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Some Triangle Centers Associated with the Circles Tangent to the Excircles

Boris Odehnal

Abstract. We study those tritangent circles of the excircles of a triangle which enclose exactly one excircle and touch the two others from the outside. It turns out that these three circles share exactly the Spieker point. Moreover we show that these circles give rise to some triangles which are in perspective with the base triangle. The respective perspectors turn out to be new polynomial triangle centers.

1. Introduction

Let $\mathbf{T} := ABC$ be a triangle in the Euclidean plane, and Γ_a, Γ_b, Γ_c its excircles, lying opposite to A, B, C respectively, with centers I_a, I_b, I_c and radii r_a, r_b, r_c. There are eight circles tangent to all three excircles: the side lines of $\mathbf{T}$ (considered as circles with infinite radius), the Feuerbach circle (see [2, 4]), the so-called Apollonius circle (enclosing all the three excircles (see for example [3, 6, 9]), and three remaining circles which will in the following be denoted by $\mathcal{K}_a$, $\mathcal{K}_b$, $\mathcal{K}_c$. The circle $\mathcal{K}_a$ is tangent to Γ_a and externally to Γ_b and Γ_c; similarly for $\mathcal{K}_b$ and $\mathcal{K}_c$. The radii of these circles are computed in [1]. These circles have the Spieker center X_{10} as a common point. In this note we study these circles in more details, and show that the triangle of contact points $K_{a,a}K_{b,b}K_{c,c}$ is perspective with $\mathbf{T}$. Surprisingly, the triangle $M_aM_bM_c$ of the centers these circles is also perspective with $\mathbf{T}$.

2. Main results

The problem of constructing the circles tangent to three given circles is well studied. Applying the ideas of J. D. Gergonne [5] to the three excircles we see that the construction of the circles $\mathcal{K}_a$ etc can be accomplished simply by a ruler. Let $K_{a,b}$ be the contact point of circle Γ_a with $\mathcal{K}_b$, and analogously define the remaining eight contact points. The contact points $K_{a,a}$, $K_{b,a}$, $K_{c,a}$ are the intersections of the excircles Γ_a, Γ_b, Γ_c with the lines joining their contact points with the sideline BC to the radical center radical center of the three excircles, namely, the Spieker point

$$X_{10} = (b + c : c + a : a + b)$$

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in homogeneous barycentric coordinates (see for example [8]). The circle $\mathcal{K}_a$ is the circle containing these points (see Figure 1). The other two circle $\mathcal{K}_b$ and $\mathcal{K}_c$ can be analogously constructed.

