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On the Inradius of a Tangential Quadrilateral

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Abstract. We give a survey of known formulas for the inradius r of a tangential quadrilateral, and derive the possibly new formula

$$r = 2\sqrt{\frac{(M-uvx)(M-vxy)(M-xyu)(M-yuv)}{uvxy(uv+xy)(ux+vy)(uy+vx)}}$$

where u, v, x and y are the distances from the incenter to the vertices, and $M = \frac{1}{2}(uvx+vxy+xyu+yuv)$.

1. Introduction

A *tangential quadrilateral* [1] is a convex quadrilateral with an incircle, that is, a circle which is tangent to all four sides. Not all quadrilaterals are tangential. The most useful characterization is that its two pairs of opposite sides have equal sums, $a+c=b+d$, where a, b, c and d are the sides in that order [1, pp. 64–67]. [2]

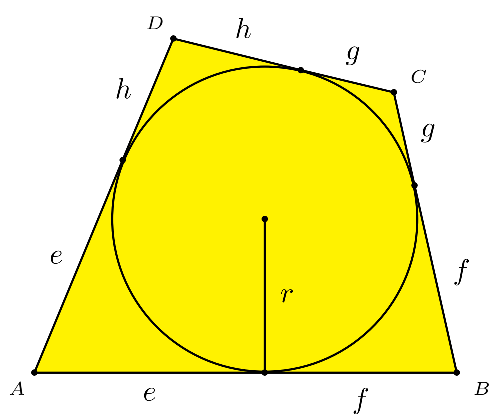

Figure 1. Klamkin's problem

It is well known that the inradius r of the incircle is given by

$$r = \frac{K}{s}$$

where K is the area of the quadrilateral and s is the semiperimeter[3]. The area of a tangential quadrilateral $ABCD$ with sides a, b, c and d is according to P. Yiu [10]

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[1]Other names for these quadrilaterals are circumscriptible quadrilateral [10], circumscribable quadrilateral [9] and circumscribed quadrilateral [4].

[2]There exists a lot of other interesting characterizations, see [9] and [7].

[3]This formula holds for all polygons with an incircle, where K in the area of the polygon.

given by[4]

$$K = \sqrt{abcd}\,\sin\frac{A+C}{2}.$$

From the formulas for the radius and area we conclude that the inradius of a tangential quadrilateral is not determined by the sides alone; there must be at least one angle given, then the opposite angle can be calculated by trigonometry.

Another formula for the inradius is

$$r = \sqrt{\frac{efg+fgh+ghe+hef}{e+f+g+h}}$$

where e, f, g and h are the distances from the four vertices to the points where the incircle is tangent to the sides (see Figure 1). This is interesting, since here the radius is only a function of four distances and no angles! The problem of deriving this formula was a quickie by M. S. Klamkin, with a solution given in [5].

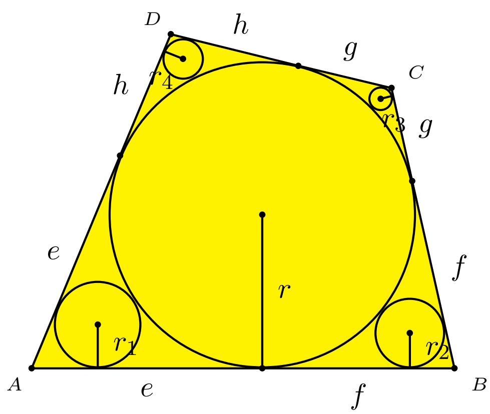

Figure 2. Minkus' 5 circles

If there are four circles with radii r_1, r_2, r_3 and r_4 inscribed in a tangential quadrilateral in such a way, that each of them is tangent to two of the sides and the incircle (see Figure 2), then the radius r of the incircle is a root of the quadratic equation

$$r^2 - (\sqrt{r_1r_2}+\sqrt{r_1r_3}+\sqrt{r_1r_4}+\sqrt{r_2r_3}+\sqrt{r_2r_4}+\sqrt{r_3r_4})\,r + \sqrt{r_1r_2r_3r_4} = 0$$

according to J. Minkus in [8, editorial comment].[5]

In [2, p.83] there are other formulas for the inradius, whose derivation was only a part of the solution of a contest problem from China. If the incircle in a tangential quadrilateral $ABCD$ is tangent to the sides at points W, X, Y and Z, and if E,

[4]A long synthetic proof can be found in [4]. Another way of deriving the formula is to use the formula $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{A+C}{2}}$ for the area of a general quadrilateral, derived in [10, pp.146–147], and the characterization $a+c=b+d$.

[5]The corresponding problem for the triangle is an old Sangaku problem, solved in [8], [3, pp. 30, 107–108.].

