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A New Proof of a "Hard But Important" Sangaku Problem

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Abstract. We present a solution to a well-known *sangaku* problem, previously solved using the Sawayama Lemma, that does not require it.

1. Introduction

The following problem has been described as "hard but important".

Problem 1 ([5, Problem 2.2.8])**.** *M is the midpoint of chord AB of circle (O). Given triangle ABC with sides a, b, c, and semiperimter s, let N be the midpoint of the arc AB opposite the incircle $I(r)$, and $MN = v$. Given circle $Q(q)$ tangent to AC, BC, and (O)* (see Figure 1)*, prove that*

$$q - r = \frac{2v(s-a)(s-b)}{cs}. \tag{1}$$

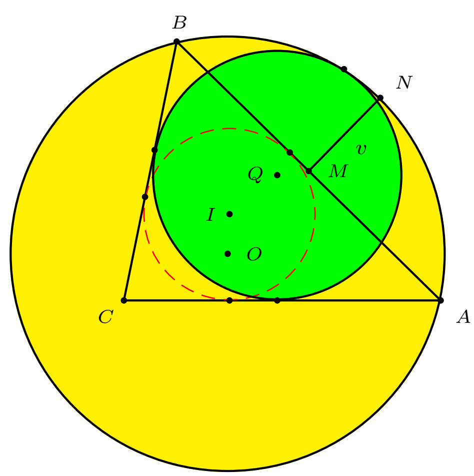

Figure 1.

Problem 1 is certainly hard. One solution [6], summarized in [8, pp. 25–26], entails a series of algebraic manipulations so lengthy that it is hard to believe anyone would have pursued it without knowing in advance that (1) holds. Another solution [3, 9] involves somewhat less algebra, but requires the Sawayama Lemma, the proof of which [1] is itself no simple matter. Neither proof gives much insight into the motivation for (1).

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Problem 1 is important in that solutions of other *sangaku* problems depend on (1). For instance, in the special case in which AC and BC are the sides of squares $ACDE$ and $BCFG$ with E and G on (O), it can be shown using (1) that $q = 2r$ [5, Example 3.2], [8, pp. 20–21]. But the solution we offer here follows from that of another *sangaku* problem, which it cannot be used to solve in a straight-forward way.

2. The source problem

Problem 2. *Triangle ABC has incircle $I(r)$, to which (O), through B and C, is internally tangent. Circle $P(p)$ is tangent to AB and AC and externally tangent to (O). The sagitta from M, the midpoint of BC, to the arc opposite the incircle has length v. Prove that $r^2 = 2pv$.*

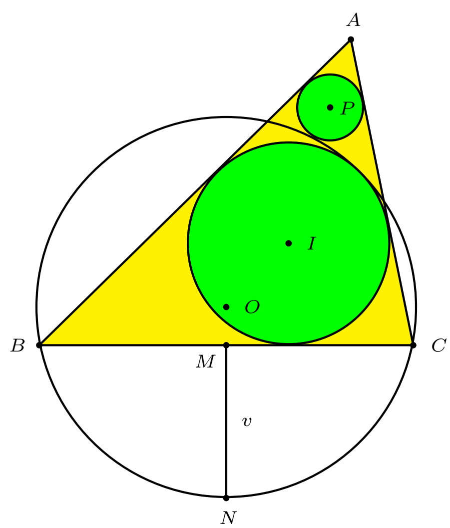

Figure 2.

In the original problem [5, Problem 2.4.2], v in Figure 2 is replaced by a circle of diameter v, which gives the figure a pleasing harmony, but is something of a red herring. The solution relies on the following theorem, which we state as it pertains to the foregoing figure.

Theorem 1. *A circle (O) passing through B, C, tangent to the circles (I)* (internally) *and (P)* (externally)*, exists if and only if one of the intangents of (P) and (I) is parallel to BC.*

Proof. Consider P in general position (see Figure 3). Since the two right triangles formed by AI, the two intangents, and radii of (I) are congruent, we see that
(1) DE, FG, and AI are concurrent in X,
(2) the triangles AEX and AGX are congruent,
(3) the triangles ADE and AFG are congruent.