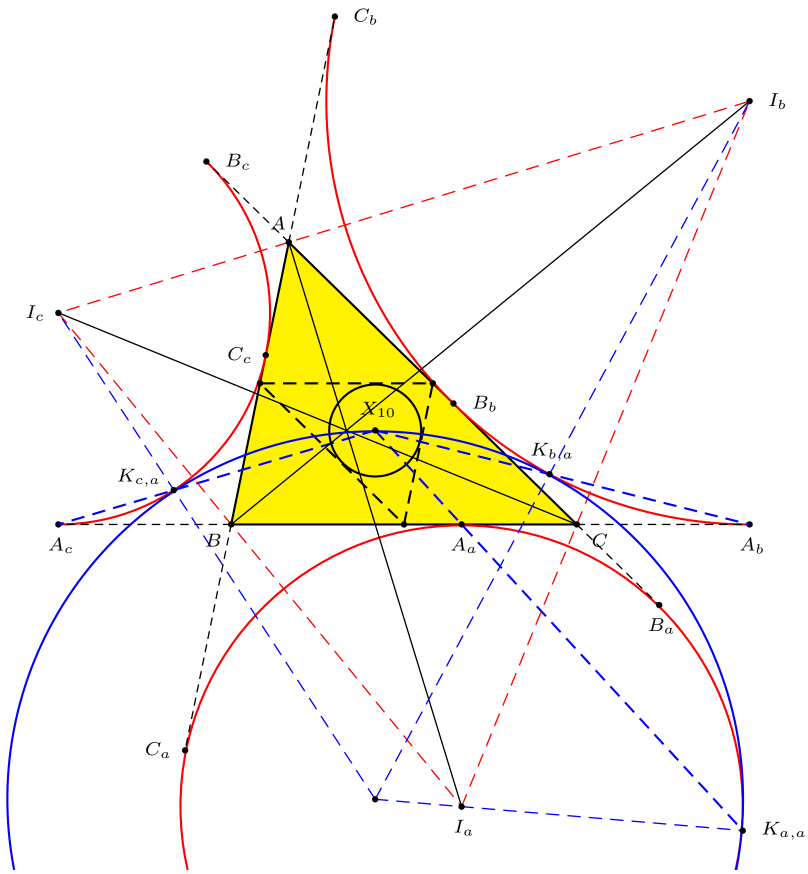

Figure 1. The circle $\mathcal{K}_a$

Let $s := \frac{1}{2}(a + b = c)$ be the semiperimeter. The contact points of the excircles with the sidelines are the points

$$\begin{aligned} A_a &= (0 : s-b : s-c), & B_a &= (-(s-b) : 0 : s), & C_a &= (-(s-c) : s : 0); \\ A_b &= (0 : -(s-a) : s), & B_b &= (s-a : 0 : s-c), & C_b &= (s : -(s-c) : 0); \\ A_c &= (0 : c : -(s-a)), & B_c &= (s : 0 : -(s-b)), & C_c &= (s-a : s-b : 0). \end{aligned}$$

A conic is be represented by an equation in the form $x^{\mathrm{T}} M x = 0$, where $x^{\mathrm{T}} = (x_0 \;\; x_1 \;\; x_2)$ is the vector collecting the homogeneous barycentric coordinates of a

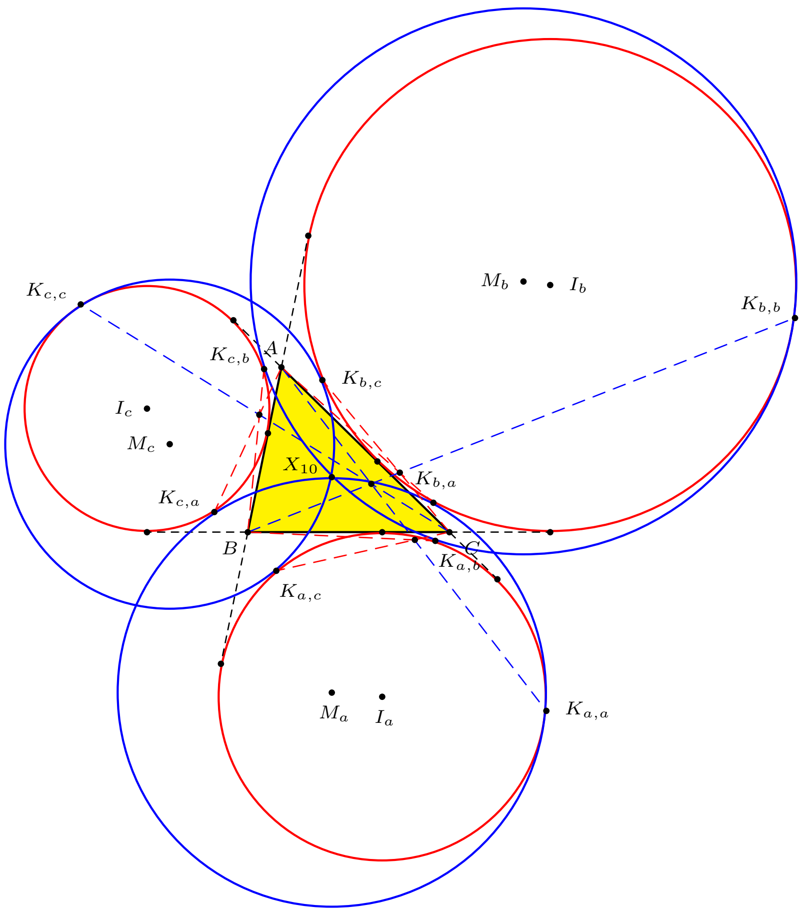

Figure 2.

point X, and M is a symmetric 3×3-matrix. For the excircles, these matrices are

$$M_a = \begin{pmatrix} s^2 & s(s-c) & s(s-b) \\ s(s-c) & (s-c)^2 & -(s-b)(s-c) \\ s(s-b) & -(s-b)(s-c) & (s-b)^2 \end{pmatrix},$$

$$M_b = \begin{pmatrix} (s-c)^2 & s(s-c) & -(s-a)(s-c) \\ s(s-c) & s^2 & s(s-a) \\ -(s-a)(s-c) & s(s-a) & (s-a)^2 \end{pmatrix},$$

$$M_c = \begin{pmatrix} (s-b)^2 & -(s-a)(s-b) & s(s-b) \\ -(s-a)(s-b) & (s-a)^2 & s(s-a) \\ s(s-b) & s(s-a) & s^2 \end{pmatrix}.$$

