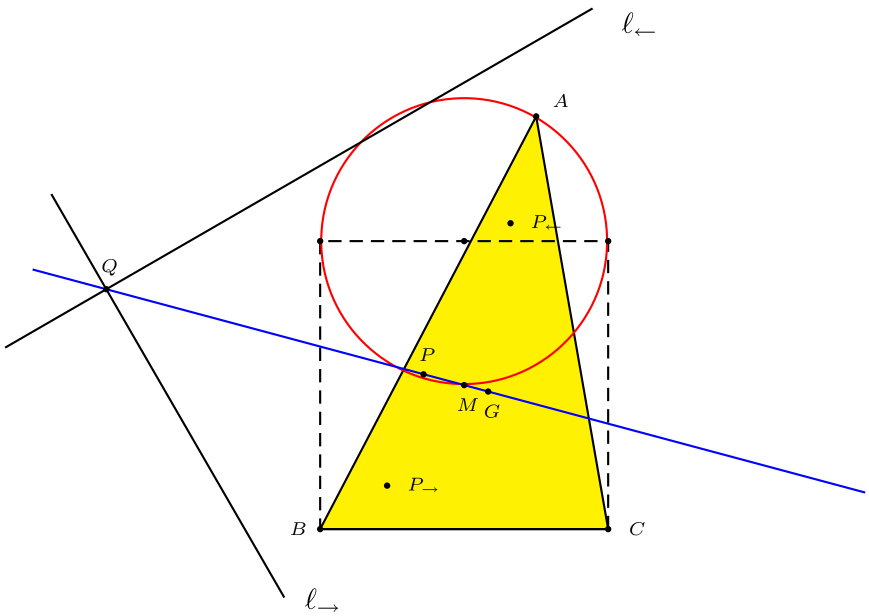

Figure 3.

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Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: `garciacapitan@gmail.com`

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Reflections in Triangle Geometry

Antreas P. Hatzipolakis and Paul Yiu

On the 10th Anniversary of Hyacinthos

Abstract. This paper is a survey of results on reflections in triangle geometry. We work with homogeneous barycentric coordinates with reference to a given triangle ABC and establish various concurrency and perspectivity results related to triangles formed by reflections, in particular the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ of a point P in the sidelines of ABC, and the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ of the vertices of ABC in their respective opposite sides. We also consider triads of concurrent circles related to these reflections. In this process, we obtain a number of interesting triangle centers with relatively simple coordinates. While most of these triangle centers have been catalogued in Kimberling's *Encyclopedia of Triangle Centers* [27] (ETC), there are a few interesting new ones. We give additional properties of known triangle centers related to reflections, and in a few cases, exhibit interesting correspondences of cubic curves catalogued in Gibert's *Catalogue of Triangle Cubics* [14] (CTC).

CONTENTS

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Notations. We adopt the usual notations of triangle geometry and work with homogeneous barycentric coordinates with reference to a given triangle ABC with sidelengths a, b, c and angle measures A, B, C. Occasionally, expressions for coordinates are simplified by using Conway's notation:

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-a^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2},$$

subject to $S_{AB} + S_{BC} + S_{CA} = S^2$, where S is twice the area of triangle ABC, and S_{BC} stands for $S_B S_C$ etc. The labeling of triangle centers follows ETC [27], except for the most basic and well known ones listed below. References to triangle cubics are made to Gibert's CTC [14].

G	X_2	centroid
H	X_4	orthocenter
E_∞	X_{30}	Euler infinity point
I	X_1	incenter
N_{a}	X_8	Nagel point
K	X_6	symmedian point
$J_\pm$	$X_{15},\ X_{16}$	isodynamic points

O	X_3	circumcenter
N	X_5	nine point center
E	X_{110}	Euler reflection point
G_{e}	X_7	Gergonne point
F_{e}	X_{11}	Feuerbach point
$F_\pm$	X_{13}, X_{14}	Fermat points
W	X_{484}	first Evans perspector

P^*	isogonal conjugate of P
$P^\bullet$	isotomic conjugate of P
P^{-1}	inverse of P in circumcircle
P/Q	cevian quotient
$P_aP_bP_c$	cevian triangle of P
$P^aP^bP^c$	anticevian triangle of P
$P_{[a]}$	pedal of P on BC
$P^{(a)}$	reflection of P in BC
E_t	Point on Euler line dividing OH in the ratio $t : 1-t$
$\mathscr{C}(P,Q)$	Bicevian conic through the traces of P and Q on the sidelines

1. The reflection triangle

Let P be a point with the homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The reflections of P in the sidelines BC, CA, AB are the points

$$\begin{aligned}
P^{(a)} &= (-a^2u : (a^2+b^2-c^2)u + a^2v : (c^2+a^2-b^2)u + a^2w),\\
P^{(b)} &= ((a^2+b^2-c^2)v + b^2u : -b^2v : (b^2+c^2-a^2)v + a^2w),\\
P^{(c)} &= ((c^2+a^2-b^2)w + b^2u : (b^2+c^2-a^2)w + c^2v : -c^2w).
\end{aligned}$$

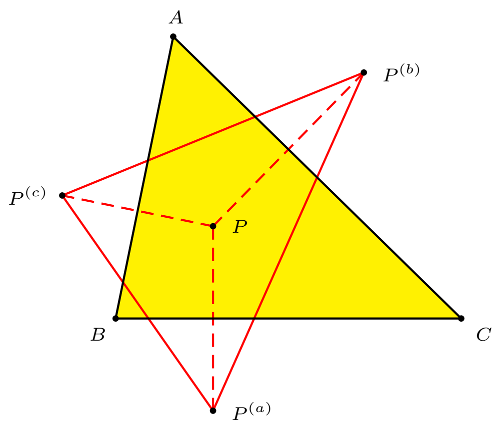


Figure 1. The reflection triangle

We call $P^{(a)}P^{(b)}P^{(c)}$ the reflection triangle of P (see Figure 1). Here are some examples.

(1) The reflection triangle of the circumcenter O is oppositely congruent to ABC at the midpoint of OH, which is the nine-point center N. This is the only reflection triangle congruent to ABC.

(2) The reflection triangle of H is inscribed in the circumcircle of ABC (see Remark (1) following Proposition 2 and Figure 3(b) below).

(3) The reflection triangle of N is homothetic at O to the triangle of reflections (see Propositioin 5 below).

Proposition 1. *The reflection triangle of P is*

(a) *right-angled if and only if P lies on one of the circles with centers* $\begin{array}{c}K^a\\ K^b\\ K^c\end{array}$ *passing through* $\begin{array}{c}B,\ C\\ C,\ A\\ A,\ B\end{array}$ *respectively,*

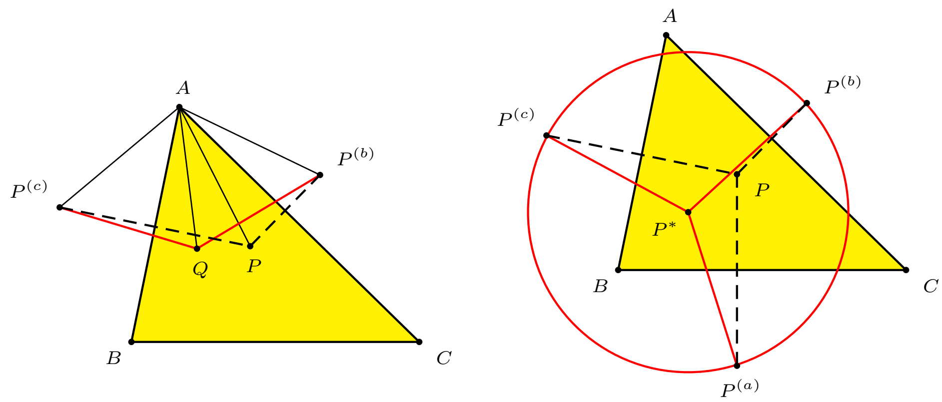
(b) *isosceles if and only if P is on one of the Apollonian circles, each with diameter the feet of the bisectors of an angle on its opposite side,*

(c) *equilateral if and only if P is one of the isodynamic points $J_\pm$,*

(d) *degenerate if and only if P lies on the circumcircle.*

1.1. *Circle of reflections.*

Proposition 2. *The circle $P^{(a)}P^{(b)}P^{(c)}$ has center P^*.*



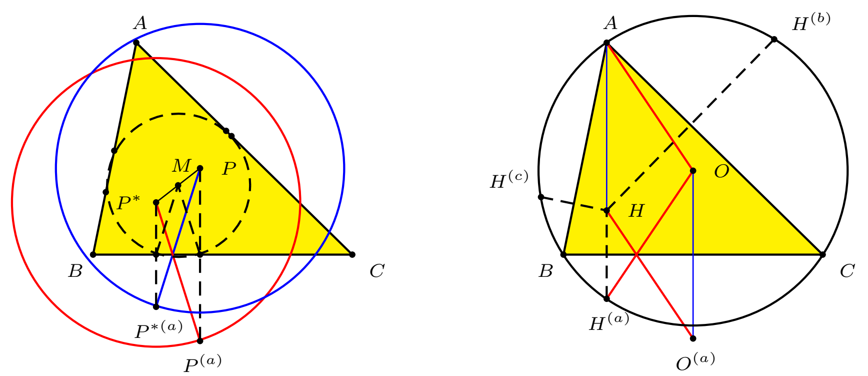
(a) Reflections and isogonal lines (b) circle of reflections

Figure 2. Circle of reflections of P with center P^*

Proof. Let Q be a point on the line isogonal to AP with respect to angle A, *i.e.*, the lines AQ and AP are symmetric with respect to the bisector of angle BAC (see Figure 2(a)). Clearly, the triangles $AQP^{(b)}$ and $AQP^{(c)}$ are congruent, so that Q is equidistant from $P^{(b)}$ and $P^{(c)}$. For the same reason, any point on a line isogonal to BP is equidistant from $P^{(c)}$ and $P^{(a)}$. It follows that the isogonal conjugate P^* is equidistant from the three reflections $P^{(a)}$, $P^{(b)}$, $P^{(c)}$. □

This simple fact has a few interesting consequences.

(1) The circle through the reflections of P and the one through the reflections of P^* are congruent (see Figure 3(a)). In particular, the reflections of the orthocenter H lie on the circumcircle (see Figure 3(b)).



(a) Congruent circles of reflection (b) Reflections of H on circumcircle

Figure 3. Congruence of circles of reflection of P and P^*

(2) The (six) pedals of P and P^* on the sidelines of triangle ABC are concyclic. The center of the common pedal circle is the midpoint of PP^* (see Figure 3(a)). For the isogonal pair O and H, this pedal circle is the nine-point circle.

1.2. *Line of reflections* .

Theorem 3. (a) *The reflections of P in the sidelines are collinear if and only if P lies on the circumcircle. In this case, the line containing the reflections passes through the orthocenter H.*
(b) *The reflections of a line ℓ in the sidelines are concurrent if and only if the line contains the orthocenter H. In this case, the point of concurrency lies on the circumcircle.*

Remarks. (1) Let P be a point on the circumcircle and ℓ a line through the orthocenter H. The reflections of P lies on ℓ if and only if the reflections of ℓ concur at P ([6, 29]). Figure 4 illustrates the case of the Euler line.

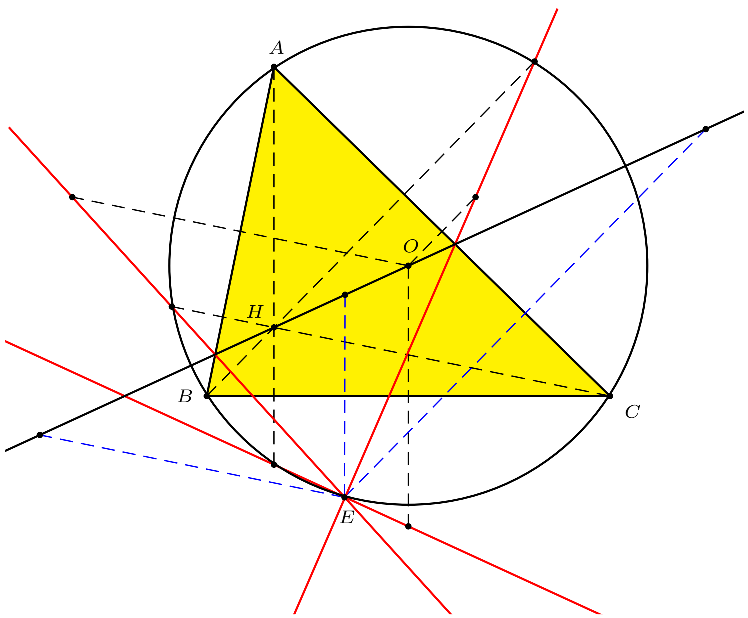


Figure 4. Euler line and Euler reflection point

(2) If $P = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}\right)$ is the isogonal conjugate of the infinite point of a line $ux + vy + wz = 0$, its line of reflections is

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0.$$

(3) Let ℓ be the line joining H to $P = (u : v : w)$. The reflections of ℓ in the sidelines of triangle ABC intersect at the point

$$r_0(P) := \left(\frac{a^2}{S_B v - S_C w} : \frac{b^2}{S_C w - S_A u} : \frac{c^2}{S_A u - S_B v}\right).$$

Clearly, $r_0(P_1) = r_0(P_2)$ if and only if P_1, P_2, H are collinear.

line HP	$r_0(P)$ = intersection of reflections
Euler line	$E = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$
HI	$X_{109} = \left(\frac{a^2}{(b-c)(b+c-a)} : \frac{b^2}{(c-a)(c+a-b)} : \frac{c^2}{(a-b)(a+b-c)}\right)$
HK	$X_{112} = \left(\frac{a^2}{(b^2-c^2)S_A} : \frac{b^2}{(c^2-a^2)S_B} : \frac{c^2}{(a^2-b^2)S_C}\right)$

Theorem 4 (Blanc [3]). *Let ℓ be a line through the circumcenter O of triangle ABC, intersecting the sidelines at X, Y, Z respectively. The circles with diameters AX, BY, CZ are coaxial with two common points and radical axis $\mathscr{L}$ containing the orthocenter H.*

(a) *One of the common points P lies on the nine-point circle, and is the center of the rectangular circum-hyperbola which is the isogonal conjugate of the line ℓ.*

(b) *The second common point Q lies on the circumcircle, and is the reflection of $r_0(P)$ in ℓ.*

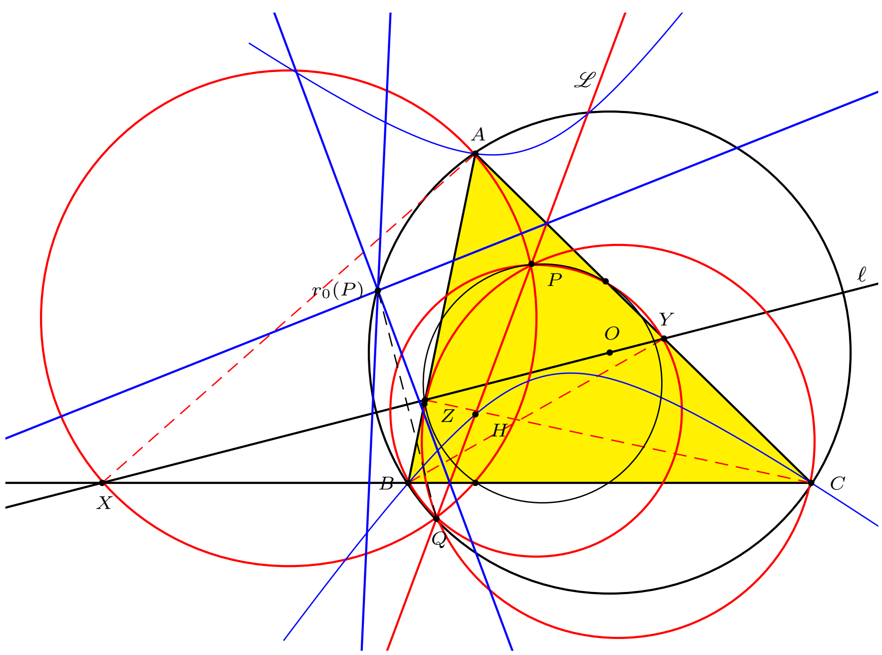


Figure 5. Blanc's theorem

Here are some examples.

