Forum Geometricorum
Volume 10 (2010) 15–20.

FORUM GEOM
ISSN 1534-1178

The Poncelet Pencil of Rectangular Hyperbolas

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Abstract. We describe a pencil of rectangular hyperbolas depending on a triangle and the relations of the hyperbolas with several triangle centers.

1. Isogonal Transforms

In this section we collect some background on the isogonal transform and its connections to our study. The results are somewhat scattered in the literature so we review some of the details necessary for our investigation.

The isogonal transformation determined by a triangle ABC is a birational quadratic transformation taking a general line to a conic passing through A, B, C. The fixed points of this transformation are the incenter and excenters of the triangle, see [2]; the vertices of the triangle are singular points. The isogonal transform of the point P is denoted P'. The quadratic nature of the transform means that a pencil of lines through P is transformed to a pencil of conics passing through A, B, C and P'.

Theorem 1. *The pencil of lines through the circumcenter O is transformed to a pencil of rectangular hyperbolas passing through the vertices and orthocenter H of triangle ABC. The centers of these rectangular hyperbolas lie on the Euler circle.*

Proof. The isogonal transform of O is $O' = H$. Thus all the conics in the pencil pass through A, B, C, H, an orthocentric set, i.e., any one of the points is the orthocenter of the other three. Hence the pencil of conics is a pencil of rectangular hyperbolas.

It is a well-known theorem of Feuerbach [2] that a pencil of rectangular hyperbolas has a circle as its locus of centers.

The pencil has three degenerate rectangular hyperbolas. These are determined by the lines through O and the vertices; the transform of, say OA is the reducible conic consisting of the opposite sides BC together with HA, the altitude. These reducible conics have centers at the feet of the altitudes (this also makes sense when the triangle is equilateral); thus since the locus of centers is a circle it is the Euler nine-point circle. □

Publication Date: February 22, 2010. Communicating Editor: Paul Yiu.

We refer to this pencil of hyperbolas as the *Poncelet pencil*. The referee has kindly pointed out that Poncelet discovered this pencil in 1822.

The points on the circumcircle of triangle ABC are sent by the isogonal transform to points on the line at infinity in the projective plane. The isogonal transformation from the circumcircle to the line at infinity halves the angle of arc on the circle, measuring the angle between the lines from O to the corresponding points at infinity. The orientation of these angles is reversed by the isogonal transformation.

For every line ℓ at O the isogonal transform is a conic of the Poncelet pencil. The intersections of ℓ with the circumcircle are transformed to the line at infinity; thus antipodal points on the circumcircle are taken to asymptotic directions which are consequently perpendicular. Hence again we see that the conics of the Poncelet pencil are rectangular hyperbolas.

2. Parametrizations

Every point of the plane (other than A, B, C, H) is on a unique conic of the Poncelet pencil. This gives a curious way of distinguishing the notable 'centers' of a triangle. We give four ways of parameterizing this pencil.