F, G and H are the midpoints of ZW, WX, XY and YZ respectively, then the inradius is given by the formulas

$$r = \sqrt{AI \cdot IE} = \sqrt{BI \cdot IF} = \sqrt{CI \cdot IG} = \sqrt{DI \cdot IH}$$

where I is the incenter (see Figure 3). The derivation is easy. Triangles IWA and IEW are similar, so $\frac{r}{AI} = \frac{IE}{r}$ which gives the first formula and the others follow by symmetry.

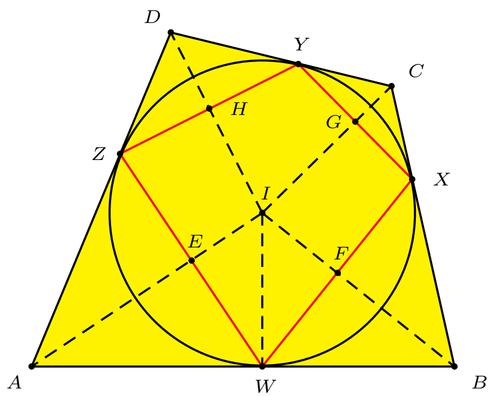

Figure 3. The problem from China

The main purpose of this paper is to derive yet another (perhaps new) formula for the inradius of a tangential quadrilateral. This formula is also a function of only four distances, which are from the incenter I to the four vertices A, B, C and D (see Figure 4).

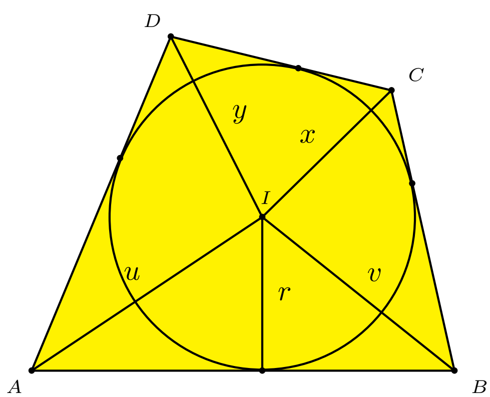

Figure 4. The main problem

Theorem 1. *If u, v, x and y are the distances from the incenter to the vertices of a tangential quadrilateral, then the inradius is given by the formula*

$$r = 2\sqrt{\frac{(M - uvx)(M - vxy)(M - xyu)(M - yuv)}{uvxy(uv + xy)(ux + vy)(uy + vx)}} \tag{1}$$

where

$$M = \frac{uvx + vxy + xyu + yuv}{2}.$$

Remark. It is noteworthy that formula (1) is somewhat similar to Parameshvaras' formula for the circumradius R of a *cyclic* quadrilateral, [6]

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

where s is the semiperimeter, which is derived in [6, p.17].

2. Preliminary results about triangles

The proof of formula (1) uses two equations that holds for all triangles. These are two cubic equations, and one of them is a sort of correspondence to formula (1). The fact is, that while it is possible to give r as a function of the distances AI, BI, CI and DI in a tangential quadrilateral, the same problem of giving r as a function of AI, BI and CI in a triangle ABC is not so easy to solve, since it gives a cubic equation. The second cubic equation is found when solving the problem, in a triangle, of finding an exradius as a function of the distances from the corresponding excenter to the vertices.

Lemma 2. *If x, y and z are the distances from the incenter to the vertices of a triangle, then the inradius r is a root of the cubic equation*

$$2xyzr^3 + (x^2y^2 + y^2z^2 + z^2x^2)r^2 - x^2y^2z^2 = 0. \tag{2}$$

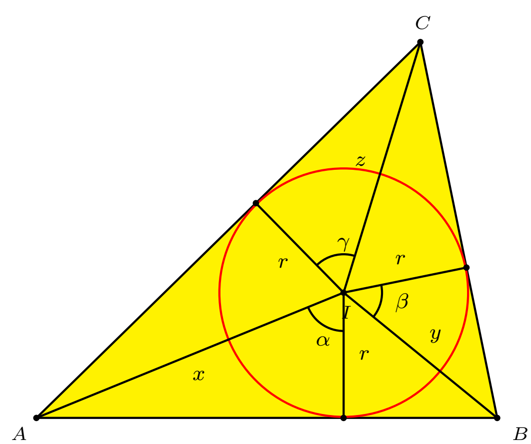

Figure 5. The incircle

[6] A quadrilateral with a circumcircle.

