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A Maximal Parallelogram Characterization of Ovals Having Circles as Orthoptic Curves

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Abstract. It is shown that ovals admitting circles as orthoptic curves are precisely characterized by the property that every one of their points is the vertex of exactly one maximal-perimeter inscribed parallelogram. This generalizes an old property of ellipses, most recently revived by Connes and Zagier in the paper *A Property of Parallelograms Inscribed in Ellipses*.

Let $\mathcal{C}$ be a centrally symmetric smooth strictly convex closed plane curve (oval with a center). It has been known for a long time, see for instance [1], that if $\mathcal{C}$ is an ellipse then among all the inscribed parallelograms those of maximal perimeter have the property that any point of $\mathcal{C}$ is the vertex of exactly one. The proof given in [1] makes it clear that this property is related to the fact that the Monge orthoptic curve, *i.e.*, the locus of all the points from where a given closed curve can be seen at a right angle, of an ellipse is a circle.

The purpose of this note is to show that the maximal-perimeter property of parallelograms inscribed in centrally symmetric ovals described above for ellipses is characteristic precisely to the class of ovals admitting circles as orthoptic curves. For a parallel result, proved by analytic methods, see [2].

Theorem 1. *Let $\mathcal{C}$ be a centrally symmetric oval. Then every point of $\mathcal{C}$ is the vertex of an unique parallelogram of maximal perimeter among those inscribed in $\mathcal{C}$ if and only if the orthoptic curve of $\mathcal{C}$ is a circle.*

The proof of Theorem 1 will be an immediate consequence of the following lemma.

Lemma 2. *Let $ABCD$ be a parallelogram of maximal perimeter among those inscribed in a given centrally symmetric oval $\mathcal{C}$. Then the tangent lines to $\mathcal{C}$ at A, B, C, and D, form with the sides of the parallelogram equal angles, respectively,* (for the 'table' $\mathcal{C}$, the parallelogram is a billiard of period 4)*, and intersect at the vertices of a rectangle $PQRS$, concentric to $\mathcal{C}$* (see Figure 1)*. Moreover, the perimeter of the parallelogram $ABCD$ equals four times the radius of the circle circumscribed about the rectangle $PQRS$.*

Proof. The strict convexity and central symmetry of $\mathcal{C}$ imply that the center of any parallelogram inscribed in $\mathcal{C}$ is the same as the center O of $\mathcal{C}$. Therefore, any parallelogram inscribed in $\mathcal{C}$ is completely determined by two consecutive vertices. The function from $\mathcal{C} \times \mathcal{C}$ to $[0, \infty)$ which gives the perimeter of the associated

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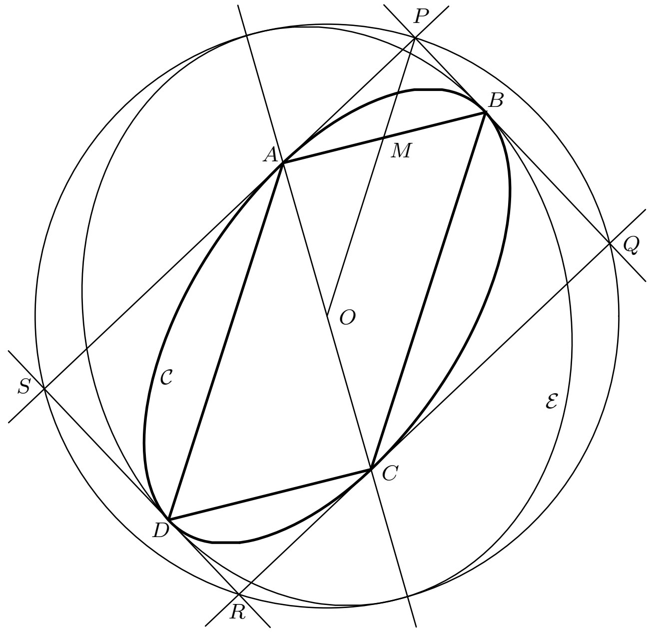

Figure 1. A maximal-perimeter inscribed parallelogram $ABCD$ and its associated circumscribed rectangle $PQRS$

parallelogram being continuous, an inscribed parallelogram of maximal perimeter always exists.