The equal vertical angles EXF and GXD are both equal to $\angle AGF - \angle ADE = \angle AED - \angle AFG$. Hence, if $\angle AFG = \angle ABC$ and $\angle AGF = \angle ACB$, then FG

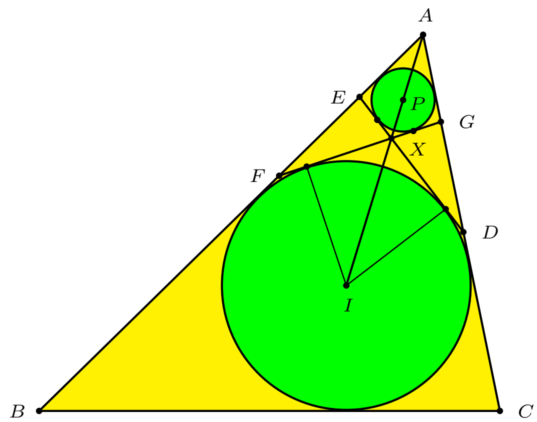

Figure 3.

and BC are parallel, and $\angle ACB - \angle ADE = \angle AED - \angle ABC$. But since $\angle ADE < \angle ACB$ and $\angle AED > \angle ABC$, this can be true only if $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$. Thus,

(4) FG (respectively DE) is parallel to BC if and only if DE (respectively FG) is its antiparallel in triangle ABC.

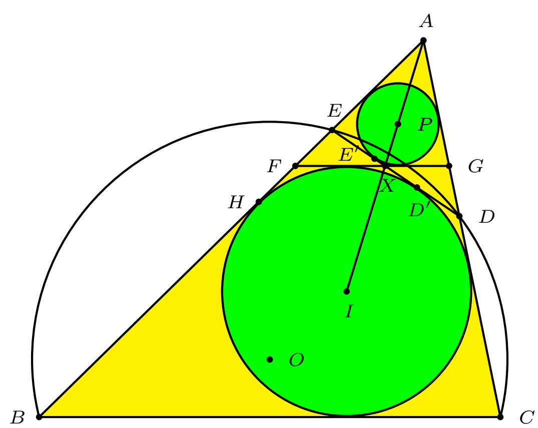

Figure 4.

Now consider the coaxial system Γ of circles with centers on the perpendicular bisector of BC. If a circle in Γ cuts AB and AC in two distinct points, then, together with B and C, they form a cyclic quadrilateral. On the other hand, if $FG||BC$, $BCDE$ is a cyclic quadrilateral because, as just shown, $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$. Say DE touches (I) and (P) in D' and E' respectively. Because chords of all circles in Γ cut off by AB and AC are parallel to DE, a circle in Γ that passes through either D' or E' passes through the other. In light of statement (4), this is the circle (O) in Problem 2 (compare Figure 2 with Figure 4). $\square$

Let H be the point on AB touched by (I). $AH = s - a$, where s is the semiperimeter of triangle ABC. Since (I) is the excircle of AFG, AH is also the semiperimeter of AFG. Hence $\frac{p}{r} = \frac{s-a}{s}$. Therefore, $2pv = \frac{2vr(s-a)}{s}$, and for this to be equal to r^2, we must have $2v(s-a) = rs =$ (area of ABC). But that is the case if and only if $2v$ is the radius of the excircle (S) touching BC at T (see Figure 5).

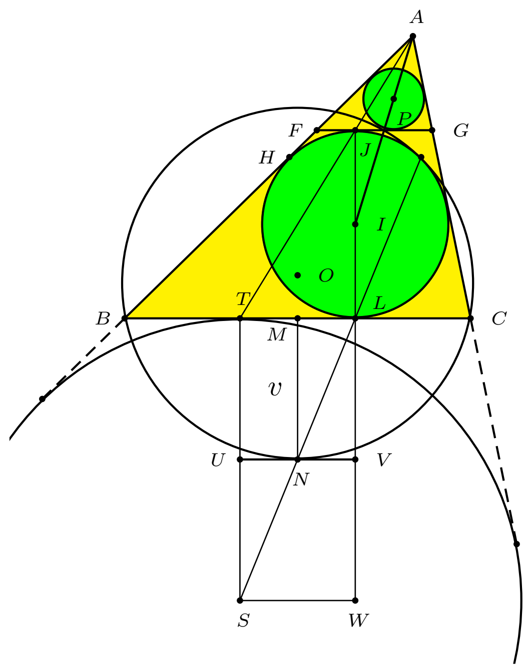

Figure 5.