Line ℓ	P	Q	$r_0(P)$
Euler line	X_{125}	$X_{476} = \left(\frac{1}{(S_B-S_C)(S^2-3S_{AA})} : \cdots : \cdots\right)$	E
Brocard axis	X_{115}	X_{112}	X_{2715}
OI	X_{11}	$X_{108} = \left(\frac{a}{(b-c)(b+c-a)S_A} : \cdots : \cdots\right)$	X_{2720}

1.3. *The triangle of reflections.* The reflections of the vertices of triangle ABC in their opposite sides are the points

$$\begin{aligned} A^{(a)} &= (-a^2 : a^2+b^2-c^2 : c^2+a^2-b^2), \\ B^{(b)} &= (a^2+b^2-c^2 : -b^2 : b^2+c^2-a^2), \\ C^{(c)} &= (c^2+a^2-b^2 : b^2+c^2-a^2 : -2c^2). \end{aligned}$$

We call triangle $A^{(a)}B^{(b)}C^{(c)}$ the triangle of reflections.

Proposition 5. *The triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ is the image of the reflection triangle of N under the homothety $\mathsf{h}(O,2)$.*

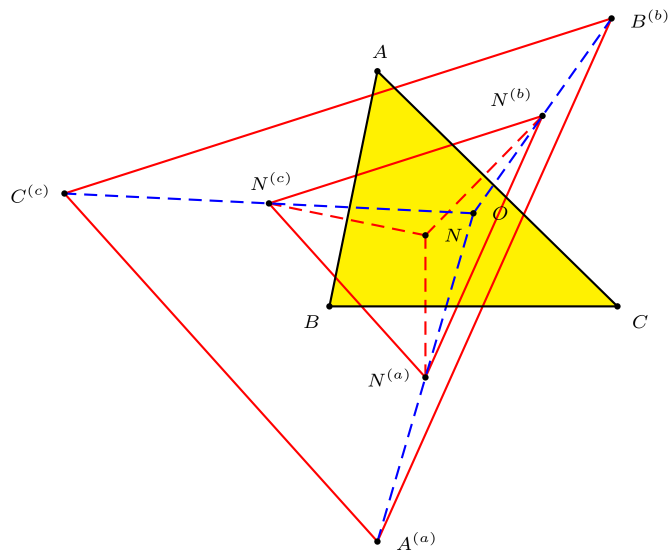


Figure 6. Homothety of triangle of reflections and reflection triangle of N

From this we conclude that
(1) the center of the circle $A^{(a)}B^{(b)}C^{(c)}$ is the point $\mathsf{h}(O,2)(N^*)$, the reflection of O in N^*, which appears as X_{195} in ETC, and
(2) the triangle of reflections is degenerate if and only if the nine-point center N lies on the circumcircle. Here is a simple construction of such a triangle (see Figure 7). Given a point N on a circle $O(R)$, construct

(a) the circle $N\left(\frac{R}{2}\right)$ and choose a point D on this circle, inside the given one (O),

(b) the perpendicular to OD at D to intersect (O) at B and C,

(c) the antipode X of D on the circle (N), and complete the parallelogram $ODXA$ (by translating X by the vector **DO**).

Then triangle ABC has nine-point center N on its circumcircle. For further results, see [4, p.77], [18] or Proposition 21 below.

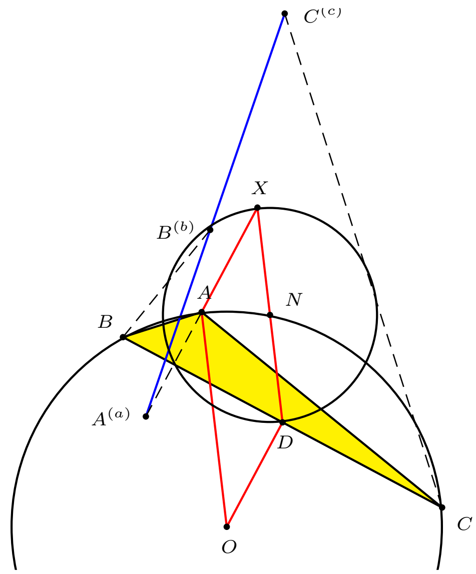


Figure 7. Triangle with degenerate triangle of reflections

2. Perspectivity of reflection triangle

2.1. *Perspectivity with anticevian and orthic triangles.*

Proposition 6 ([10])**.** *The reflection triangle of P is perspective with its anticevian triangle at the cevian quotient $Q = H/P$, which is also the isogonal conjugate of P in the orthic triangle.*

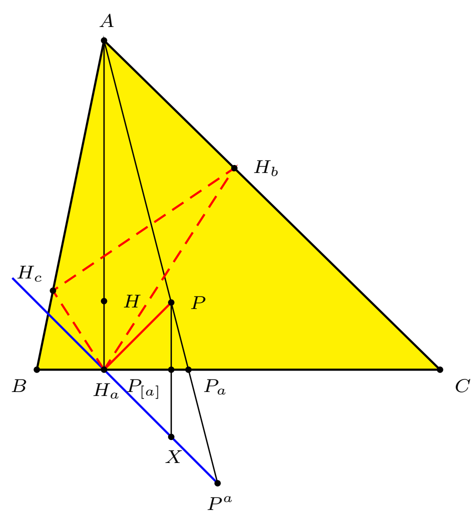


Figure 8. H_aP and $H_aP^{(a)}$ isogonal in orthic triangle

Proof. Let $P_aP_bP_c$ be the cevian triangle of P, and $P^aP^bP^c$ the anticevian triangle. Since P and P^a divide AP_a harmonically, we have $\frac{1}{AP^a}+\frac{1}{AP}=\frac{2}{AP_a}$. If the perpendicular from P to BC intersects the line P^aH_a at X, then

$$\frac{PX}{AH_a}=\frac{PP^a}{AP^a}=\frac{PP_a+P_aP^a}{AP^a}=\frac{PP_a}{AP^a}+\frac{PP_a}{AP}=\frac{2PP_a}{AP_a}=\frac{2PP_{[a]}}{AH_a}.$$

Therefore, $PX=2PP_{[a]}$, and $X=P^{(a)}$. This shows that $P^{(a)}$ lies on the line P^aH_a. Similarly, $P^{(b)}$ and $P^{(c)}$ lie on P^bH_b and P^cH_c respectively. Since the anticevian triangle of P and the orthic triangle are perspective at the cevian quotient H/P, these triangles are perspective with the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ at the same point.

The fact that $P^{(a)}$ lies on the line H_aP^a means that the lines H_aP^a and H_aP are isogonal lines with respect to the sides H_aH_b and H_aH_c of the orthic triangle; similarly for the pairs H_bP^b, H_bP and H_cP^c, H_cP. It follows that H/P and P are isogonal conjugates in the orthic triangle. □

If $P=(u:v:w)$ in homogeneous barycentric coordinates, then

$$H/P=(u(-S_Au+S_Bv+S_Cw):v(-S_Bv+S_Cw+S_Au):w(-S_Cw+S_Au+S_Bv)).$$

Here are some examples of $(P,\ H/P)$ pairs.

P	I	G	O	H	N	K
H/P	X_{46}	X_{193}	X_{155}	H	X_{52}	X_{25}

2.2. *Perspectivity with the reference triangle.*

Proposition 7. *The reflection triangle of a point P is perspective with ABC if and only if P lies on the Neuberg cubic*

$$\sum_{\text{cyclic}}(S_{AB}+S_{AC}-2S_{BC})u(c^2v^2-b^2w^2)=0. \tag{1}$$

As P traverses the Neuberg cubic, the locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}}\frac{S_Ax}{S^2-3S_{AA}}\left(\frac{y^2}{S^2-3S_{CC}}-\frac{z^2}{S^2-3S_{BB}}\right)=0. \tag{2}$$

The first statement can be found in [30]. The cubic (1) is the famous Neuberg cubic, the isogonal cubic p$K(K,\ E_\infty)$ with pivot the Euler infinity point. It appears as K001 in CTC, where numerous locus properties of the Neuberg cubic can be found; see also [5]. The cubic (2), on the other hand, is the pivotal isocubic p$K(X_{1989},\ X_{265})$, and appears as K060. Given Q on the cubic (2), the point P on the Neuberg cubic can be constructed as the perspector of the cevian and reflection triangles of Q (see Figure 9). Here are some examples of (P,Q) with P on Neuberg cubic and perspector Q of the reflection triangle.

P	O	H	I	W	X_{1157}
Q	N	H	X_{79}	X_{80}	X_{1141}

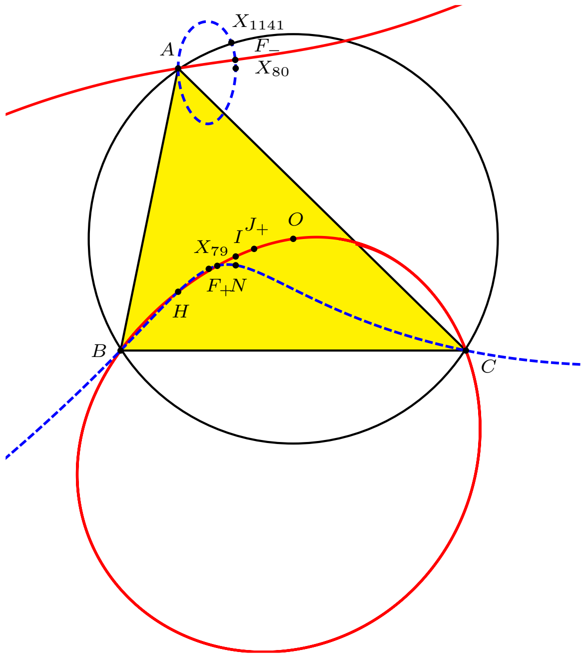


Figure 9. The Neuberg cubic and the cubic K060

Remarks. (1) $X_{79} = \left(\frac{1}{b^2+c^2-a^2+bc} : \frac{1}{c^2+a^2-b^2+ca} : \frac{1}{a^2+b^2-c^2+ab}\right)$ is also the perspector of the triangle formed by the three lines each joining the perpendicular feet of a trace of the incenter on the other two sides (see Figure 10).

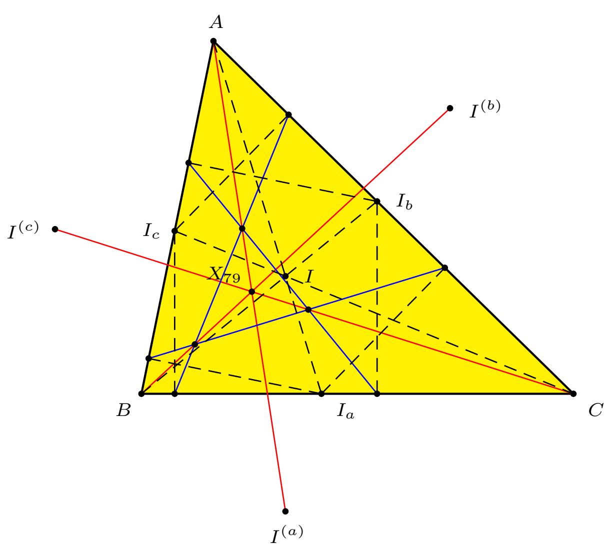


Figure 10. Perspector of reflection triangle of I

(2) For the pair (W, X_{80}),
(i) $W = X_{484} = (a(a^3+a^2(b+c)-a(b^2+bc+c^2)-(b+c)(b-c)^2) : \cdots : \cdots)$ is the first Evans perspector, the perspector of the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ and the excentral triangle $I^aI^bI^c$ (see [45]),
(ii) $X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$ is the reflection conjugate of I (see §3 below).
(3) For the pair $(X_{1157},\ X_{1141})$,
(i) $X_{1157} = \left(\frac{a^2(a^6-3a^4(b^2+c^2)+a^2(3b^4-b^2c^2+3c^4)-(b^2-c^2)^2(b^2+c^2))}{a^2(b^2+c^2)-(b^2-c^2)^2} : \cdots : \cdots\right)$ is the inverse of N^* in the circumcircle,
(ii) $X_{1141} = \left(\frac{1}{(S^2+S_{BC})(S^2-3S_{AA})} : \cdots : \cdots\right)$ lies on the circumcircle.

The Neuberg cubic also contains the Fermat points and the isodynamic points. The perspectors of the reflection triangles of
(i) the Fermat points $F_\varepsilon = \left(\frac{1}{\sqrt{3}S_A+\varepsilon S} : \frac{1}{\sqrt{3}S_B+\varepsilon S} : \frac{1}{\sqrt{3}S_C+\varepsilon S}\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{(S_A+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_A+\varepsilon S)^2} : \frac{(S_B+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_B+\varepsilon S)^2} : \frac{(S_C+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_C+\varepsilon S)^2}\right),$$

(ii) the isodynamic points $J_\varepsilon = \left(a^2(\sqrt{3}S_A+\varepsilon S) : b^2(\sqrt{3}S_B+\varepsilon S) : c^2(\sqrt{3}S_C+\varepsilon S)\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{1}{(S_A+\varepsilon\sqrt{3}S)(\sqrt{3}S_A+\varepsilon S)} : \frac{1}{(S_B+\varepsilon\sqrt{3}S)(\sqrt{3}S_B+\varepsilon S)} : \frac{1}{(S_C+\varepsilon\sqrt{3}S)(\sqrt{3}S_C+\varepsilon S)}\right).$$

The cubic (2) also contains the Fermat points. For these, the corresponding points on the Neuberg cubic are

$$\left(a^2(2(b^2+c^2-a^2)^3-5(b^2+c^2-a^2)b^2c^2-\varepsilon\cdot 2\sqrt{3}b^2c^2S) : \cdots : \cdots\right).$$

2.3. *Perspectivity with cevian triangle and the triangle of reflections.*

Proposition 8. *The reflection triangle of P is perspective with the triangle of reflections if and only if P lies on the cubic* (2). *The locus of the perspector Q is the Neuberg cubic* (1).

Proof. Note that $A^{(a)}$, $P^{(a)}$ and P_a are collinear, since they are the reflections of A, P and P_a in BC. Similarly, $B^{(b)}$, $P^{(b)}$, P_b are collinear, so are $C^{(c)}$, $P^{(c)}$, P_c. It follows that the reflection triangle of P is perspective with the triangle of reflections if and only if it is perspective with the cevian triangle of P. □

Remark. The correspondence (P, Q) in Proposition 8 is the inverse of the correspondence in Proposition 7 above.

2.4. *Perspectivity of triangle of reflections and anticevian triangles.*

Proposition 9. *The triangle of reflections is perspective to the anticevian triangle of P if and only if P lies on the Napoleon cubic,* i.e., *the isogonal cubic $pK(K,N)$*

$$\sum_{\text{cyclic}}(a^2(b^2+c^2)-(b^2-c^2)^2)u(c^2v^2-b^2w^2)=0. \tag{3}$$

The locus of the perspector Q is the Neuberg cubic (1).

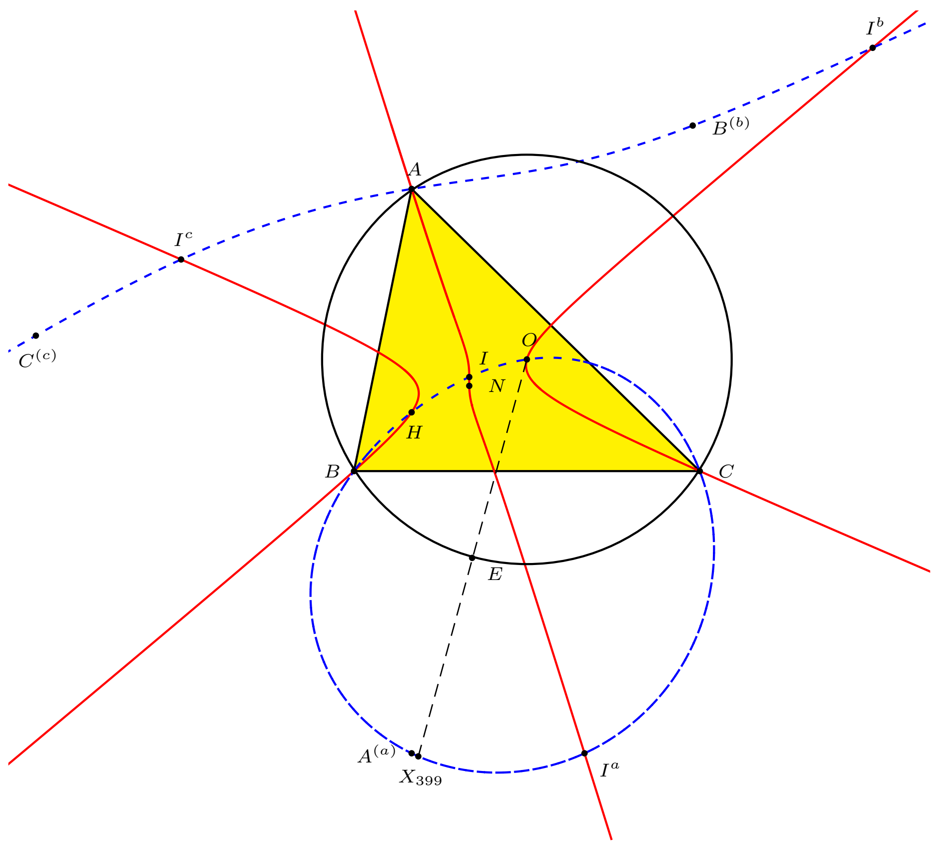


Figure 11. The Napoleon cubic and the Neuberg cubic

P	I	O	N	N^*	X_{195}
Q	W	X_{399}	E_∞	X_{1157}	O

Remarks. (1) For the case of O, the perspector is the Parry reflection point, the triangle center X_{399} which is the reflection of O in the Euler reflection point E. It is also the point of concurrency of reflections in sidelines of lines through vertices parallel to the Euler line (see [34, 35]). In other words, it is the perspector of the triangle of reflections and the cevian triangle of E_∞. The Euler line is the only direction for which these reflections are concurrent.