Let now $ABCD$ be such a parallelogram. In the family of ellipses with foci A and C there is an unique ellipse $\mathcal{E}$ such that $\mathcal{C} \cap \mathcal{E} \neq \emptyset$, but $\mathcal{C} \cap \mathcal{E}' = \emptyset$ for any ellipse $\mathcal{E}'$ in the family whose major axis is greater than the major axis of $\mathcal{E}$. In other words, $\mathcal{E}$ is the largest ellipse with foci A and C intersecting $\mathcal{C}$. Consequently, $\mathcal{C}$ and $\mathcal{E}$ have the same tangent lines at any point in $\mathcal{C} \cap \mathcal{E}$. Notice that $\mathcal{C} \cap \mathcal{E}$ contains at least two points, symmetric with respect to the center O of $\mathcal{C}$. The maximality of $ABCD$ implies that B and D belong to $\mathcal{C} \cap \mathcal{E}$ and the familiar reflective property of ellipses guarantees the reflective property with respect to the 'mirror' $\mathcal{C}$ of the parallelogram $ABCD$ at the vertices B and D. A similar argument applied to the family of ellipses with foci B and D yields the reflective property of the parallelogram $ABCD$ at the other two vertices, A and C.

Assume now that the tangent lines to $\mathcal{C}$ at A, B, C, and D, meet at P, Q, R, and S (see Figure 1). By symmetry, $PQRS$ is a parallelogram with the same center as $\mathcal{C}$. The reflective property of $ABCD$ and symmetry imply that, say $\triangle APB$ and $\triangle CQB$ are similar triangles. Consequently, the parallelogram $PQRS$ has two adjacent angles congruent, so it must be a rectangle.

Referring to Figure 1, let M be the midpoint of $\overline{AB}$. Similarity inside $\triangle ABC$ shows that the mid-segment $\overline{MO}$ is parallel to $\overline{BC}$ and $MO = \dfrac{BC}{2}$. Now, in the right triangle APB, $PM = BM = \dfrac{AB}{2}$, since the segment $\overline{PM}$ is the median relative to the hypotenuse $\overline{AB}$. It follows that $\angle MBP$ is congruent to $\angle MPB$, as base angles in the isosceles triangle MBP. Since by the reflective property of the parallelogram $ABCD$, $\angle MBP$ is congruent to $\angle CBQ$, the transversal $\overleftrightarrow{BM}$ cuts on $\overleftrightarrow{BC}$ and $\overleftrightarrow{PM}$ congruent corresponding angles, and so $\overline{BC}$ is parallel to $\overline{PM}$. In conclusion, the points P, M, and O are collinear and $PO = \dfrac{AB + BC}{2}$, or equivalently the perimeter of $ABCD = 4PO =$ four times the radius of the circle circumscribed about the rectangle $PQRS$. Being inscribed in the rectangle $PQRS$, clearly the oval $\mathcal{C}$ is completely contained inside the circle circumscribed about $PQRS$.

As a byproduct of Lemma 2, in a centrally symmetric oval a vertex can be prescribed to at most one parallelogram of maximal perimeter, in which case the other vertices belong to the unique rectangle circumscribed about $\mathcal{C}$ and sharing a side with the tangent line to $\mathcal{C}$ at the prescribed point. Such a construction can be performed for any point of $\mathcal{C}$ but it does not always yield maximal-perimeter parallelograms. □

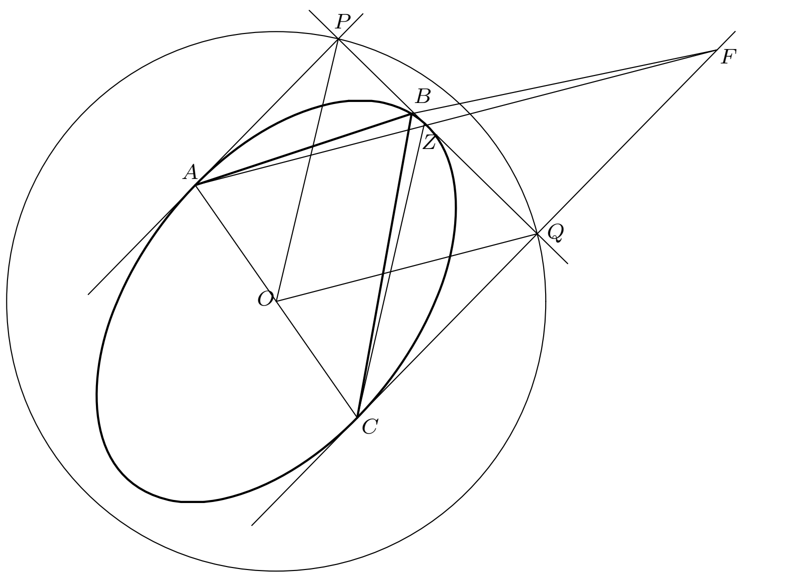

Figure 2. Ovals whose orthoptic curves are circles satisfy the maximal parallelogram property