AI is concurrent in S with the bisectors of the exterior angles at B and C, and $ST \perp BC$ just as $JL \perp BC$. Therefore, drawing parallels to BC through N and S, we obtain rectangles $LTUV$, $LTSW$, and $SUVW$. By construction, $MN = v = TU = LV$. Since $BT = CL = s - c$, $MT = ML$. Hence triangles LMN and NUS are congruent: LNS is a straight angle, and $US = VW = v$. Therefore $2v$ is indeed the radius of (S).

3. Extension to Problem 1

We have just proved $\frac{2v(s-a)}{s} = r$ in case (O) is tangent to (I). To extrapolate to Problem 1, we add circle (Q) tangent to AB, AC, and (O) (see Figure 6).

Say (Q) touches AB in H', and (I) touches AC in Y. Since triangles AIH and AQH' are similar, $\frac{q-r}{r} = \frac{AH'-AH}{AH} = \frac{HH'}{AH}$. But $AH = s - a$. Hence

$$q - r = \frac{r \cdot HH'}{s-a} = \frac{2v(s-a)}{s} \cdot \frac{HH'}{s-a} = \frac{2v \cdot HH'}{s}. \qquad (2)$$

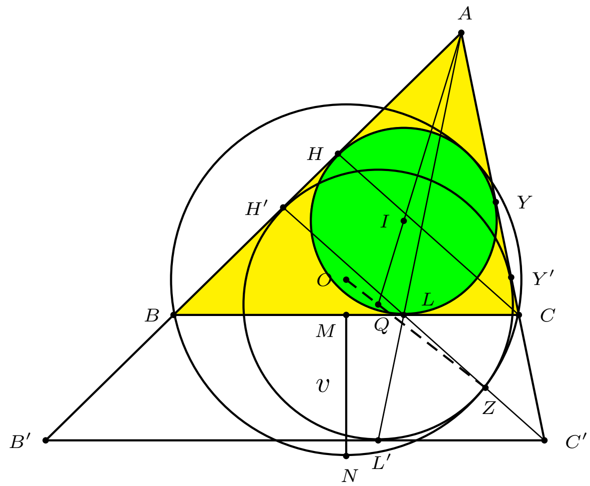

Figure 6.

This gives $q - r = \frac{2v(s-b)(s-c)}{as}$ if $\frac{HH'}{s-b} = \frac{s-c}{a}$. Since $BH = s - b$ and $CL = s - c$, this proportion is equivalent to the similarity of triangles BLH' and BCH, *i.e.*, $LH'||CH$. Extend AB and AC to intersect the tangent to (Q) at L' parallel to BC on the side opposite (I). This makes triangles ABC and $AB'C'$ similar and (Q) the incircle of the latter. Since LY is a side of the intouch triangle of ABC, $CH \perp LY$. Likewise $C'H' \perp L'Y'$. Since $L'Y'||LY$, $C'H'||CH$. Thus $C'LH'$ is a straight angle and $LH'||CH$. Indeed, for this choice of O, L is the Gergonne point of $AB'C'$. Now as O varies on the perpendicular bisector of BC, points B', C', H', L', N, Y' and Z vary relative to A, B, C, I, L, M, H, and Y, which are unaffected, but the relations $B'C'||BC$, $H'C'||HC$, and $L'Y'||LY$ remain the same. We still have the similarity of triangles AIH and AQH'; hence (2) is valid, and, although the Gergonne point of $AB'C'$ no longer coincides with L, we continue to have $\frac{HH'}{s-b} = \frac{s-c}{a}$ as before. Therefore, $q - r = \frac{2v(s-b)(s-c)}{as}$. With the change of notation $a, b, c \to c, a, b$, we obtain (1).