(2) N^* is the triangle center X_{54} in ETC, called the Kosnita point. It is also the perspector of the centers of the circles OBC, OCA, OAB (see Figure 12).

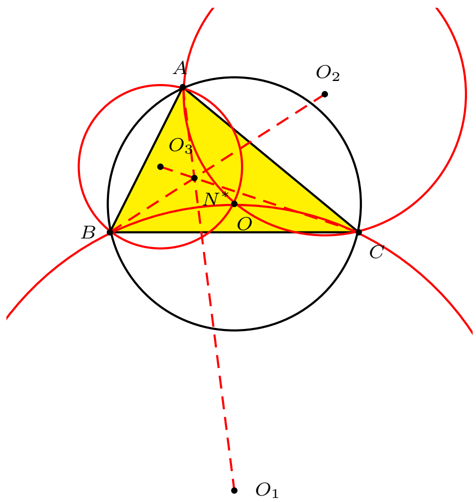


Figure 12. Perspectivity of the centers of the circles OBC, OCA, OAB

3. Reflection conjugates

Proposition 10. *The three circles $P^{(a)}BC$, $P^{(b)}CA$, and $P^{(c)}AB$ have a common point*

$$r_1(P) = \left(\frac{u}{(b^2+c^2-a^2)u(u+v+w)-(a^2vw+b^2wu+c^2uv)} : \cdots : \cdots\right). \tag{4}$$

It is easy to see that $r_1(P) = H$ if and only if P lies on the circumcircle. If $P \neq H$ and P does not lie on the circumcircle, we call $r_1(P)$ the reflection conjugate of P; it is the antipode of P in the rectangular circum-hyperbola $\mathscr{H}(P)$ through P (and the orthocenter H). It also lies on the circle of reflections $P^{(a)}P^{(b)}P^{(c)}$ (see Figure 13).

P	$r_1(P)$	midpoint	hyperbola
I	$X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$	X_{11}	Feuerbach
G	$X_{671} = \left(\frac{1}{b^2+c^2-2a^2} : \cdots : \cdots\right)$	X_{115}	Kiepert
O	$X_{265} = \left(\frac{S_A}{S^2-3S_{AA}} : \cdots : \cdots\right)$	X_{125}	Jerabek
K	$X_{67} = \left(\frac{1}{b^4+c^4-a^4-b^2c^2} : \cdots : \cdots\right)$	X_{125}	Jerabek
X_7	$X_{1156} = \left(\frac{a}{-2a^2+a(b+c)+(b-c)^2} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_8	$X_{1320} = \left(\frac{a(b+c-a)}{b+c-2a} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_{13}	X_{14}	X_{115}	Kiepert

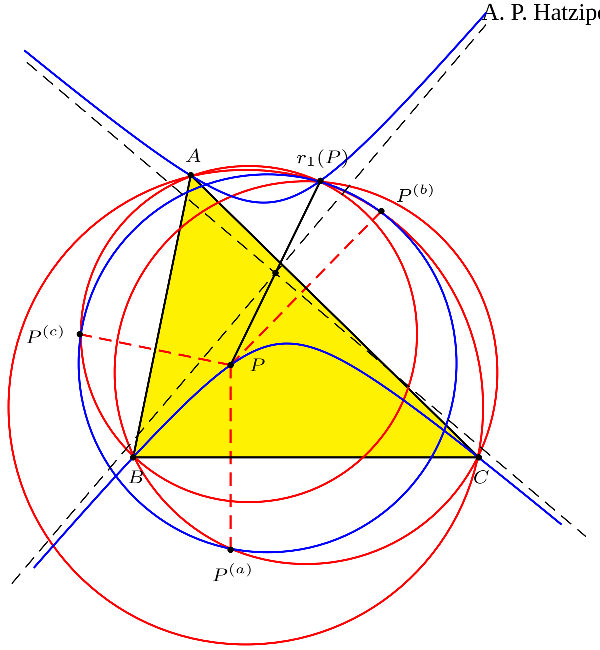


Figure 13. $r_1(P)$ and P are antipodal in $\mathscr{H}(P)$

Remark. $r_1(I) = X_{80}$ is also the perspector of the reflections of the excenters in the respective sidelines (see [42] and Figure 14). In §2.2, we have shown that $r_1(I)$ is the perspector of the reflection triangle of W.

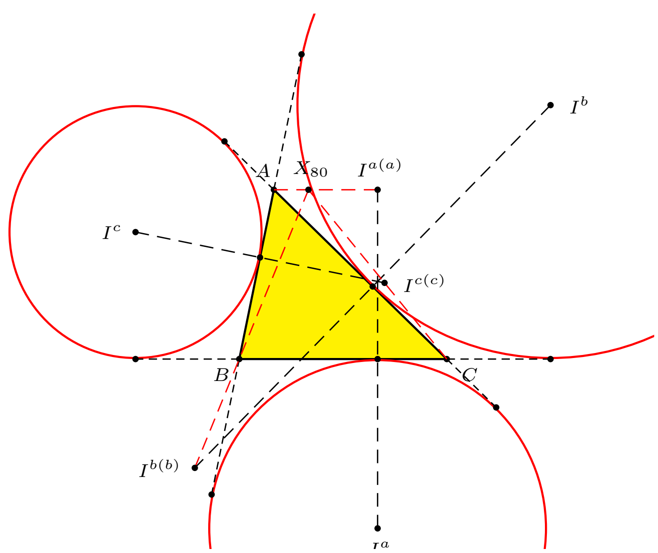


Figure 14. $r_1(I)$ as perspector of reflections of excenters

Proposition 11. *Let $P^{[a]}P^{[b]}P^{[c]}$ be the antipedal triangle of $P = (u : v : w)$. The reflections of the circles $P^{[a]}BC$ in BC, $P^{[b]}CA$ in CA and $P^{[c]}AB$ in AB all contain the reflection conjugate $r_1(P)$.*

Proof. Since B, P, C, and $P^{[a]}$ are concyclic, so are their reflections in the line BC. The circle $P^{[a]}BC$ is identical with the reflection of the circle $P^{(a)}BC$ in BC; similarly for the other two circles. The triad of circles therefore have $r_1(P)$ for a common point. □

Proposition 12. *Let $P_{[a]}P_{[b]}P_{[c]}$ be the pedal triangle of $P = (u : v : w)$. The reflections of the circles $AP_{[b]}P_{[c]}$ in $P_{[b]}P_{[c]}$, $BP_{[c]}P_{[a]}$ in $P_{[c]}P_{[a]}$, and $CP_{[a]}P_{[b]}$ in $P_{[a]}P_{[b]}$ have a common point*

$$\begin{aligned} r_2(P) = \; & (a^2(2a^2b^2c^2u + c^2((a^2+b^2-c^2)^2 - 2a^2b^2)v + b^2((c^2+a^2-b^2)^2 - 2c^2a^2)w) \\ & \cdot (b^2c^2u^2 - c^2(c^2-a^2)uv + b^2(a^2-b^2)uw - a^2(b^2+c^2-a^2)vw) : \cdots : \cdots). \end{aligned}$$

P	$r_2(P)$
G	$(a^2(b^4+c^4-a^4-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2) : \cdots : \cdots)$
I	$(a(b^2+c^2-a^2-bc)(a^3(b+c)-a^2(b^2+c^2)-a(b+c)(b-c)^2+(b^2-c^2)^2) : \cdots : \cdots)$
O	circles coincide with nine-point circle
H	$X_{1986} = \left(\frac{a^2((b^2+c^2-a^2)^2-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2)}{b^2+c^2-a^2} : \cdots : \cdots\right)$
X_{186}	X_{403}

Remarks. (1) For the case of $(H,\ X_{1986})$, see [22].

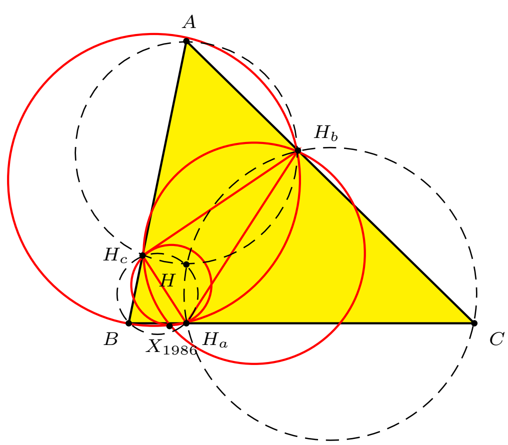


Figure 15. X_{1986} as the common point of reflections of circumcircles of residuals of orthic triangle

(2) For the pair $(X_{186},\ X_{403})$,
(i) X_{186} is the inverse of H in the circumcircle,
(ii) X_{403} is the inverse of H in the nine-point circle (see §4 below).

4. Inversion in the circumcircle

The inverse of P in the circumcircle is the point

$$\begin{aligned} P^{-1} = & \ (a^2(b^2c^2u^2 + b^2(a^2-b^2)wu + c^2(a^2-c^2)uv - a^2(b^2+c^2-a^2)vw) \\ : & \ b^2(c^2a^2v^2 + a^2(b^2-a^2)vw - b^2(c^2+a^2-b^2)wu + c^2(b^2-c^2)uv) \\ : & \ c^2(a^2b^2w^2 + a^2(c^2-a^2)vw + b^2(c^2-b^2)wu - c^2(a^2+b^2-c^2)uv)). \end{aligned}$$

4.1. *Bailey's theorem.*

Theorem 13 (Bailey [1, Theorem 5])**.** *The isogonal conjugates of P and $r_1(P)$ are inverse in the circumcircle.*

Proof. Let $P = (u : v : w)$, so that $P^* = (a^2vw : b^2wu : c^2uv)$. From the above formula,

$$\begin{aligned} (P^*)^{-1} = & \ \left(a^2vw(a^2vw + (a^2-b^2)uv + (a^2-c^2)wu - (b^2+c^2-a^2)u^2) : \cdots : \cdots\right) \\ = & \ \left(a^2vw(-(b^2+c^2-a^2)u(u+v+w) + a^2vw + b^2wu + c^2uv) : \cdots : \cdots\right). \end{aligned}$$

This clearly is the isogonal conjugate of $r_1(P)$ by a comparison with (4). $\square$

4.2. *The inverses of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.*

Proposition 14. *The inversive images of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$ in the circumcircle are perspective with ABC at N^*.*

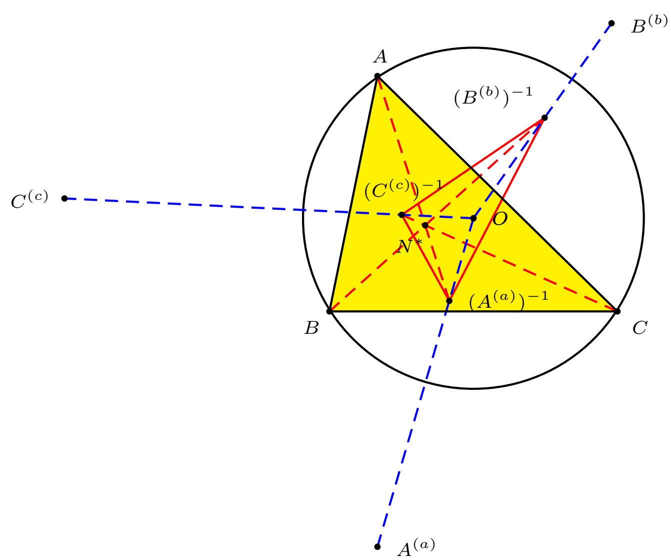


Figure 16. N^* as perspector of inverses of reflections of vertices in opposite sides

Proof. These inversive images are

$$\begin{aligned}(A^{(a)})^{-1} &= (-a^2(S^2 - 3S_{AA}) : b^2(S^2 + S_{AB}) : c^2(S^2 + S_{CA})),\\ (B^{(b)})^{-1} &= (a^2(S^2 + S_{AB}) : -b^2(S^2 - 3S_{BB}) : c^2(S^2 + S_{BC})),\\ (C^{(c)})^{-1} &= (a^2(S^2 + S_{CA}) : b^2(S^2 + S_{BC}) : -c^2(S^2 - 3S_{CC})).\end{aligned}$$

From these, the triangles ABC and $(A^{(a)})^{-1}(B^{(b)})^{-1}(C^{(c)})^{-1}$ are perspective at

$$N^* = \left(\frac{a^2}{S^2 + S_{BC}} : \frac{b^2}{S^2 + S_{CA}} : \frac{c^2}{S^2 + S_{AB}}\right).$$

□

Corollary 15 (Musselman [32]). *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are coaxial with common points O and $(N^*)^{-1}$.*

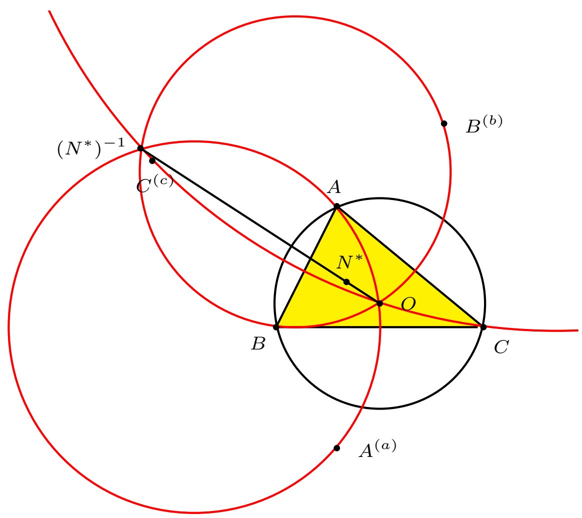


Figure 17. Coaxial circles $APA^{(a)}$, $BOB^{(a)}$, $COC^{(c)}$

Proof. Invert the configuration in Proposition 14 in the circumcircle. □

A generalization of Corollary 15 is the following.

Proposition 16 (van Lamoen [28]). *The circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ are coaxial if and only if P lies on the Neuberg cubic.*

Remarks. (1) Another example is the pair (I, W).

(2) If P is a point on the Neuberg cubic, the second common point of the circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ is also on the same cubic.

4.3. *Perspectivity of inverses of cevian and anticevian triangles.*

Proposition 17. *The inversive images of P_a, P_b, P_c in the circumcircle form a triangle perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P lies on the circumcircle, the perspector is the isogonal conjugate of the inferior of P. The locus is the isogonal conjugate of the nine-point circle* (see Figure 18).

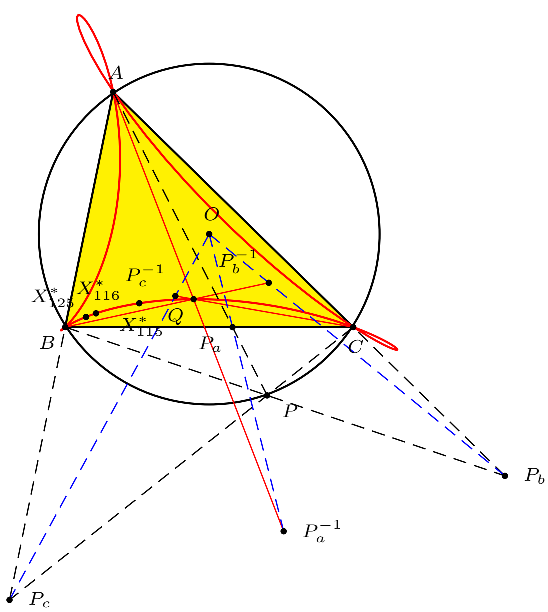


Figure 18. Isogonal conjugate of the nine-point circle

(b) *If P lies on the Euler line, the locus of the perspector is the bicevian conic through the traces of the isogonal conjugates of the Kiepert and Jerabek centers* (see Figure 19).