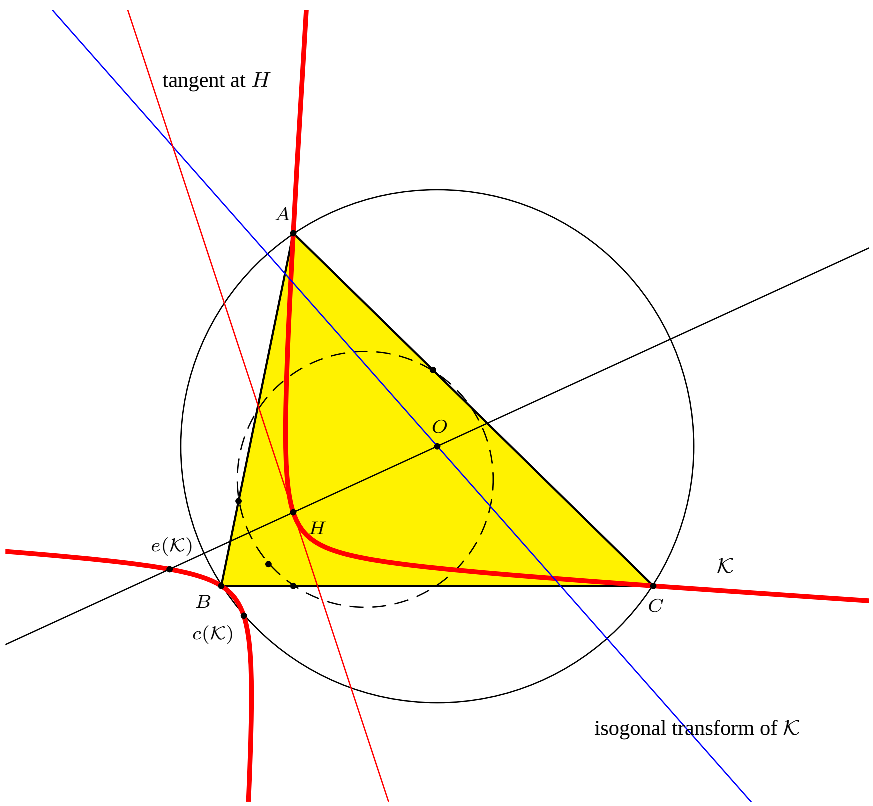

Figure 1. Parametrizations of hyperbolas in the Poncelet pencil

2.1. *Lines at O.* Our first parametrization (via isogonal transform) uses the pencil of lines passing through O; the isogonal transform of this pencil yields the Poncelet pencil of rectangular hyperbolas.

2.2. *Euler Point.* A conic $\mathcal{K}$ of the Poncelet pencil meets the Euler line at H and a second time at $e(K)$, the ***Euler point***. Intersections of a conic in a pencil with a line affords an involution in general (by Desargues Involution Theorem), except when the line passes through a common point of the pencil; in this case the Euler line passes through H so $e(\mathcal{K})$ is linear in $\mathcal{K}$.

2.3. *Tangents.* Our third parametrization of the Poncelet pencil uses the tangents at H, a pencil of lines at H.

2.4. *Circumcircle Point.* The intersection of the circumcircle with any hyperbola of the Poncelet pencil consists of the vertices of triangle ABC and a fourth intersection $c(\mathcal{K})$ called the circumcircle point of the hyperbola K. Our fourth parametrization uses the circumcircle point.

For a point S on the circumcircle we determine S' on the line at infinity. Then the isogonal transform of OS' is the hyperbola in the Poncelet pencil passsing through S.

3. Notable Hyperbolas

We next give some notable members in the Poncelet pencil of rectangular hyperbolas.

3.1. *Kiepert's hyperbola* $\mathcal{K}_{\mathrm{t}}$. The first Euler point is the centroid G. Let L denote the (Lemoine) symmedian point, $L = G'$. The isogonal transform of the line OL is the Kiepert hyperbola. Let S be the center of Spieker's circle; OLS' lie on a line [3] so S lies on this hyperbola.

3.2. *Jerabek's hyperbola* $\mathcal{J}_{\mathrm{k}}$. This is the isogonal transform of the Euler line OH. Hence $O = H'$ and the symmedian point $L = G'$ lie on this hyperbola. Thus $c(\mathcal{J}_{\mathrm{k}})'$ is the end (at infinity) of the Euler line.

Given a line m through O meeting Jerabek's hyperbola at X, the isogonal conjugate of X lies on the Euler line and on the isogonal of $m = OX$, hence $X' = e(m')$.

3.3. *Feuerbach's hyperbola* $\mathcal{F}_{\mathrm{h}}$. Transform line OI where I denotes the incenter to obtain this hyperbola. Since $OIM'K'$ lie on a line where M is the Nagel point and K is the Gergonne point [3], the Nagel and Gergonne points lie on Feuerbach's hyperbola. Feuerbach's hyperbola is tangent to OI at I since I is a fixed point of the isogonal transformation

The three *excentral Feuerbach* hyperbolas pass through the excenters, and hence are tangent to the lines through O there, since the excenters are also fixed points of the isogonal transformation (see Figure 2).