4. Historical remarks

The problem presented in Figure 7 is found on p. 26 fasc. 3 of *Seiyō sanpō* (‘0‘8‘0‘2‘0‘1∼‘0‘3‘0‘3‘0‘9‘0‘3) [4] (1781) of Fujita Sadasuke (‘0‘4‘0‘5‘0‘3‘0‘2‘0‘2⊥‘0‘3‘0‘9), sometimes called Fujita Teishi, who lived from 1734 to 1807. Like most *sangaku* problems, this one is presented as a concrete example: one is told that the lengths of the sagitta, chord, and left and right diagonal lines are $v = 5$, $c = 30$, $a = 8$, and $b = 26$ inches (*sun* ‘0‘8‘0‘5), respectively, and asked for the diameter of the inner circle. The answer 9 inches is then given, following which the method of calculation is presented. It uses the labels East, West, South, and North for intermediate results; freely translated into Western notation, if

$$E := a + b + c, \quad W := E(a + b - c), \quad S := 4ab - W, \quad N := \sqrt{WS},$$

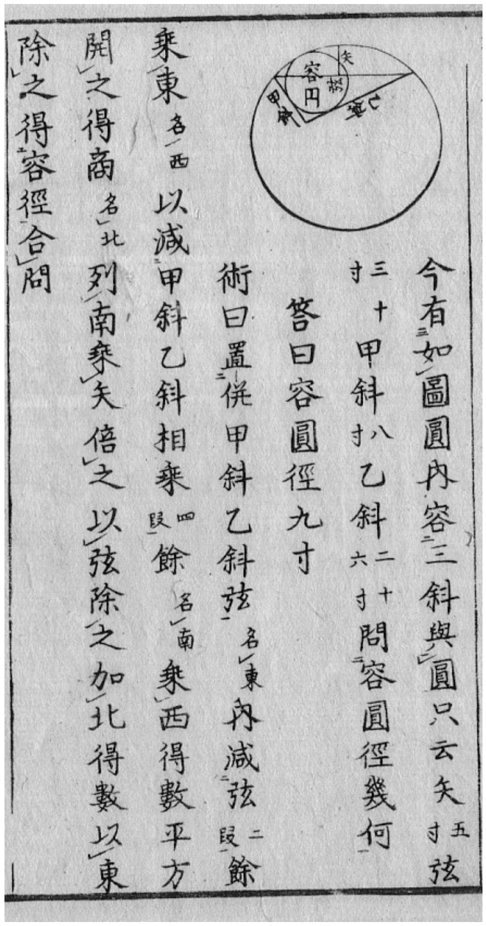

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Figure 7.

then the diameter sought is $\frac{N+\frac{2vS}{c}}{E}$. This reduces to equation (1) because $N = 4rs$. There is, however, no explicit mention of (I) or r in the text. Indeed, the sides a and b of the triangle are treated as arbitrary line segments emanating from the endpoints of c, and the fact that N is four times the area of the triangle goes unmentioned. Looking at just this excerpt, we get no idea of how Fujita arrived at this method, but it is doubtful that he had the Sawayama Lemma in mind. Another special choice of O to consider is the circumcenter of ABC. In that case, Q is particularly easy to construct, but standard proofs of the construction [2], [10, pp. 56–57] involve trigonometric equations not seen in Edo period Japanese mathematics [7]. One runs into a snag if one assumes (1) and tries to solve Problem 2 by adding circle (Q). If, in Figure 2, (P) is tangent to AB at K, this leads in short order to $HH' \cdot HK = 2pv$, but proving $HH' \cdot HK = r^2$ is equivalent to showing that $H'IK$ is a right triangle. If there is a way to do this other than by solving Problem

2 independently, it must be quite difficult. At any rate, because FG is not a cevian of ABC, the Sawayama Lemma does not help when taking this approach.

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