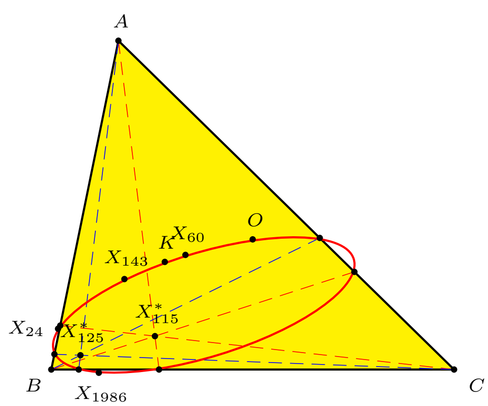


Figure 19. The bicevian conic $\mathscr{C}(X_{115}^*, X_{125}^*)$

The conic in Proposition 17(b) has equation

$$\sum_{\text{cyclic}} b^4c^4(b^2-c^2)^4(b^2+c^2-a^2)x^2 - 2a^6b^2c^2(c^2-a^2)^2(a^2-b^2)^2yz = 0.$$

P	O	G	H	N	X_{21}	H^{-1}
Q	O	K	X_{24}	X_{143}	X_{60}	X_{1986}

Remarks. (1) X_{21} is the Schiffler point, the intersection of the Euler lines of IBC, ICA, IAB (see [21]). Here is another property of X_{21} relating to reflections discovered by L. Emelyanov [11]. Let X be the reflection of the touch point of A-excircle in the line joining the other two touch points; similarly define Y and Z. The triangles ABC and XYZ are perspective at the Schiffler point (see Figure 20).

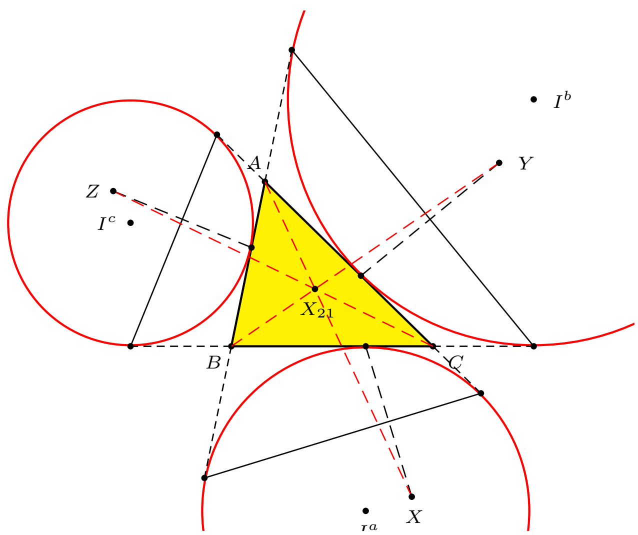


Figure 20. Schiffler point and reflections

(2) $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$ is the perspector of the orthic-of-orthic triangle (see [26]).

(3) X_{143} is the nine-point center of the orthic triangle.

(4) $X_{60} = \left(\frac{a^2(b+c-a)}{(b+c)^2} : \frac{b^2(c+a-b)}{(c+a)^2} : \frac{c^2(a+b-c)}{(a+b)^2}\right)$ is the isogonal conjugate of the outer Feuerbach point X_{12}.

Proposition 18. *The inversive images of P^a, P^b, P^c in the circumcircle form a triangle perspective with ABC if and only if P lies on*
(1) *the isogonal conjugate of the circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$, or*
(2) *the conic*

$$b^2c^2(b^2-c^2)x^2 + c^2a^2(c^2-a^2)y^2 + a^2b^2(a^2-b^2)z^2 = 0.$$

Remarks. (1) The circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$ is real only when ABC contains an obtuse angle. In this case, it is the circle with center H orthogonal to the circumcircle.

(2) The conic in (2) is real only when ABC is acute. It has center E and is homothetic to the Jerabek hyperbola, with ratio $\sqrt{\frac{1}{2\cos A\cos B\cos C}}$.

5. Dual triads of concurrent circles

Proposition 19. *Let* $\begin{matrix} X, Y, Z \\ X', Y' Z' \end{matrix}$ *be two triads of points. The triad of circles* $XY'Z'$, $YZ'X'$ *and* $ZX'Y'$ *have a common point if and only if the triad of circles* $X'YZ$, $Y'ZX$ *and* $Z'XY$ *have a common point.*

Proof. Let Q be a common point of the triad of circles $XY'Z'$, $YZ'X'$, $ZX'Y'$. Inversion with respect to a circle, center Q transforms the six points X, Y, Z, X', Y', Z' into x, y, z, x', y', z' respectively. Note that $xy'z'$, $yz'x'$ and $zx'y'$ are lines bounding a triangle $x'y'z'$. By Miquel's theorem, the circles $x'yz$, $y'zx$ and $z'xy$ have a common point q'. Their inverses $X'YZ$, $Y'ZX$ and $Z'XY$ have the inverse Q' of q' as a common point. □

Proposition 20 (Musselman [31])**.** *The circles* $AP^{(b)}P^{(c)}$, $BP^{(c)}P^{(a)}$, $CP^{(a)}P^{(b)}$ *intersect at the point* $r_0(P)$ *on the circumcircle.*

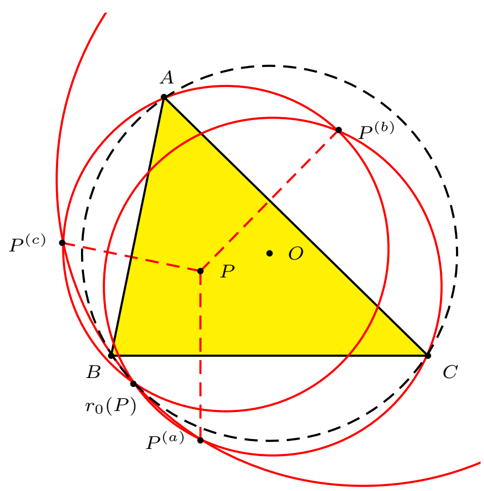


Figure 21. The circles $AP^{(b)}P^{(c)}$, $BP^{(c)}P^{(a)}$, $CP^{(a)}P^{(b)}$ intersect on the circumcircle

5.1. *Circles containing* $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.

5.1.1. *The triad of circles* $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$. Since the circles $A^{(a)}BC$, $AB^{(b)}C$ and $ABC^{(c)}$ all contain the orthocenter H, it follows that that the circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$ and $A^{(a)}B^{(b)}C$ also have a common point. This is the point $X_{1157} = (N^*)^{-1}$ (see [41, 18]). The radical axes of the circumcircle with each of these circles bound the anticevian triangle of N^* (see Figure 22).

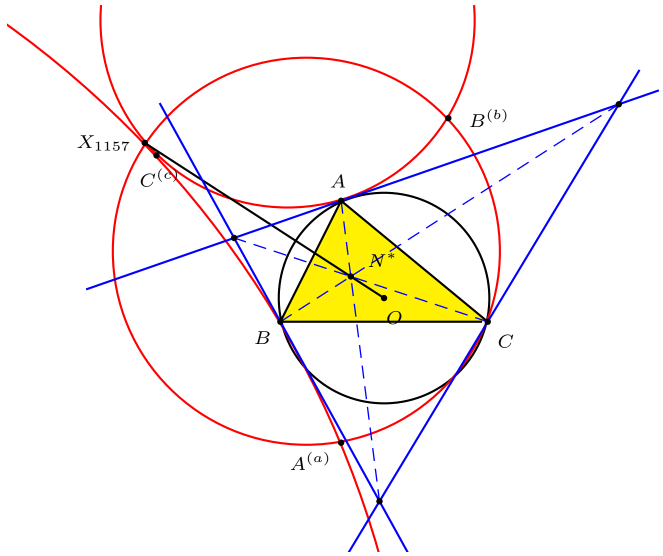


Figure 22. Concurrency of circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$

5.1.2. *The tangential triangle*. The circles $K^aB^{(b)}C^{(c)}$, $A^{(a)}K^bC^{(c)}$, $A^{(a)}B^{(b)}K^c$ have X_{399} the Parry reflection point as a common point. On the other hand, the circles $A^{(a)}K^bK^c$, $B^{(b)}K^cK^a$, $C^{(c)}K^aK^b$ are concurrent. (see [35]).

5.1.3. *The excentral triangle*. The circles $A^{(a)}I^bI^c$, $I^aB^{(b)}I^c$, $I^aI^bC^{(c)}$ also have the Parry reflection point X_{399} as a common point (see Figure 23).

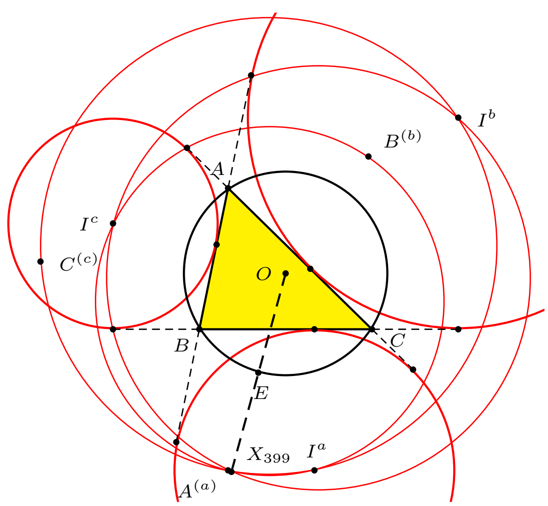


Figure 23. The Parry reflection point X_{399}

The Parry reflection point X_{399}, according to Evans [12], is also the common point of the circles $II^aA^{(a)}$, $II^bB^{(b)}$ and $II^cC^{(c)}$.

By Proposition 19, the circles $I^aB^{(b)}C^{(c)}$, $A^{(a)}I^bC^{(c)}$ and $A^{(a)}B^{(b)}I^c$ have a common point as well. Their centers are perspective with ABC at the point

$$(a(a^2(a+b+c)-a(b^2-bc+c^2)-(b+c)(b-c)^2):\cdots:\cdots)$$

on the OI line.

5.1.4. *Equilateral triangles on the sides* . For $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1. Now, for $\varepsilon = \pm 1$, the circles $A^{(a)}B_\varepsilon C_\varepsilon$, $B^{(b)}C_\varepsilon A_\varepsilon$, $C^{(c)}A_\varepsilon B_\varepsilon$ are concurrent at the superior of the Fermat point $F_{-\varepsilon}$ (see [36]).

5.1.5. *Degenerate triangle of reflections* .

Proposition 21 ([18, Theorem 4])**.** *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*
(1) *The reflection triangle $A^{(a)}B^{(b)}C^{(c)}$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB^{(b)}C^{(c)}$, $BC^{(c)}A^{(a)}$, $CA^{(a)}B^{(b)}$ pass through O.*

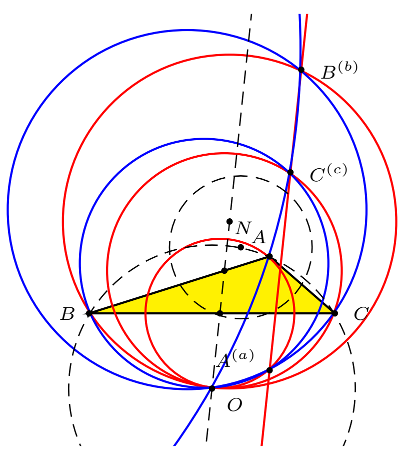


Figure 24. Triangle with degenerate triangle of reflections

5.2. *Reflections in a point.*

Proposition 22. *Given $P = (u : v : w)$, let X, Y, Z be the reflections of A, B, C in P.*

(a) *The circles AYZ, BZX, CXY have a common point a point*

$$r_3(P) = \left(\frac{1}{c^2v(w+u-v)-b^2w(u+v-w)}:\cdots:\cdots\right)$$

which is also the fourth intersection of the circumcircle and the circumconic with center P (see Figure 25).

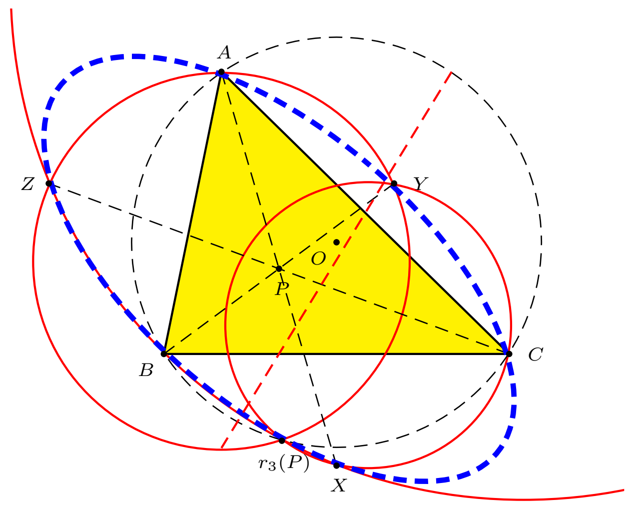


Figure 25. Circles AYZ, BZX, CXY through $r_3(P)$ on circumcircle and circumconic with center P

(b) *The circles* XBC, YCA *and* ZAB *intersect have a common point*

$$r_4(P) = \left(\frac{v+w-u}{2a^2vw-(v+w-u)(bw+cv)} : \cdots : \cdots\right)$$

which is the antipode of $r_3(P)$ *on the circumconic with center* P (see Figure 26). *It is also the reflection conjugate of the superior of* P.

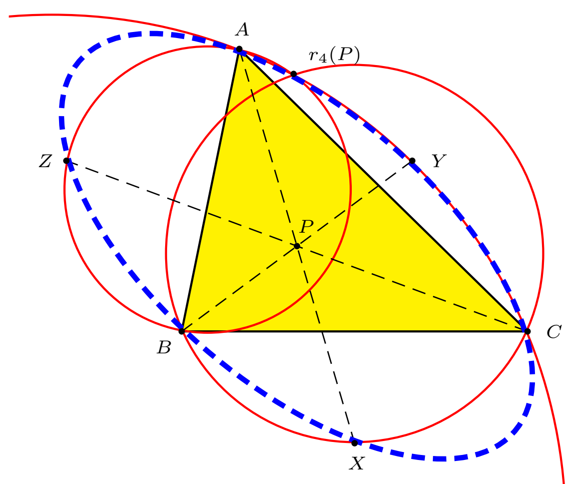


Figure 26. Circles XBC, YCA, ZAB through $r_4(P)$ circumconic with center P

(c) *For a given Q on the circumcircle, the locus of P for which $r_3(P) = Q$ is the bicevian conic $\mathscr{C}(G, Q)$.*

Here are some examples of $r_3(P)$ and $r_4(P)$.

P	G	I	N	K	X_9	X_{10}	X_{2482}	X_{214}	X_{1145}
$r_3(P)$	X_{99}	X_{100}	E	E	X_{100}	X_{100}	X_{99}	X_{100}	X_{100}
$r_4(P)$	$r_1(G)$	X_{1320}	$r_1(O)$	X_{895}	X_{1156}	X_{80}	G	I	N_a

6. Reflections and Miquel circles

6.1. *The reflection of I in O.* If X, Y, Z are the points of tangency of the excircles with the respective sides, the Miquel point of the circles AYZ, BZX, CXY is the reflection of I in O, which is X_{40} in ETC. It is also the circumcenter of the excentral triangle.