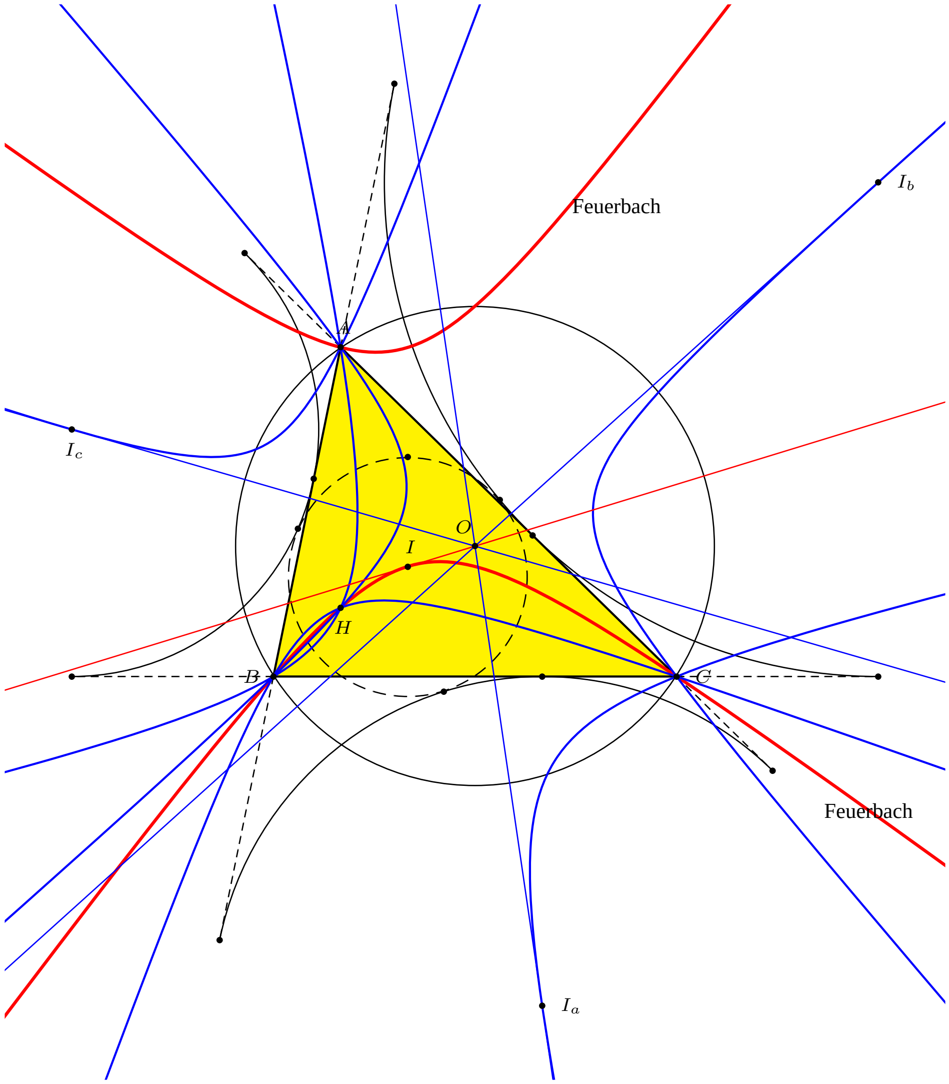

Figure 2. The Feuerbach hyperbolas in the Poncelet pencil

The names for the next two conics are chosen for convenience. Huygens did use a hyperbola however in his solution of Alhazen's problem.

3.4. *Huygens' Hyperbola* $\mathcal{H}_{\rm n}$. We define the isogonal transform of the tangent line to Jerabek at O as Huygens' hyperbola. Hence Huygens' rectangular hyperbola is tangent to the Euler line at H since $O' = H$.

3.5. *Euler's Hyperbola* $\mathcal{E}_{\rm r}$. This hyperbola corresponds to the end of the Euler line using the parametrization by its second intersection with the Euler line. It has the property that the isogonal transform is line OW for $W = c(\mathcal{J}_{\rm k})$ on the circumcircle. The tangent at H meets circumcircle at W.

Remark. Here are the Euler points for the different hyperbolas mentioned:

$$e(\mathcal{H}_{\rm n}) = H, \quad e(\mathcal{K}_{\rm t}) = G, \quad e(\mathcal{J}_{\rm k}) = O, \quad e(\mathcal{E}_{\rm r}) = \infty.$$

The Euler point of Feuerbach's hyperbola is the Schiffler point, which is the common point of the Euler lines of the four triangles IBC, ICA, IAB, and ABC (see [4]).

4. Orthic Poncelet pencil

Consider the locus $\mathcal{K}$ of duals of the Euler line of triangle ABC in each conic of the Poncelet pencil. We show next that this locus is a rectangular hyperbola in the Poncelet pencil of the orthic triangle.