Proof of Theorem 1. Let p be the common perimeter of all the maximal parallelograms inscribed in the oval $\mathcal{C}$ of center O. By Lemma 2, the vertices of the rectangle $PQRS$ associated to some maximal parallelogram $ABCD$ inscribed in $\mathcal{C}$ all belong to the orthoptic curve of $\mathcal{C}$ and, at the same time, to the circle centered at O and of radius $\frac{p}{4}$. So, when the orthoptic curve of $\mathcal{C}$ is a circle it must be the circle of center O and radius $\frac{p}{4}$.

Assume now the existence of maximal-perimeter inscribed parallelograms for all the points of $\mathcal{C}$. If X is a point on the orthoptic curve of $\mathcal{C}$ and $\overleftrightarrow{XA}$ and $\overleftrightarrow{XB}$,

$A, B \in \mathcal{C}$, are the two tangent lines from X to $\mathcal{C}$, then $\overline{AB}$ must be a side of the maximal parallelogram inscribed in $\mathcal{C}$ and based at A, by Lemma 2. Again by Lemma 2, $XO = \frac{p}{4}$, or X belongs to the circle centered at O and of radius $\frac{p}{4}$. Conversely, let X be a point on the circle. Then X is located outside $\mathcal{C}$ and if $\overleftrightarrow{XA}$ is one of the two tangent lines from X to $\mathcal{C}$, $A \in \mathcal{C}$, and $ABCD$ is the maximal-perimeter inscribed parallelogram with associated circumscribed rectangle $PQRS$, $\overleftrightarrow{PS} = \overleftrightarrow{XA}$, then P belongs to $\overrightarrow{AX}$, $PO = SO = XO = \frac{p}{4}$, and so $X = P$, since $S \notin \overrightarrow{AX}$. Thus, X belongs to the orthoptic curve of $\mathcal{C}$.

Assume now the orthoptic curve of $\mathcal{C}$ is the circle with center O and radius $\frac{p}{4}$. For a given point $A \in \mathcal{C}$, consider the inscribed parallelogram $ABCD$ such that the tangent lines to $\mathcal{C}$ at A, B, C, and D, intersect at the vertices of circumscribed rectangle $PQRS$. By hypothesis, $PO = QO = \frac{p}{4}$. We claim that the perimeter of $ABCD$ equals p, so $ABCD$ will be the maximal-perimeter inscribed parallelogram based at A. On the one hand, $AB + BC \leq \frac{p}{2}$. On the other hand,

$$AB + BC \geq \min_{Y \in \overline{PQ}} (AY + YC)$$

This minimum occurs exactly at the point $Z \in \overline{PQ}$ where $\overline{AF}$, F the symmetric point of C with respect to $\overline{PQ}$, intersects $\overline{PQ}$ (see Figure 2). By construction, $\angle AZP$ is congruent to $\angle CZQ$, and then an argument similar to that given in Lemma 2 shows that $\overline{CZ}$ is parallel to $\overline{PO}$ and $AZ + CZ = 2PO = \frac{p}{2}$. As a result, $AB + BC \geq \frac{p}{2}$, which concludes the proof of the Theorem. □

We end this note with an application to the Theorem. It gives a 'quarter-oval' geometric description of the centrally symmetric ovals whose orthoptic curves are circles.