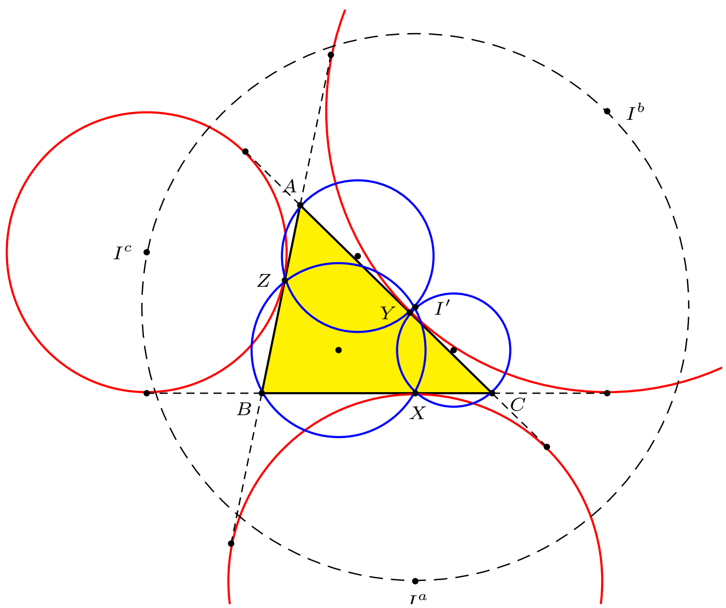


Figure 27. Reflection of I in O as a Miquel point

6.2. *Miquel circles.* For a real number t, we consider the triad of points

$$X_t = (0 : 1-t : t), \quad Y_t = (t : 0 : 1-t), \quad Z_t = (1-t : t : 0)$$

on the sides of the reference triangle. The circles AY_tZ_t, BZ_tX_t and CX_tY_t intersect at the Miquel point

$$\begin{aligned} M_t = & \left(a^2(b^2t^2 + c^2(1-t)^2 - a^2t(1-t))\right. \\ & : b^2(c^2t^2 + a^2(1-t)^2 - b^2t(1-t)) \\ & : \left. c^2(a^2t^2 + b^2(1-t)^2 - c^2t(1-t))\right). \end{aligned}$$

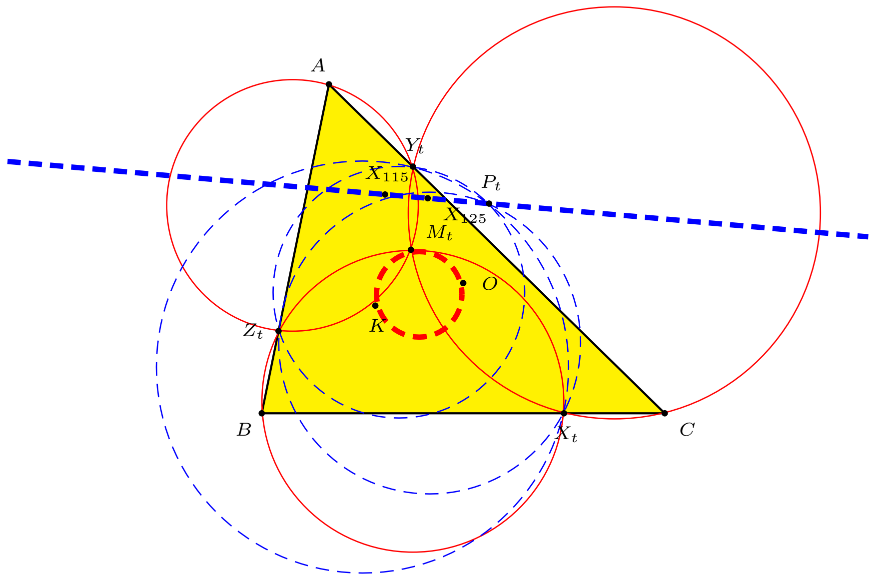


Figure 28. Miquel circles and their reflections

The locus of M_t is the Brocard circle with diameter OK, as is evident from the data in the table below; see Figure 28 and [37, 17].

t	M_t	P_t
0	$\Omega = \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}$	$\frac{1}{c^2-a^2} : \frac{1}{a^2-b^2} : \frac{1}{b^2-c^2}$
$\frac{1}{2}$	O	$X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$
1	$\Omega' = \frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}$	$\frac{1}{a^2-b^2} : \frac{1}{b^2-c^2} : \frac{1}{c^2-a^2}$
∞	K	$((b^2-c^2)(b^2+c^2-2a^2) : \cdots : \cdots)$
$\frac{a^2b^2-c^4}{(b^2-c^2)(a^2+b^2+c^2)}$	$B_1 = a^2 : c^2 : b^2$	$-(b^4-c^4) : b^2(c^2-a^2) : c^2(a^2-b^2)$
$\frac{c^2a^2-b^4}{(c^2-a^2)(a^2+b^2+c^2)}$	$B_2 = c^2 : b^2 : a^2$	$a^2(b^2-c^2) : -(c^4-a^4) : c^2(a^2-b^2)$
$\frac{a^2b^2-c^4}{(a^2-b^2)(a^2+b^2+c^2)}$	$B_3 = b^2 : a^2 : c^2$	$a^2(b^2-c^2) : b^2(c^2-a^2) : -(a^4-b^4)$

6.3. *Reflections of Miquel circles.* Let A_t, B_t, C_t be the reflections of A in Y_tZ_t, B in Z_tX_t, C in X_tY_t. The circles $A_tY_tZ_t$, $B_tZ_tX_t$ and $C_tX_tY_t$ also have a common point

$$\begin{aligned} P_t = \; & ((b^2-c^2)((c^2-a^2)t + (a^2-b^2)(1-t)) \\ & : (c^2-a^2)((a^2-b^2)t + (b^2-c^2)(1-t)) \\ & : (a^2-b^2)((b^2-c^2)t + (c^2-a^2)(1-t))). \end{aligned}$$

For $t = \frac{1}{2}$, all three reflections coincide with the nine-point circle. However, P_t approaches the Kiepert center $X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$ as $t \to \frac{1}{2}$. The locus of P_t is the line

$$\frac{x}{b^2-c^2} + \frac{y}{c^2-a^2} + \frac{z}{a^2-b^2} = 0,$$

which clearly contains both the Kiepert center X_{115} and the Jerabek center X_{125} (see Figure 28). This line is the radical axis of the nine-point circle and the pedal circle of G. These two centers are the common points of the two circles (see Figure 29).

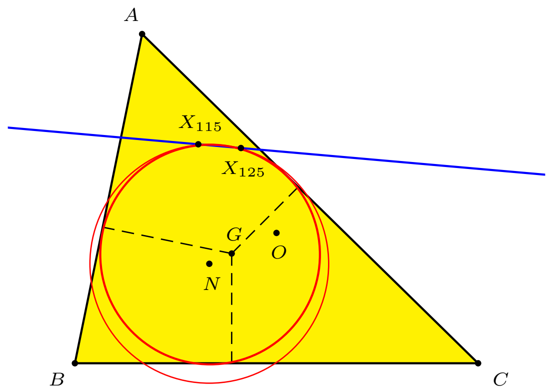


Figure 29. X_{115} and X_{125} as the intersections of nine-point circle and pedal circle of G

6.4. *Reflections of circles of anticevian residuals.* Consider points X^t, Y^t, Z^t such that A, B, C divide Y^tZ^t, Z^tX^t, X^tY^t respectively in the ratio $1-t:t$. Figure 30 shows the construction of these points from X_t, Y_t, Z_t and the midpoints of the sides. Explicitly,

$$\begin{aligned} X^t &= (-t(1-t) : (1-t)^2 : t^2), \\ Y^t &= (t^2 : -t(1-t) : (1-t)^2), \\ Z^t &= ((1-t)^2 : t^2 : -t(1-t)). \end{aligned}$$

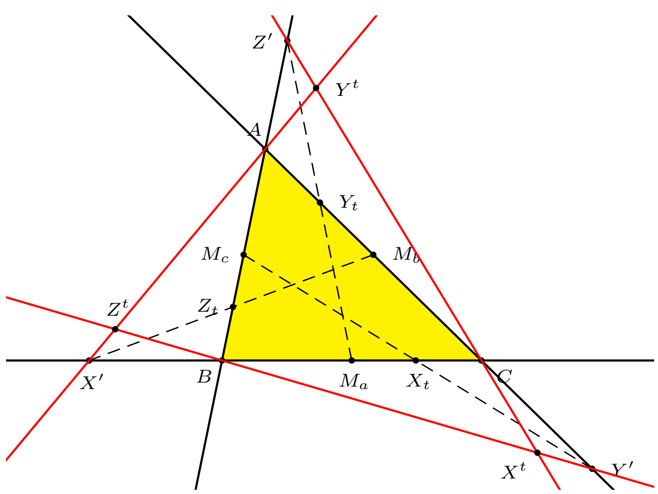


Figure 30. Construction of $X^tY^tZ^t$ from $X_tY_tZ_t$

The circles X^tBC, Y^tCA, Z^tAB intersect at the isogonal conjugate of M_t. The locus of the intersection is therefore the isogonal conjugate of the Brocard circle. On the other hand, the reflections of the circles X_tBC, Y_tCA, Z_tAB intersect at the point

$$\left(\frac{1}{(b^2+c^2-2a^2)t+(a^2-b^2)}:\frac{1}{(c^2+a^2-b^2)t+(b^2-c^2)}:\frac{1}{(a^2+b^2-c^2)t+(c^2-a^2)}\right),$$

which traverses the Steiner circum-ellipse.

7. Reflections of a point in various triangles

7.1. *Reflections in the medial triangle.* If $P=(u:v:w)$, the reflections in the sides of the medial triangle are

$$\begin{aligned}
X' &= ((S_B+S_C)(v+w):S_Bv-S_C(w-u):S_Cw+S_B(u-v)),\\
Y' &= (S_Au+S_C(v-w):(S_C+S_A)(w+u):S_Cw-S_A(u-v)),\\
Z' &= (S_Au-S_B(v-w):S_Bv+S_A(w-u):(S_A+S_B)(u+v)).
\end{aligned}$$

Proposition 23. *The reflection triangle of P in the medial triangle is perspective with ABC if and only if P lies on the Euler line or the nine-point circle of ABC.*

(a) *If P lies on the Euler line, the perspector traverses the Jerabek hyperbola.*

(b) *If P lies on the nine-point circle, the perspector is the infinite point which is the isogonal conjugate of the superior of P.*

Remarks. (1) If $P=E_t$, then the perspector $Q=E^*_{t'}$, where

$$t'=\frac{a^2b^2c^2(1-t)}{a^2b^2c^2(1-t)-4S_{ABC}(1-2t)}.$$

P	G	O	H	N	X_{25}	X_{403}	X_{427}	X_{429}	X_{442}	E_∞
Q	$H^\bullet$	X_{68}	H	O	X_{66}	X_{74}	K	X_{65}	X_{72}	X_{265}

(2) For $P=G$, these reflections are the points

$$X'=(2a^2:S_B:S_C),\quad Y'=(S_A:2b^2:S_C),\quad Z'=(S_A:S_B:2c^2).$$

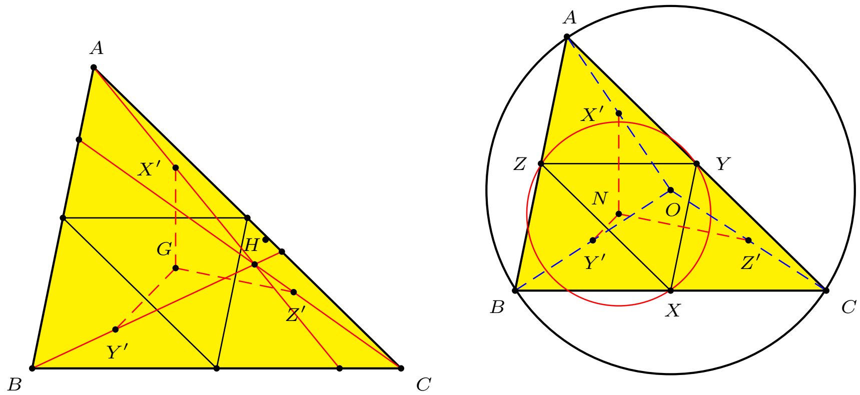
They are trisection points of the corresponding $H^\bullet-$cevian (see Figure 31(a)). The perspector of $X'Y'Z'$ is $X_{69}=H^\bullet$.

(3) If $P=N$, the circumcenter of the medial triangle, the circle through its reflections in the sides of the medial triangle is congruent to the nine-point circle and has center at the orthocenter of the medial triangle, which is the circumcenter O of triangle ABC. These reflections are therefore the midpoints of the circumradii OA, OB, OC (see Figure 31(b)).

(4) $X_{25}=\left(\frac{a^2}{b^2+c^2-a^2}:\frac{b^2}{c^2+a^2-b^2}:\frac{c^2}{a^2+b^2-c^2}\right)$ is the homothetic center of the tangential and orthic triangles. It is also the perspector of the tangential triangle and the reflection triangle of K. In fact,

$$A'X':X'H_a=a^2:S_A,\quad B'Y':Y'H_b=b^2:S_B,\quad C'Z':Z'H_c=c^2:S_C.$$

(5) $X_{427}=\left(\frac{b^2+c^2}{b^2+c^2-a^2}:\frac{c^2+a^2}{c^2+a^2-b^2}:\frac{a^2+b^2}{a^2+b^2-c^2}\right)$ is the inverse of X_{25} in the orthocentroidal circle. It is also the homothetic center of the orthic triangle and the



(a) Reflections of G (b) Reflections of N

Figure 31. Reflections in the medial triangle

triangle bounded by the tangents to the nine-point circle at the midpoints of the sidelines (see [7]).

(6) If P is on the nine-point circle, it is the inferior of a point P' on the circumcircle. In this case, the perspector Q is the infinite point which is the isogonal conjugate of P'. In particular, for the Jerabek center $J = X_{125}$ (which is the inferior of the Euler reflection point $E = X_{110}$), the reflections are the pedals of the vertices on the Euler line. The perspector is the infinite point of the perpendicular to the Euler line (see Figure 32).

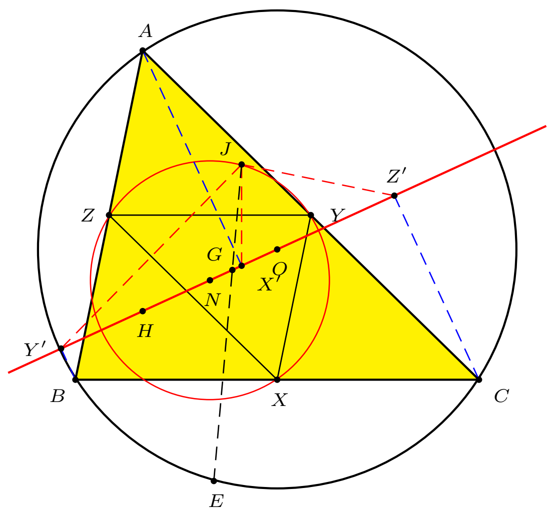


Figure 32. Reflections of Jerabek center in medial triangle

Proposition 24. *The reflections of AP, BP, CP in the respective sidelines of the medial triangle are concurrrent* (i.e., triangle $X'Y'Z'$ is perspective with the orthic triangle) *if and only if P lies on the Jerabek hyperbola of ABC. As P traverses the Jerabek hyperbola, the locus of the perspector is the Euler line* (see Figure 33).

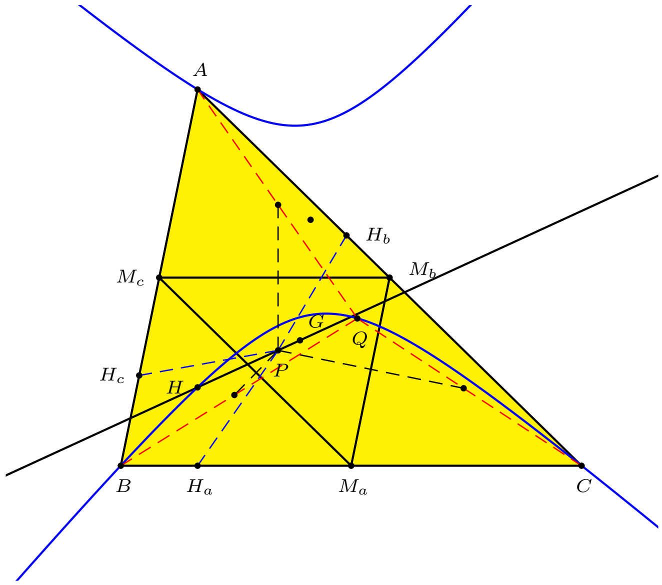


Figure 33. Reflections in medial triangle

Remark. The correspondence is the inverse of the correspondence in Proposition 23(a).

7.2. *Reflections in the orthic triangle.*

Proposition 25. *The reflection triangle of P in the orthic triangle $H_aH_bH_c$ is perspective with ABC if and only if P lies on the cubic*

$$\sum_{\text{cyclic}} \frac{u}{b^2+c^2-a^2}\left(f(c,a,b)v^2 - f(b,c,a)w^2\right) = 0. \tag{5}$$

where

$$f(a,b,c) = a^4(b^2+c^2) - 2a^2(b^4-b^2c^2+c^4) + (b^2+c^2)(b^2-c^2)^2.$$

The locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}} \frac{a^2(S^2-3S_{AA})x}{b^2+c^2-a^2}\left(c^4(S^2-S_{CC})y^2 - b^4(S^2-S_{BB})z^2\right) = 0. \tag{6}$$

Remarks. (1) The cubic (5) is the isocubic p$K(X_{3003},\ H)$, labeled K339 in TCT.
(2) The cubic (6) is the isocubic p$K(X_{186},\ X_{571})$ (see Figure 34).
(3) Here are some correspondences of

P	H	O	X_{1986}
Q	X_{24}	O	X_{186}

The reflection triangle of H in the orthic triangle is homothetic to ABC at $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$.