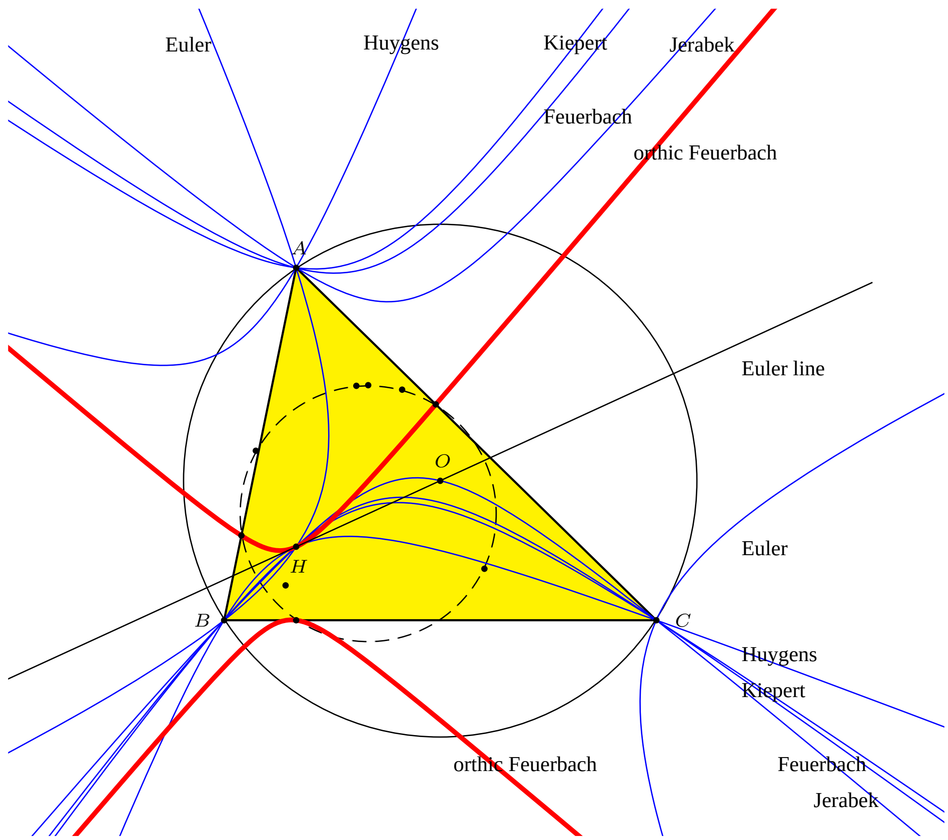

Figure 3. Poncelet pencil with orthic Feuerbach

To see this we recall the close connection between the duality relation and conjugacy: every point P of a line ℓ is conjugate to the dual $Q = \ell^*$ of the line, since the dual of P is a line through Q. But the conjugacy relation is the same as isogonal conjugacy, [1, §4]; thus the dual of the line ℓ in every conic of the pencil is the same as the isogonal conjugate of the line with respect to the triangle of diagonal points. In our situation here the quadrangle is A, B, C, H and its diagonal triangle is the orthic triangle of triangle ABC. Thus the locus K is a conic passing through the vertices of the orthic triangle and it is the isogonal conjugate of the Euler line of triangle ABC with respect to the orthic triangle.

Notice that the isogonal conjugate (with respect to the orthic triangle) of the center of nine-point circle N is the orthocenter h of the orthic triangle, since the circumcircle of the orthic triangle is the nine-point circle. Since N lies on the Euler line then h lies on $\mathcal{K}$ since $\mathcal{K}$ is the isogonal conjugate of the Euler line. Thus the conic $\mathcal{K}$ is a rectangular hyperbola since it contains an orthocentric set. Since H, A, B, C are the incenter and excenters of the orthic triangle, $\mathcal{K}$ is tangent to the Euler line at H (see Figure 3). Thus this conic is a Feuerbach hyperbola of the orthic triangle. If the triangle is acute, then the incenter is H.

Theorem 2. *The locus of duals of the Euler line in the conics in Poncelet pencil is the Feuerbach hyperbola of the orthic triangle.*

As a consequence if we consider the excentral triangle, the original triangle is its orthic triangle. We obtain the following result.

Corollary 3. *Feuerbach's hyperbola is the locus of duals of the Euler line of the excentral triangle in the Poncelet pencil of the excentral triangle.*

Remark. The Euler line of the excentral triangle is the line OI.

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