Proof. If α, β and γ are the angles between these distances and the inradius (see Figure 5), we have $\alpha+\beta+\gamma=\pi$, so $\cos(\alpha+\beta)=\cos(\pi-\gamma)$ and it follows that $\cos\alpha\cos\beta-\sin\alpha\sin\beta=-\cos\gamma$. Using the formulas $\cos\alpha=\frac{r}{x}$, $\cos\beta=\frac{r}{y}$ and $\sin^2\alpha+\cos^2\alpha=1$, we get

$$\frac{r}{x}\cdot\frac{r}{y}-\sqrt{1-\frac{r^2}{x^2}}\sqrt{1-\frac{r^2}{y^2}}=-\frac{r}{z}$$

or

$$\frac{r^2}{xy}+\frac{r}{z}=\frac{\sqrt{(x^2-r^2)(y^2-r^2)}}{xy}.$$

Multiplying both sides with xyz, reducing common factors and squaring, we get

$$(zr^2+xyr)^2=z^2(x^2-r^2)(y^2-r^2)$$

which after expansion and simplification reduces to (2). □

Lemma 3. *If u, v and z are the distances from an excenter to the vertices of a triangle, then the corresponding exradius r_c is a root of the cubic equation*

$$2uvzr_c^3-(u^2v^2+v^2z^2+z^2u^2)r_c^2+u^2v^2z^2=0. \tag{3}$$

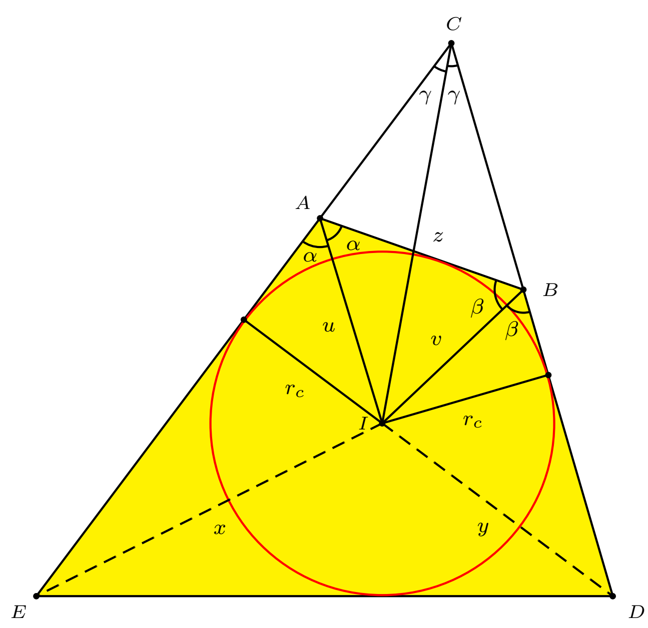

Figure 6. Excircle to triangle ABC and incircle to $ABDE$

Proof. Define angles α, β and γ to be between u, v, z and the sides of the triangle ABC or their extensions (see Figure 6). Then $2\alpha+A=\pi$, $2\beta+B=\pi$ and $2\gamma=C$. From the sum of angles in a triangle, $A+B+C=\pi$, this simplifies to $\alpha+\beta=\frac{\pi}{2}+\gamma$. Hence $\cos(\alpha+\beta)=\cos\left(\frac{\pi}{2}+\gamma\right)$ and it follows that $\cos\alpha\cos\beta-$

$\sin\alpha\sin\beta = -\sin\gamma$. For the exradius r_c, we have $\sin\alpha = \frac{r_c}{u}$, $\sin\beta = \frac{r_c}{v}$, $\sin\gamma = \frac{r_c}{z}$, and so

$$\sqrt{1-\frac{r_c^2}{u^2}}\sqrt{1-\frac{r_c^2}{v^2}}-\frac{r_c}{u}\cdot\frac{r_c}{v}=-\frac{r_c}{z}.$$

This can, in the same way as in the proof of Lemma 2, be rewritten as

$$z^2(u^2-r_c^2)(v^2-r_c^2)=(zr_c^2-uvr_c)^2$$

which after expansion and simplification reduces to (3). □

3. Proof of the theorem

Given a tangential quadrilateral $ABDE$ where the distances from the incenter to the vertices are u, v, x and y, we see that if we extend the two sides DB and EA to meet at C, then the incircle in $ABDE$ is both an incircle in triangle CDE and an excircle to triangle ABC (see Figure 6). The incircle and the excircle therefore have the same radius r, and from (2) and (3) we get that

$$2xyzr^3+(x^2y^2+y^2z^2+z^2x^2)r^2-x^2y^2z^2 = 0, \tag{4}$$

$$2uvzr^3-(u^2v^2+v^2z^2+z^2u^2)r^2+u^2v^2z^2 = 0. \tag{5}$$

We shall use these two equations to eliminate the common variable z. To do so, equation (4) is multiplied by uv and equation (5) by xy, giving