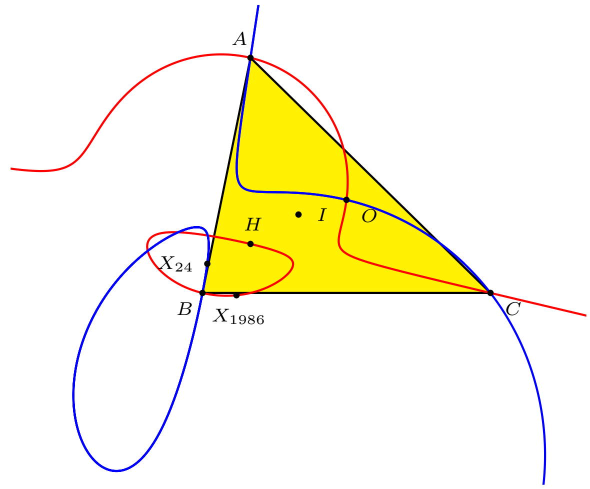


Figure 34. The cubics K339 and p$K(X_{186}, X_{571})$

7.3. *Reflections in the pedal triangle.*

Proposition 26. *The reflection triangle of P in its pedal triangle are perspective with*
(a) ABC *if and only if P lies on the orthocubic cubic*

$$\sum_{\text{cyclic}} S_{BC}x(c^2y^2 - b^2z^2) = 0, \tag{7}$$

(b) *the pedal triangle if and only if P lies on the Neuberg cubic* (1).

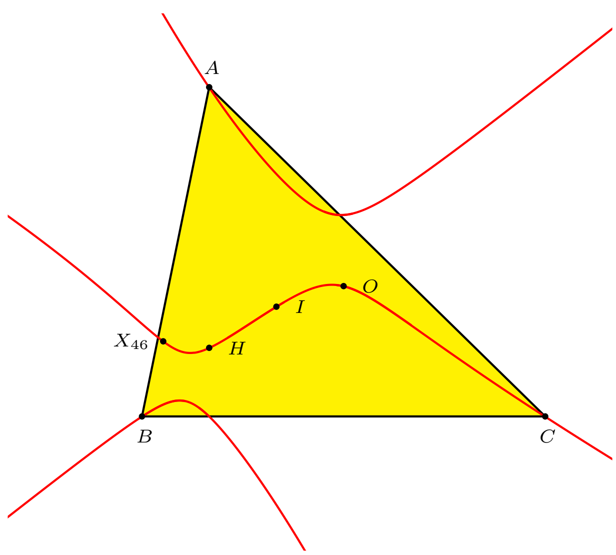


Figure 35. The orthocubic cubic

Remarks. (1) The orthocubic defined by (7) is the curve K006 in CTC.

(2) Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{68}	X_{24}
perspector with pedal triangle	I	O	

The missing entry is the perspector of the orthic triangle and the reflection triangle of H in the orthic triangle; it is the triangle center

$$(a^2S_{BC}(3S^2 - S_{AA})(a^2b^2c^2 + 2S_A(S^2 + S_{BC})) : \cdots : \cdots).$$

7.4. *Reflections in the reflection triangle.*

Proposition 27. *The reflections of P in the sidelines of its reflection triangle are perspective with*
(a) ABC *if and only if P lies on the Napoleon cubic* (3).
(b) *the reflection triangle if and only if P lies on the Neuberg cubic* (1).

Remark. Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{265}	X_{186}
perspector with reflection triangle	I	O	

The missing entry is the perspector of $H^{(a)}H^{(b)}H^{(c)}$ and the reflection triangle of H in $H^{(a)}H^{(b)}H^{(c)}$; it is the triangle center

$$(a^2S_{BC}(a^2b^2c^2(3S^2 - S_{AA}) + 8S_A(S^2 + S_{BC})(S^2 - S_{AA})) : \cdots : \cdots).$$

8. Reflections in lines

8.1. *Reflections in a line.*

Proposition 28. *Let ℓ be a line through the circumcenter O, and $A'B'C'$ be the reflection of ABC in ℓ. $A'B'C'$ is orthologic to ABC at the fourth intersection of the circumcircle and the rectangular circum-hyperbola which is the isogonal conjugate of ℓ* (see Figure 36).

Remarks. (1) By symmetry, if $A'B'C'$ is orthologic to ABC at Q, then ABC is orthologic to $A'B'C'$ at the reflection of Q in the line ℓ.

Line ℓ	Q	Q'
Euler line	$X_{74} = \left(\frac{a^2}{S^2-3S_{BC}} : \ : \ \right)$	X_{477}
Brocard axis	$X_{98} = \left(\frac{1}{S_{BC}-S_{AA}} : \cdots : \cdots\right)$	X_{2698}
OI	$X_{104} = \left(\frac{a}{a^2(b+c)-2abc-(b+c)(b-c)^2} : \cdots : \cdots\right)$	X_{953}

(2) The orthology is valid if ℓ is replaced by an arbitrary line.

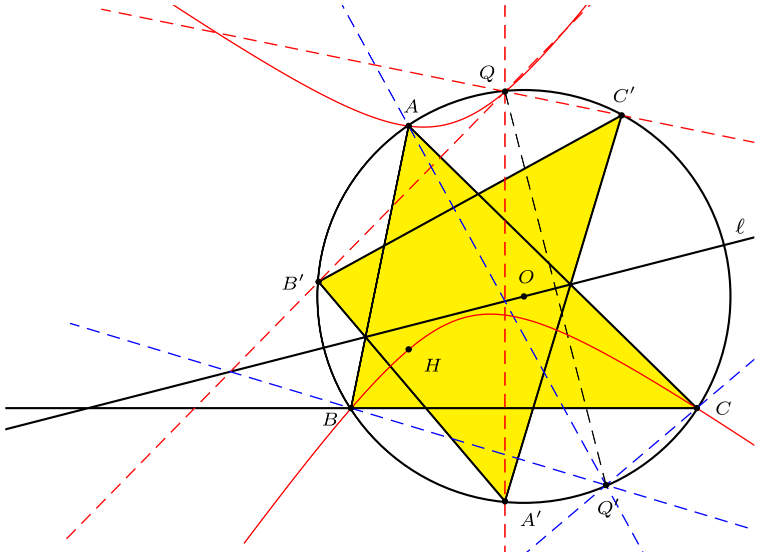


Figure 36. Orthology of triangles symmetric in ℓ

Proposition 29. *Let ℓ be a line through a given point P, and A', B', C' the reflections of A, B, C in ℓ. The lines $A'P$, $B'P$, $C'P$ intersect the sidelines BC, CA, AB respectively at X, Y, Z. The points X, Y, Z are collinear, and the line $\mathscr{L}$ containing them envelopes the inscribed conic with P as a focus* (see Figure 37).

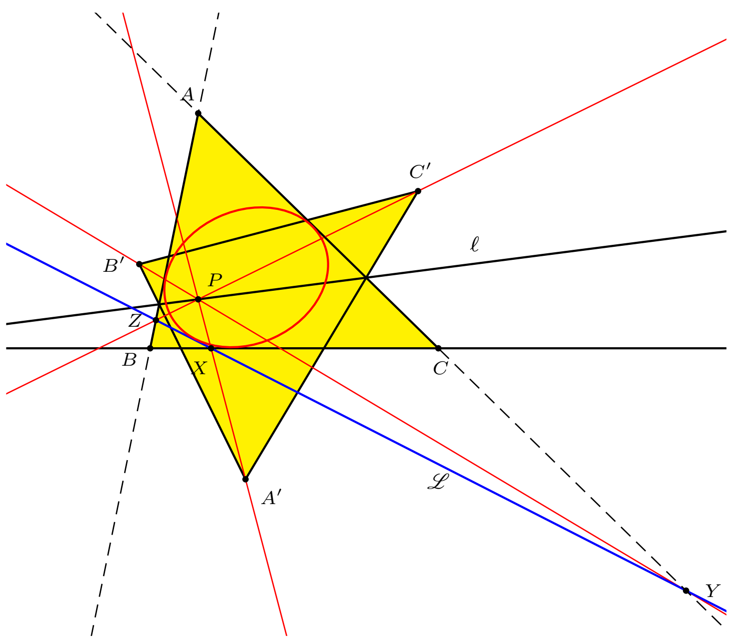


Figure 37. Line $\mathscr{L}$ induced by reflections in ℓ

Proof. Let ℓ be the line joining $P = (u : v : w)$ and $Q = (x : y : z)$. The line $\mathscr{L}$ containing X, Y, Z is

$$\sum_{\text{cyclic}} \frac{u\mathbb{X}}{(b^2u^2 + 2S_Cuv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_Bwu + c^2u^2)(vx - uy)^2} = 0,$$

equivalently with line coordinates

$$\left(\frac{u}{(b^2u^2 + 2S_Cuv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_Bwu + c^2u^2)(vx - uy)^2} : \cdots : \cdots\right).$$

Now, the inscribed conic $\mathscr{C}$ with a focus at $P = (u : v : w)$ has center the midpoint between P and P^* and perspector

$$\left(\frac{1}{u(c^2v^2 + 2S_Avw + b^2w^2)} : \frac{1}{v(a^2w^2 + 2S_Bwu + c^2u^2)} : \frac{1}{w(b^2u^2 + 2S_Cuv + a^2w^2)}\right).$$

Its dual conic is the circumconic

$$\sum_{\text{cyclic}} \frac{u(c^2v^2 + 2S_Avw + b^2w^2)}{\mathbb{X}} = 0,$$

which, as is easily verified, contains the line $\mathscr{L}$ (see [38, §10.6.4]). This means that $\mathscr{L}$ is tangent to the inscribed conic $\mathscr{C}$. □

Remarks. (1) For the collinearity of X, Y, Z, see [23].

(2) The line $\mathscr{L}$ touches the inscribed conic $\mathscr{C}$ at the point

$$\left(\frac{1}{u(c^2v^2 + 2S_Avw + b^2w^2)}\left(\frac{(uz - wx)^2}{a^2w^2 + 2S_Bwu + c^2u^2} - \frac{(vx - uy)^2}{b^2u^2 + 2S_Cuv + a^2v^2}\right)^2 : \cdots : \cdots\right).$$

(i) If $P = I$, then the line $\mathscr{L}$ is tangent to the incircle. For example, if ℓ is the OI-line, then $\mathscr{L}$ touches the incircle at

$$X_{3025} = (a^2(b - c)^2(b + c - a)(a^2 - b^2 + bc - c^2) : \cdots : \cdots).$$

(ii) If P is a point on the circumcircle, then the conic $\mathscr{C}$ is an inscribed parabola, with focus P and directrix the line of reflections of P (see §1.2). If we take ℓ to be the diameter OP, then the line $\mathscr{L}$ touches the parabola at the point

$$(a^4(b^2 - c^2)(S^2 - 3S_{AA})^2 : \cdots : \cdots).$$

(3) Let ℓ be the Euler line. The two lines $\mathscr{L}$ corresponding to O and H intersect at

$$X_{3258} = ((b^2 - c^2)^2(S^2 - 3S_{BC})(S^2 - 3S_{AA}) : \cdots : \cdots)$$

on the nine-point circle, the inferior of X_{476}, the reflection of E in the Euler line (see [15]). More generally, for isogonal conjugate points P and P^* on the Macay cubic K003, *i.e.*, p$K(K, O)$, the two corresponding lines $\mathscr{L}$ with respect to the line PP^* intersect at a point on the common pedal circle of P and P^*. For other results, see [24, 16].

8.2. *Reflections of lines in cevian triangle.*

Proposition 30 ([9])**.** *The reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at*

$$r_5(P) = \left(u \left(-\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw} \right) : \cdots : \cdots \right). \qquad (8)$$

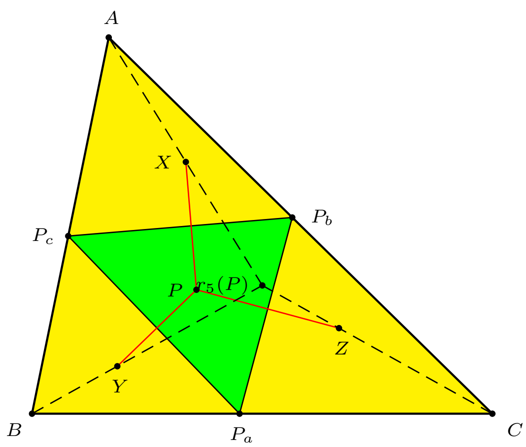


Figure 38. Reflections in sides of cevian triangle

Proof. Relative to the triangle $P_aP_bP_c$, the coordinates of P are $(v + w : w + u : u + v)$. Similarly, those of A, B, C are

$$(-(v+w) : w+u : u+v), \quad (v+w : -(w+u) : u+v), \quad (v+w : w+u : -(u+v)).$$

Triangle ABC is the anticevian triangle of P relative to $P_aP_bP_c$. The perspectivity of ABC and the reflection triangle of P in $P_aP_bP_c$ follows from Proposition 6.

The reflection of P in the line P_bP_c is the point

$$\begin{aligned} X = &\left(u \left(\frac{3a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} - \frac{b^2+c^2-a^2}{vw} + \frac{2(c^2+a^2-b^2)}{wu} + \frac{2(a^2+b^2-c^2)}{uv} \right) \right. \\ &\left. : v \left(\frac{a^2}{u^2} - \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{c^2+a^2-b^2}{wu} \right) : \; w \left(\frac{a^2}{u^2} + \frac{b^2}{v^2} - \frac{c^2}{w^2} + \frac{a^2+b^2-c^2}{uv} \right) \right). \end{aligned}$$

Similarly, the coordinates of the reflections Y of P in P_cP_a, and Z of P in P_aP_b can be written down. From these, it is clear that the lines AX, BY, CZ intersect at the point with coordinates given in (8). □

The triangle XYZ is clearly orthologic with the cevian triangle $P_aP_bP_c$, since the perpendiculars from X to P_bP_c, Y to P_cP_a, and Z to P_aP_b intersect at P. It follows that the perpendiculars from P_a to YZ, P_b to ZX, and P_c to XY are also concurrent. The point of concurrency is

$$r_6(P) = \left(u \left(\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw} \right) : \cdots : \cdots \right).$$

In fact, P_a, P_b, P_c lie respectively on the perpendicular bisectors of YZ, ZX, XY. The point $r_6(P)$ is the center of the circle XYZ (see Figure 39). As such, it is the isogonal conjugate of P in its own cevian triangle.

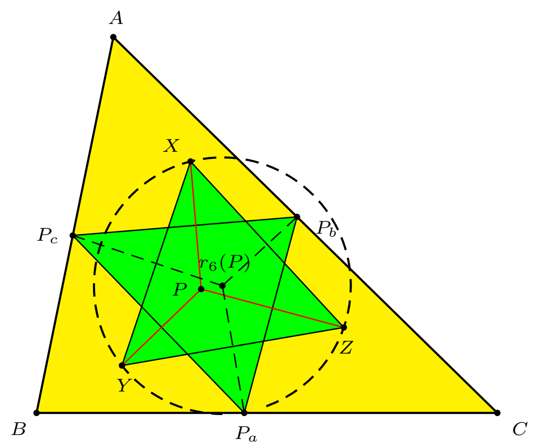


Figure 39. Circumcircle of reflections in cevian triangle

P	I	G	H	G_e	X_{99}	X_{100}	E
$r_5(P)$	X_{35}	$H^\bullet$	X_{24}	X_{57}	X_{115}^*	F_e^*	X_{125}^*
$r_6(P)$		X_{141}	H	X_{354}		X_{1618}	

Remarks. (1) In ETC, $r_5(P)$ is called the Orion transform of P.

(2) $X_{35} = (a^2(b^2+c^2-a^2+bc) : b^2(c^2+a^2-b^2+ca) : c^2(a^2+b^2-c^2+ab))$ divides OI in the ratio $R : 2r$. On the other hand,

$$r_6(I) = (a^2(b^2+c^2-a^2+3bc) : b^2(c^2+a^2-b^2+3ca) : c^2(a^2+b^2-c^2+3ab))$$

divides OI in the ratio $3R : 2r$ (see also Remark (3) following Proposition 31 below).

8.3. *Reflections of sidelines of cevian triangles.* Let P be a point with cevian triangle $P_aP_bP_c$. It is clear that the lines BC, P_bP_c, and their reflections in one another concur at a point on the trilinear polar of P (see Figure 40).

This is the same for line CA, P_cP_a and their reflections in one another; similarly for AB and P_aP_b. Therefore, the following four triangles are line-perspective at the trilinear polars of P:

(i) ABC,

(ii) the cevian triangle of P,

(iii) the triangle bounded by the reflections of P_bP_c in BC, P_cP_a in CA, P_aP_b in AB,

(iv) the triangle bounded by the reflections of BC in P_bP_c, CA in P_cP_a, AB in P_aP_b.

It follows that these triangles are also vertex-perspective (see [25, Theorems 374, 375]. Clearly if P is the centroid G, these triangles are all homothetic at G.