Corollary 3. *Let $\mathcal{C}$ be an oval with center O, having a circle Γ centered at O as orthoptic curve. By continuity, there is a point $W \in \mathcal{C}$ such that the associated maximal-perimeter inscribed parallelogram is a rhombus, $WNES$* (see Figure 3). *Then $\mathcal{C}$ is completely determined by the quarter-oval $\overgroup{WN}$ according to the following recipe:*

(a) *Consider the coordinate system centered at O, with axes $\overleftrightarrow{WO}$ and $\overleftrightarrow{ON}$, and such that the quarter-oval $\overgroup{WN}$ is situated in the third quadrant and the circle Γ has radius $\sqrt{OW^2 + ON^2}$.*

(b) *For an arbitrary point $A \in \overgroup{WN}$, the tangent line l to $\overgroup{WN}$ at A and the parallel line to l through C, the symmetric point of A with respect to O, intersect the circle Γ in two points, P, respectively Q, situated on the same side of the line $\overleftrightarrow{AC}$ as N.*

(c) *The line $\overleftrightarrow{AF}$, F being the symmetric point of C with respect to $\overline{PQ}$, intersects $\overline{PQ}$ in a point $T(A)$.*

Then the transformation $\overgroup{WN} \ni A \longmapsto T(A)$ sends bijectively and clockwise increasingly the quarter-oval $\overgroup{WN}$ onto the portion of $\mathcal{C}$ situated in the first quadrant of the coordinate system, in fact another quarter-oval, $\overgroup{NE}$. Moreover, symmetry with respect to O of the half-oval $\overgroup{WNE}$ completes the oval $\mathcal{C}$.

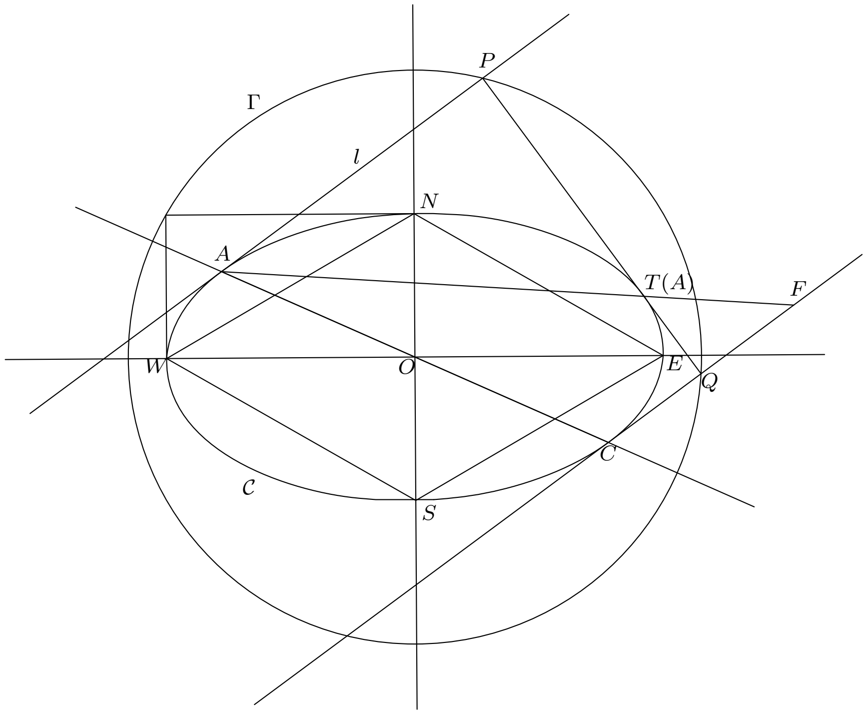

Figure 3. Construction of oval whose orthoptic curve is a circle, from quarter-oval

The proof of the Corollary is a simple consequence of all the facts considered in the proof of the Theorem.

An obvious question is this: 'Under what circumstances can a smooth quarter-oval situated in the third quadrant of a coordinate system be completed, as in the Corollary, to a full oval whose orthoptic curve is a circle?' The answer to this question has two components:

(a) The quarter-oval must globally satisfy a certain curvature growth-condition, which amounts to the fact that the transformation 'abscissa of $A \longmapsto$ abscissa of $T(A)$' must be strictly increasing.

(b) The curvatures of the quarter-oval at the end-points must be related by a transmission condition guaranteeing the smoothness of the full oval at those points.

These matters are better addressed by analytic methods and in keeping with the strictly geometric character of this note they will not be considered here.

References

[1] A. Connes and D. Zagier, A property of parallelograms inscribed in ellipses, *Amer. Math. Monthly*, 114 (2007) 909–914.

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