$$2uvxyzr^3+uv(x^2y^2+y^2z^2+z^2x^2)r^2-uvx^2y^2z^2 = 0,$$

$$2uvxyzr^3-xy(u^2v^2+v^2z^2+z^2u^2)r^2+xyu^2v^2z^2 = 0.$$

Subtracting the second of these from the first gives

$$\left(uv(x^2y^2+y^2z^2+z^2x^2)+xy(u^2v^2+v^2z^2+z^2u^2)\right)r^2-uvx^2y^2z^2-xyu^2v^2z^2=0$$

from which it follows

$$uvxyr^2(xy+uv)+z^2\left((uvy^2+uvx^2+xyv^2+xyu^2)r^2-uvxy(xy+uv)\right)=0.$$

Solving for z^2,

$$z^2=\frac{uvxy(uv+xy)r^2}{uvxy(uv+xy)-r^2(uvy^2+uvx^2+xyv^2+xyu^2)}. \tag{6}$$

Now we multiply (4) by u^2v^2 and (5) by x^2y^2, which gives

$$2u^2v^2xyzr^3+u^2v^2(x^2y^2+y^2z^2+z^2x^2)r^2-u^2v^2x^2y^2z^2 = 0,$$

$$2x^2y^2uvzr^3-x^2y^2(u^2v^2+v^2z^2+z^2u^2)r^2+u^2v^2x^2y^2z^2 = 0.$$

Adding these we get

$$2uvxyz(uv+xy)r^3+(u^2v^2y^2+u^2v^2x^2-x^2y^2v^2-x^2y^2u^2)z^2r^2=0$$

and since $zr^2\neq 0$ this reduces to

$$2uvxy(uv+xy)r+(u^2v^2y^2+u^2v^2x^2-x^2y^2v^2-x^2y^2u^2)z=0.$$

Solving for z, we get

$$z = -\frac{2uvxy(uv+xy)r}{u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2}.$$

Squaring and substituting z^2 from (6), we get the equality

$$\begin{aligned}&\frac{uvxy(uv+xy)r^2}{uvxy(uv+xy) - r^2(uvy^2 + uvx^2 + xyv^2 + xyu^2)}\\ &= \frac{4\left(uvxy(uv+xy)\right)^2 r^2}{(u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)^2},\end{aligned}$$

which, since $uvxy(uv+xy)r^2 \neq 0$, rewrites as

$$\begin{aligned}&4uvxy(uv+xy)\left(ux(vx+uy) + vy(uy+vx)\right)r^2\\ &= (2uvxy(uv+xy))^2 - (u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)^2.\end{aligned}$$

What is left is to factor this equation. Using the basic algebraic identities $a^2 - b^2 = (a+b)(a-b)$, $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$ we get

$$\begin{aligned}&4uvxy(uv+xy)(uy+vx)(ux+vy)r^2\\ &= \left(2uvxy(uv+xy) + (uvy)^2 + (uvx)^2 - (xyv)^2 - (xyu)^2\right)\\ &\quad\cdot\left(2uvxy(uv+xy) - (uvy)^2 - (uvx)^2 + (xyv)^2 + (xyu)^2\right)\\ &= \left((uvy+uvx)^2 - (xyv-xyu)^2\right)\left((xyv+xyu)^2 - (uvy-uvx)^2\right)\\ &= ((uvy+uvx+xyv-xyu)(uvy+uvx-xyv+xyu))\\ &\quad\cdot((xyv+xyu+uvy-uvx)(xyv+xyu-uvy+uvx)).\end{aligned} \tag{7}$$

Now using $M = \frac{1}{2}(uvx + vxy + xyu + yuv)$ we get

$$uvy + uvx + xyv - xyu = (uvx + vxy + xyu + yuv) - 2xyu = 2(M - xyu)$$

and in the same way

$$\begin{aligned}uvy + uvx - xyv + xyu &= 2(M - vxy),\\ xyv + xyu + uvy - uvx &= 2(M - uvx),\\ xyv + xyu - uvy + uvx &= 2(M - uvy).\end{aligned}$$

Thus, (7) is equivalent to

$$\begin{aligned}&4uvxy(uv+xy)(uy+vx)(ux+vy)r^2\\ =&2(M-uxy)\cdot 2(M-vxy)\cdot 2(M-uvx)\cdot 2(M-uvy).\end{aligned}$$

Hence

$$r^2 = \frac{4(M-uxy)(M-vxy)(M-uvx)(M-uvy)}{uvxy(uv+xy)(uy+vx)(ux+vy)}.$$

Extracting the square root of both sides finishes the derivation of formula (1).

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