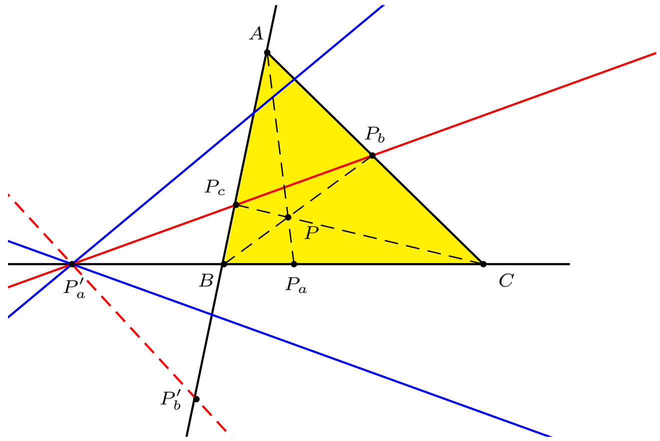


Figure 40. Reflections of sidelines of cevian triangle

Proposition 31. *Let $P_aP_bP_c$ be the cevian triangle of $P = (u : v : w)$.*
(a) *The reflections of P_bP_c in BC, P_cP_a in CA, and P_aP_b in AB bound a triangle perspective with ABC at*

$$r_7(P) = \left(\frac{a^2}{u((c^2+a^2-b^2)v+(a^2+b^2-c^2)w)} : \cdots : \cdots\right).$$

(b) *The reflections of BC in P_bP_c, CA in P_cP_a, and AB in P_aP_b bound a triangle perspective with ABC at*

$$r_8(P) = \left(\frac{a^2vw + u(S_Bv + S_Cw)}{-3a^2v^2w^2 + b^2w^2u^2 + c^2u^2v^2 - 2uvw(S_Au + S_Bv + S_Cw)} : \cdots : \cdots\right).$$

Here are some examples.

P	I	O	H	K	X_{19}	E	X_{393}
$r_7(P)$	X_{21}	X_{1105}	O	$N^\bullet$	X_{1444}	X_{925}	$H^\bullet$

Remarks. (1) The pair $(X_{19},\ X_{1444})$.
(i) $X_{19} = \left(\frac{a}{S_A} : \frac{b}{S_B} : \frac{c}{S_C}\right)$ is the Clawson point. It is the perspector of the triangle bounded by the common chords of the circumcircle with the excircles.
(ii) $X_{1444} = \left(\frac{aS_A}{b+c} : \frac{bS_B}{c+a} : \frac{cS_C}{a+b}\right)$ is the intersection of X_3X_{69} and X_7X_{21}.

(2) $X_{393} = \left(\frac{1}{S_{AA}} : \frac{1}{S_{BB}} : \frac{1}{S_{CC}}\right)$ is the barycentric square of the orthocenter. Let $H_aH_bH_c$ be the orthic triangle, and A_b, A_c the pedals of H_a on CA and AB respectively, and $A' = BA_c \cap CA_b$. Similarly define B' and C'. The lines AA', BB', CC' intersect at X_{393} (see [40]).

(3) The coordinates of $r_8(P)$ are too complicated to list here. For $P = I$, the incenter, note that
(i) $r_8(I) = X_{942} = (a(a^2(b+c) + 2abc - (b+c)(b-c)^2) : \cdots : \cdots)$, and
(ii) the reflections of BC in P_aP_b, CA in P_cP_a, and AB in P_aP_b form a triangle perspective with $P_aP_bP_c$ at $r_6(I)$ which divides OI in the ratio $3R : 2r$.

8.4. *Reflections of H in cevian lines.*

Proposition 32 (Musselman [33]). *Given a point P, let X, Y, Z be the reflections of the orthocenter H in the lines AP, BP, CP respectively. The circles APX, BPY, CPZ have a second common point*

$$r_9(P) = \left(\frac{1}{-2S^2vw + S_A(a^2vw + b^2wu + c^2uv)} : \cdots : \cdots\right).$$

Remark. $r_9(P)$ is also the second intersection of the rectangular circum-hyperbola $\mathscr{H}(P)$ (through H and P) with the circumcircle (see Figure 41).

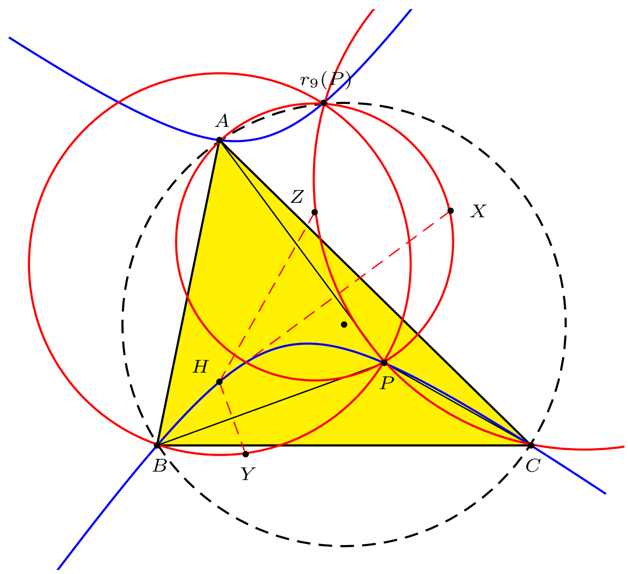


Figure 41. Triad of circles through reflections of H in three cevian lines

8.5. *Reflections in perpendicular bisectors.*

Proposition 33 ([8]). *Given a point P with reflections X, Y, Z in the perpendicular bisectors of BC, CA, AB respectively, the triangle XYZ is perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P is on the circumcircle, the lines AX, BY, CZ are parallel. The perspector is the isogonal conjugate of P* (see Figure 42).

(b) *If $P = E_t$ on the Euler line, then the perspector is $E^*_{t'}$ on the Jerabek hyperbola, where*

$$t' = \frac{a^2b^2c^2(1+t)}{a^2b^2c^2(1+t) - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)t}$$

(see Figure 43).

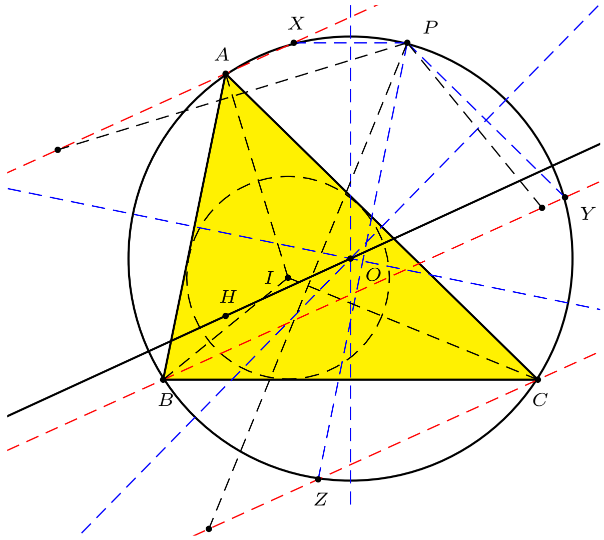


Figure 42. Reflections of P on circumcircle in perpendicular bisectors

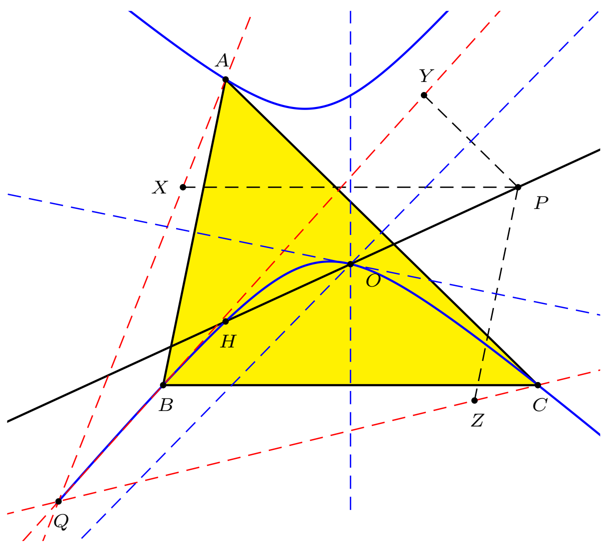


Figure 43. Reflections of P on Euler line in perpendicular bisectors

8.6. *Reflections in altitudes.* Let X, Y, Z be the reflections of P in the altitudes of triangle ABC. The lines AX, BY, CZ are concurrent (at a point Q) if and only if P lies on the reflection conjugate of the Euler line. The perspector lies on the same cubic curve (see Figure 44). This induces a conjugation on the cubic.

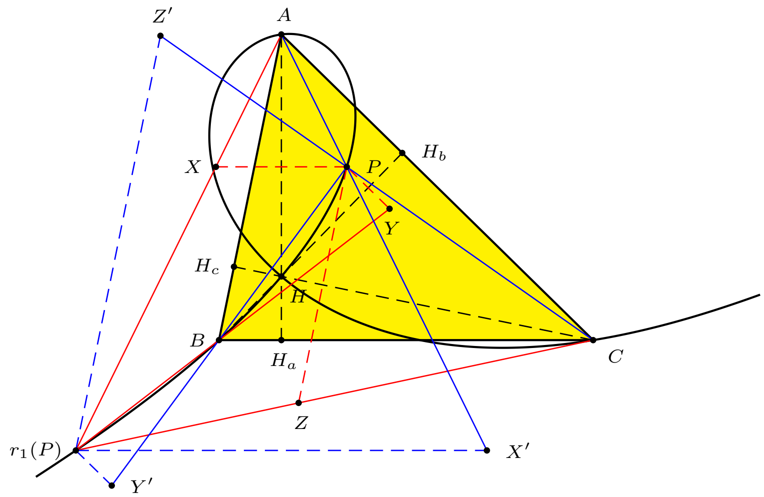


Figure 44. Reflections in altitudes and the reflection conjugate of the Euler line

Proposition 34. *The reflections of $r_1(E_t)$ in the altitudes are perspective with ABC at $r_1(E_{t'})$ if and only if*

$$tt' = \frac{a^2b^2c^2}{a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

9. Reflections of lines in the cevian triangle of incenter

Let $I_aI_bI_c$ be the cevian triangle of I.

Proposition 35 ([20, 44])**.** *The reflections of I_bI_c in AI_a, I_cI_a in BI_b, and I_aI_b in CI_c bound a triangle perspective with ABC at*

$$X_{81} = \left(\frac{a}{b+c} : \frac{b}{c+a} : \frac{c}{a+b}\right)$$

(see Figure 45).

Proof. The equations of these reflection lines are

$$\begin{aligned} -bcx + c(c+a-b)y + b(a+b-c)z =&\ 0, \\ c(b+c-a)x - cay + a(a+b-c)z =&\ 0, \\ b(b+c-a)x + a(c+a-b)y - abz =&\ 0. \end{aligned}$$

The last two lines intersect at the point

$$(-a(b^2+c^2-a^2-bc) : b(a+b)(b+c-a) : c(c+a)(b+c-a)).$$

With the other two points, this form a triangle perspective with ABC at X_{81} with coordinates indicated above. □

Remark. X_{81} is also the homothetic center of ABC and the triangle bounded by the three lines each joining the perpendicular feet of a trace of an angle bisector on the other two angle bisectors ([39]).

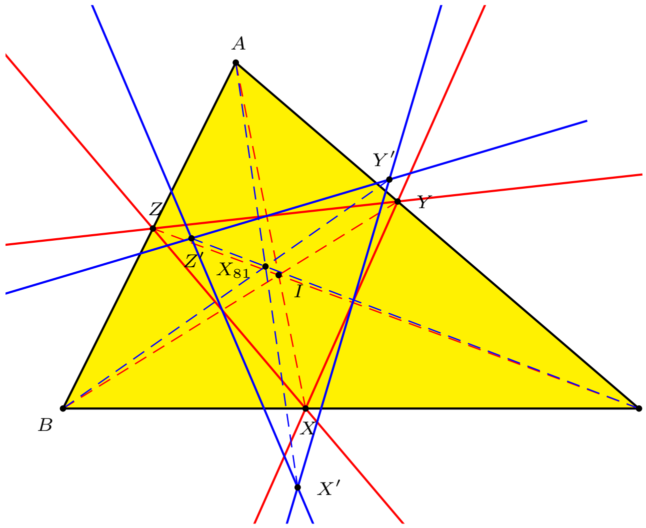


Figure 45. Reflections in the cevian triangle of incenter

Proposition 36. *The reflections of BC in AI_a, CA in BI_b, and AB in CI_c bound a triangle perspective with $I_aI_bI_c$ at*

$$X_{55} = (a^2(b+c-a) : b^2(c+a-b) : c^2(a+b-c)).$$

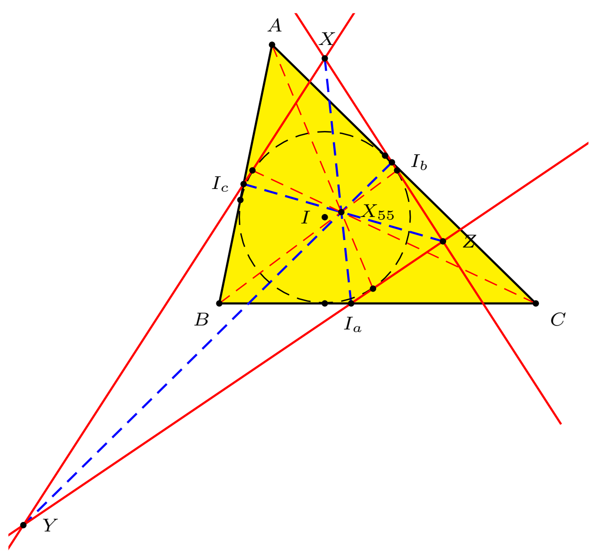


Figure 46. Reflections in angle bisectors

Proposition 37 ([43]). *The reflections of AI_a in I_bI_c, BI_b in I_cI_a, and CI_c in I_aI_b are concurrent at a point with coordinates*

$$\begin{aligned}&(a(a^6+a^5(b+c)-4a^4bc-a^3(b+c)(2b^2+bc+2c^2)\\&\quad -a^2(3b^4-b^2c^2+3c^4)+a(b+c)(b-c)^2(b^2+3bc+c^2)+2(b-c)^2(b+c)^4)\\&:\cdots:\cdots)\end{aligned}$$

(see Figure 47).

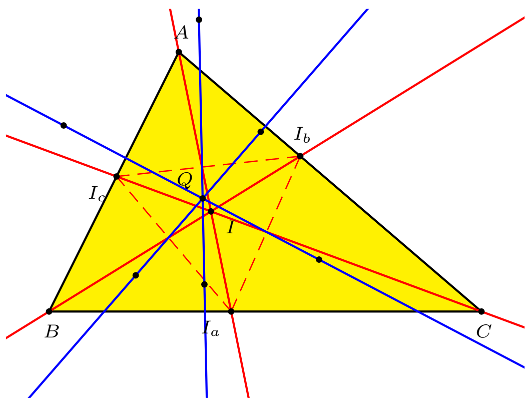


Figure 47. Reflections of angle bisectors in the sidelines of cevian triangle of incenter

10. Reflections in a triangle of feet of angle bisectors

Let P be a given point. Consider the bisectors of angles BPC, CPA, APB, intersecting the sides BC, CA, AB at D_a, D_b, D_c respectively (see Figure 48).

Proposition 38. *The reflections of the lines AP in D_bD_c, BP in D_cD_a, and CP in D_aD_b are concurrent.*

Proof. Denote by x, y, z the distances of P from A, B, C respectively. The point D_a divides BC in the ratio $y : z$ and has homogeneous barycentric coordinates $(0 : z : y)$. Similarly, $D_b = (z : 0 : x)$ and $D_c = (y : x : 0)$. These can be regarded as the traces of the isotomic conjugate of the point $(x : y : z)$. Therefore, we consider a more general situation. Given points $P = (u : v : w)$ and $Q = (x : y : z)$, let $D_aD_bD_c$ be the cevian triangle of $Q^\bullet$, the isotomic conjugate of Q. Under what condition are the reflections of the cevians AP, BP, CP in the lines D_bD_c, D_cD_a, D_aD_b concurrent?

The line D_bD_c being $-x\mathbb{X}+y\mathbb{Y}+z\mathbb{Z}=0$, the equation of the reflection of the cevian AP in D_bD_c is

$$\begin{aligned}&(-x((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+x((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{X}\\&+(y((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)w)\mathbb{Y}\\&-((a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)v+z((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{Z}\\&=0.\end{aligned}$$

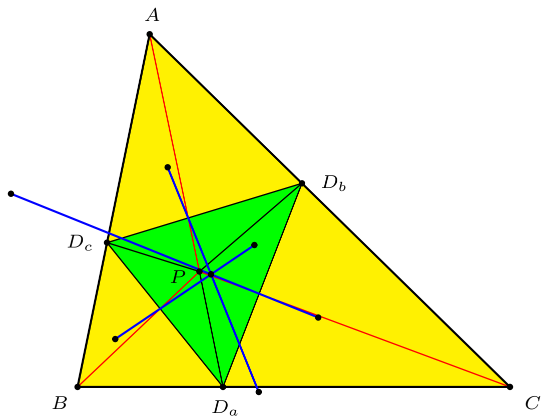


Figure 48. Reflections in a triangle of feet of angle bisectors

By permutating cyclically u, v, w; x, y, z; $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, we obtain the equations of the reflections of BP in D_cD_a and CP in D_aD_b. The condition for the concurrency of the three lines is $F = 0$, where F is a cubic form in u, v, w with coefficients which are sextic forms in x, y, z given in the table below.

term	coefficient
vw^2	$a^2zx(-a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz)\cdot$ $(a^2x^2-3b^2y^2+c^2z^2+(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx-(a^2+b^2-c^2)xy)$
v^2w	$-a^2xy(-a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz)\cdot$ $(a^2x^2+b^2y^2-3c^2z^2+(b^2+c^2-a^2)yz-(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
wu^2	$b^2xy(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)\cdot$ $(a^2x^2+b^2y^2-3c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
w^2u	$-b^2yz(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)\cdot$ $(-3a^2x^2+b^2y^2+c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx-(a^2+b^2-c^2)xy)$
uv^2	$c^2yz(a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)\cdot$ $(-3a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz-(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
u^2v	$-c^2zx(a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)\cdot$ $(a^2x^2-3b^2y^2+c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
uvw	$\sum_{\text{cyclic}} a^4u^5((a^2+b^2-c^2)v-(c^2+a^2-b^2)w)$ $+\sum_{\text{cyclic}} a^2u^4((a^2+b^2-c^2)^2v^2-(c^2+a^2-b^2)^2w^2)$ $+\sum_{\text{cyclic}} a^2u^3vw(((c^2-a^2)^2+3b^2(c^2+a^2)-4b^4)v$ $-((a^2-b^2)^2+3c^2(a^2+b^2)-4c^4)w)$

By substituting
x^2 by $c^2v^2+(b^2+c^2-a^2)vw+b^2w^2$,
y^2 by $a^2w^2+(c^2+a^2-b^2)wu+c^2u^2$, and
z^2 by $b^2u^2+(a^2+b^2-c^2)uv+a^2v^2$,
which are proportional to the squares of the distances AP, BP, CP respectively, with the help of a computer algebra system, we verify that $F = 0$. Therefore we conclude that the reflections of AP, BP, CP in the sidelines of $D_aD_bD_c$ do concur. $\square$

In the proof of Proposition 38, if we take $Q = G$, the centroid, this yields Proposition 24. On the other hand, if $Q = X_8$, the Nagel point, we have the following result.

Proposition 39. *The locus of P for which the reflections of the cevians AP, BP, CP in the respective sidelines of the intouch triangle is the union of the circumcircle and the line OI:*

$$\sum_{\text{cyclic}} bc(b-c)(b+c-a)\mathbb{X} = 0.$$

(a) *If P is on the circumcircle, the cevians are parallel, with infinite point the isogonal conjugate of P* (see Figure 49).

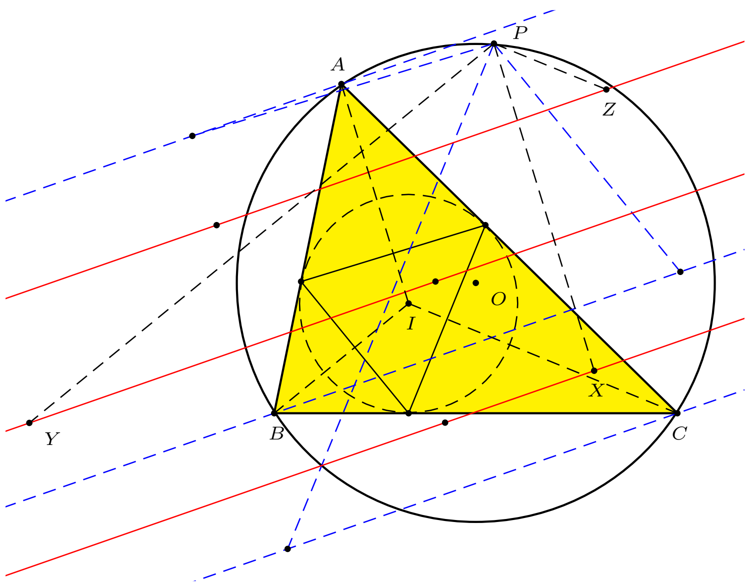


Figure 49. Reflections of cevians of P in the sidelines of the intouch triangle

(b) *If P is on the line OI, the point of concurrency traverses the conic*

$$\sum_{\text{cyclic}} (b-c)(b+c-a)^2x^2 + (b-c)(c+a-b)(a+b-c)yz = 0,$$

which is the Jerabek hyperbola of the intouch triangle (see Figure 50). *It has center*

$$(a(c+a-b)(a+b-c)(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3)) : \cdots : \cdots).$$

Finally, if we take $Q = \left(\frac{u}{a^2} : \frac{v}{b^2} : \frac{w}{c^2}\right)$ in the proof of Proposition 38, we obtain the following result.

Proposition 40. *Let $P_a^*P_b^*P_c^*$ be the cevian triangle of the isogonal conjugate of P. The reflections of AP in $P_b^*P_c^*$, BP in $P_c^*P_a^*$, CP in $P_a^*P_b^*$ are concurrent* (see Figure 51).

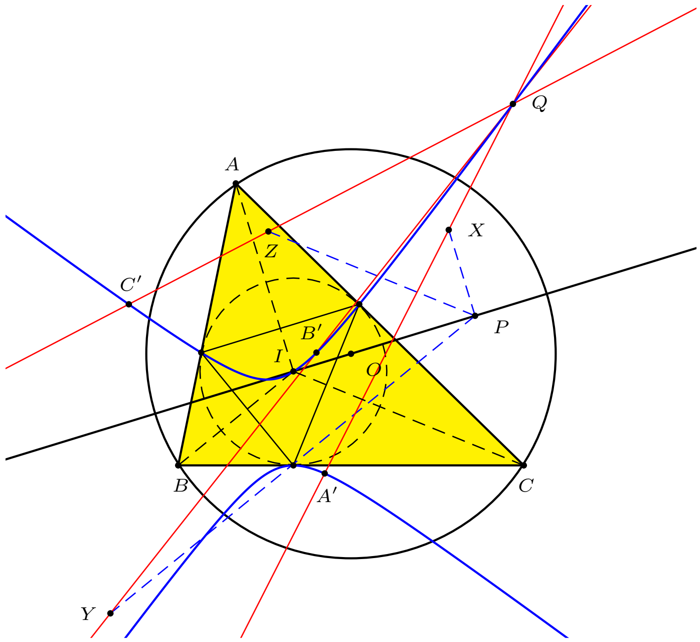


Figure 50. Reflections of cevians of P in the sidelines of the intouch triangle

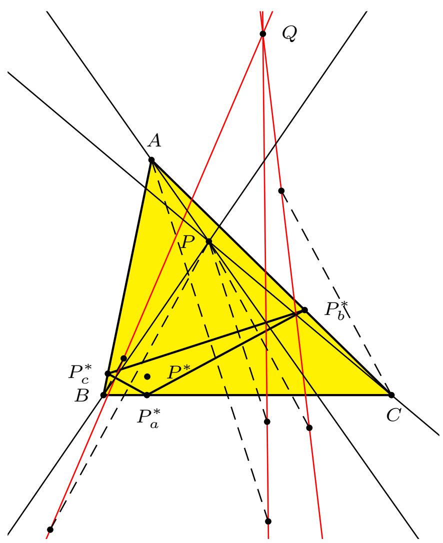


Figure 51. Reflections of cevians of P in cevian triangle of P^*

A special case is Proposition 37 above. For $P = X_3 = O$, the common point is $X_3 = O$. This is because the cevian triangle of $O^* = H$ is the orthic triangle, and the radii OA, OB, OC are perpendicular to the respective sides of the orthic triangle. Another example is $(P, Q) = (K,\ X_{427})$.

Synopsis

Triangle centers	References	**Triangle centers**	References
F_e	Table following Thm. 4	X_{79}	Rmk (1) following Prop. 7
	Table following Prop. 10	X_{80}	Rmk (2) following Prop. 7
$F_\pm$	End of §2.2		Rmk following Thm. 10
	Table following Prop. 10		Table following Prop. 22
$J_\pm$	Prop. 1(c); end of §2.2	X_{81}	Prop. 35
Ω, Ω'	Table in §6.3	$X_{95} = N^\bullet$	Table following Prop. 31
E	Rmk (3) following Thm. 3; Figure 4	X_{98}	Table in Rmk (1) following Prop. 28
	Table following Thm. 4	X_{99}	Table following Prop. 22
	Rmk (1) following Prop. 9		Table in §8.2
	Table following Prop. 22	X_{100}	Table following Prop. 22
	Table in §8.2		Table in §8.2
	Table following Prop. 31	X_{104}	Table in Rmk (1) following Prop. 28
E_∞	Rmk following Prop. 7	X_{108}	Table following Thm. 4
	Rmk (2) following Prop. 9	X_{109}	Rmk (3) following Thm. 3
$W = X_{484}$	Rmk (2) following Prop. 7	X_{112}	Rmk (3) following Thm. 3
	Rmk following Prop. 16		Table following Thm. 4
N^*	Rmk (1) following Prop. 5	X_{115}	Table following Thm. 4
	Rmk (2) following Prop. 9		Table following Prop. 10
	Prop. 14		§§6.2, 6.3
X_{19}	Rmk (1) following Prop. 31	X_{125}	Table following Thm. 4
X_{21}	Rmk (1) following Prop. 17		Table following Prop. 10;
	Table following Prop. 31		§6.3
X_{24}	Rmk (2) following Prop. 17		Rmk (6) following Prop. 23
	Rmk (3) following Prop. 25	X_{141}	Table in §8.2
	Table in Rmk (2) following Prop. 12	X_{143}	Rmk (3) Prop. following 17
	Table in §8.2	$X_{155} = H/O$	Table following Prop. 6
$X_{25} = H/K$	Table following Prop. 6	X_{186}	Rmk (2) followingProp. 12
	Rmk (4) following Prop. 23		Rmk (2) following Prop. 25
X_{35}	Rmk (2) at the end of §8.2		Table in Rmk following Prop. 27
X_{40}	§6.1	$X_{193} = H/G$	Table following Prop. 6
$X_{46} = H/I$	Table following Prop. 6	X_{195}	Rmk (1) following Prop. 5
$X_{52} = H/N$	Table following Prop. 6		Rmk (2) following Prop. 9
$X_{55} = G_\mathrm{e}^*$	Prop. 36	X_{214}	Table following Prop. 22
X_{57}	Table in §8.2	$X_{249} = X_{115}^*$	Prop. 17(b); Table in §8.2
$X_{59} = F_\mathrm{e}^*$	Table in §8.2	$X_{250} = X_{125}^*$	Prop. 17(b); Table in §8.2
X_{60}	Rmk (4) following Prop. 17	$X_{265} = r_1(O)$	Table following Prop. 10
X_{65}	Table in Rmk (1) following Prop. 23		Tables following Prop. 22, 23, 27
X_{66}	Table in Rmk (1) following Prop. 23	X_{354}	Table in §8.2
X_{67}	Table following Prop. 10	X_{393}	Rmk (2) following Prop. 31
X_{68}	Table in Rmk (1) following Prop. 23	X_{399}	Rmk (1) following Prop. 9
	Table in Rmk (2) following Prop. 12		§5.1.2; §5.1.3
$X_{69} = H^\bullet$	Table in Rmk (1) following Prop. 23	X_{403}	Rmk (2) following Prop. 12
	Table in §8.2		Table in Rmk (1) Prop. 23
	Table following Prop. 31	X_{427}	Rmk (5) following Prop. 23
X_{72}	Table in Rmk (1) following Prop. 23		Rmk following Prop. 40
X_{74}	Table in Rmk (1) following Prop. 23	X_{429}	Table in Rmk (1) Prop. 23
	Table in Rmk (1) following Prop. 28	X_{442}	Table in Rmk (1) Prop. 23

Triangle centers	References	Triangle centers	References
X_{476}	Table following Thm. 4	X_{1986}	Rmk (1) following Prop. 12
X_{477}	Table in Rmk (1) following Prop. 28		Table following Prop. 17
X_{571}	Rmk (2) following Prop. 25		Rmk (3) following Prop. 25
X_{671}	Tables following Prop. 10, 22	X_{2698}	Table in Rmk (1) following Prop. 28
X_{895}	Table following Prop. 22	X_{2715}	Table following Thm. 4
X_{942}	Rmk (3) following Prop. 31	X_{2720}	Table following Thm. 4
X_{925}	Table Prop. 31	X_{2482}	Table following Prop. 22
X_{953}	Table in Rmk (1) following Prop. 28	X_{3003}	Rmk (1) following Prop. 25
X_{1105}	Table Prop. 31	X_{3025}	Rmk (2) following Prop. 29
X_{1141}	Rmk (3) following Prop. 7	X_{3528}	Rmk (3) following Prop. 29
X_{1145}	Table following Prop. 22	superiors of	
X_{1156}	Tables following Prop. 10, 22	Fermat points	§5.1.4
X_{1157}	Rmk (3) following Prop. 7	**new**	End of §2.2
$= (N^*)^{-1}$	Table following Prop. 9		Rmk (2) following Prop. 12
	Corollary 15; §5.1.1		§5.1.3
X_{1320}	Table following Prop. 10		Rmk following Prop. 27
	Table following Prop. 22		Rmk (2) following Prop. 29
X_{1444}	Rmk (1) following Prop. 31		Rmk (2) following Prop. 30
X_{1618}	Table in §8.2		Prop. 37, 39

Reflection triangles	References
O	§1
H	Rmk (1) following Prop. 12; Rmk following Prop. 27
N	§1, Prop. 5
K	Rmk (4) following Prop. 23

Cevian triangles	References
G (medial)	§7.1
I (incentral)	Rmk (1) following Prop. 7; §9
H (orthic)	Prop. 6; §5.1.1, §7.2; Rmk (5) following Prop. 23

Anticevian triangles	References
I (excentral)	Figure 11; §5.1.3
K (tangential)	Prop. 1(a); §5.1.2; Rmk (4) following Prop. 23
N^*	§5.1.1

Lines	References
Euler line	Figure 4; Prop. 17, 23, 24, 33; Rmk (2) following Prop. 29
OI	Prop. 39

Circles	References
Circumcircle	Prop. 1(d); Thm. 3; Prop. 17, 33, 39
Incircle	Rmk (2) following Prop. 29
Nine-point circle	Rmk 2 Prop. 2; §6.3; Prop. 23
Apollonian circles	Prop. 1(b)
Brocard circle	§6.2
Pedal circle of G	§6.3
$P^{(a)}P^{(b)}P^{(c)}$	Prop. 2; Rmk following Prop. 10
Circles containing $A^{(a)}$, $B^{(b)}$, $C^{(c)}$	passim

Conics	References
Steiner circum-ellipse	§6.4
Jerabek hyperbola	Prop. 23, 24, 33
bicevian conic $\mathscr{C}(G, Q)$	Prop. 22
bicevian conic $\mathscr{C}(X^*_{115},\ X^*_{125})$	Prop. 17
Jerabek hyperbola of intouch triangle	Prop. 39
circumconic with center P	Prop. 22
Inscribed parabola with focus E	Rmk (2) following Prop. 29
rectangular circum-hyperbola through P	Rmk following Prop. 10; Rmk following Prop. 32
Inscribed conic with a given focus P	Prop. 29

Cubics	References
Neuberg cubic K001	Prop. 7, 8, 9, 16, 26, 27
Macay cubic K003	Rmk (3) following Prop. 29
Napoleon cubic K005	Prop. 9, 27
Orthocubic K006	Prop. 26
p$K(X_{1989},\ X_{265})$ = K060	Prop. 7, 8
p$K(X_{3003},\ H)$ = K339	Prop. 25
p$K(X_{186},\ X_{571})$	Prop. 25
Reflection conjugate of Euler line	§8.6

Quartics	References
Isogonal conjugate of nine-point circle	Prop. 17
Isogonal conjugate of Brocard circle	§6.4

Constructions	References
H/P	Prop. 6
$r_0(P)$	Rmk (3) following Thm. 3; Thm. 4; Prop. 20
$r_1(P)$	Prop. 10, Prop. 11
$r_2(P)$	Prop. 12
$r_3(P)$	Prop. 22
$r_4(P)$	Prop. 22
$r_5(P)$	Prop. 30
$r_6(P)$	§8.2
$r_7(P)$	Prop. 31
$r_8(P)$	Prop. 31
$r_9(P)$	Prop. 32

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Antreas P. Hatzipolakis: 81 Patmou Street, Athens 11144, Greece
E-mail address: anopolis72@gmail.com

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA
E-mail address: yiu@fau.edu

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