It is elementary to verify that the homogeneous barycentrics of the contact points are given by:

$$\begin{array}{rcl}
K_{a,a} & = & (-(b+c)^2(s-b)(s-c) : c^2s(s-b) : b^2s(s-c)), \\
K_{a,b} & = & (-c^2s(s-b) : (c+a)^2(s-b)(s-c) : (as+bc)^2), \\
K_{a,c} & = & (-b^2s(s-c) : (as+bc)^2 : (a+b)^2(s-b)(s-c)); \\
K_{b,a} & = & ((b+c)^2(s-a)(s-c) : -c^2s(s-a) : (bs+ac)^2), \\
K_{b,b} & = & (c^2s(s-a) : -(c+a)^2(s-a)(s-c) : a^2s(s-c)), \\
K_{b,c} & = & ((bs+ac)^2 : -a^2s(s-c) : (a+b)^2(s-a)(s-c)); \\
K_{c,a} & = & ((b+c)^2(s-a)(s-b) : (cs+ab)^2 : -b^2s(s-a)), \\
K_{c,b} & = & ((cs+ab)^2 : (c+a)^2(s-a)(s-c) : -a^2s(s-b)), \\
K_{c,c} & = & (b^2s(s-a) : a^2s(s-b) : -(a+b)^2(s-a)(s-b)).
\end{array} \tag{1}$$

Theorem 1.
The triangle $K_{a,a}K_{b,c}K_{c,c}$ of contact points is perspective with $\mathbf{T}$ at a point with homogeneous barycentric coordinates

$$\left(\frac{s-a}{a^2} : \frac{s-b}{b^2} : \frac{s-c}{c^2}\right). \tag{2}$$

Proof. The coordinates of $K_{a,a}$, $K_{b,b}$, $K_{c,c}$ can be rewritten as

$$\begin{array}{rcl}
K_{a,a} & = & \left(-\frac{(b+c)^2(s-b)(s-c)}{b^2c^2s} : \frac{s-b}{b^2} : \frac{s-c}{c^2}\right), \\
K_{b,b} & = & \left(\frac{s-a}{a^2} : -\frac{(c+a)^2(s-a)(s-c)}{c^2a^2s} : \frac{s-c}{c^2}\right), \\
K_{c,c} & = & \left(\frac{s-a}{a^2} : \frac{s-b}{b^2} : -\frac{(a+b)^2(s-a)(s-b)}{a^2b^2s}\right).
\end{array} \tag{3}$$

From these, it is clear that the lines $AK_{a,a}$, $BK_{b,b}$, $CK_{c,c}$ meet in the point given in (2). □

Remark. The triangle center P_K is not listed in [7].

Theorem 2.
The lines $AK_{a,a}$, $BK_{a,b}$, and $CK_{a,c}$ are concurrent.

Proof. The coordinates of the points $K_{a,a}$, $K_{a,b}$, $K_{a,c}$ can be rewritten in the form

$$\begin{array}{rcl}
K_{a,a} & = & \left(-\frac{(b+c)^2(s-b)(s-c)}{b^2c^2s} : \frac{s-b}{b^2} : \frac{s-c}{c^2})\right), \\
K_{a,b} & = & \left(-\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{(c+a)^2(s-b)(s-c)^2}{c^2(as+b^2)^2} : \frac{s-c}{c^2}\right), \\
K_{a,c} & = & \left(-\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{s-b}{b^2} : \frac{(a+b)^2(s-b)^2(s-c)}{b^2(as+bc)^2}\right).
\end{array} \tag{4}$$

From these, the lines $AK_{a,a}$, $BK_{a,b}$, and $CK_{a,c}$ intersect at the point

$$\left(-\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{s-b}{b^2} : \frac{s-c}{c^2}\right).$$

□

Let M_i be the center of the circle $\mathcal{K}_i$.

Theorem 3.
The triangle $M_1M_2M_3$ is perspective with $\mathbf{T}$ at the point

$$\left(\frac{1}{-a^5 - a^4(b+c) + a^3(b-c)^2 + a^2(b+c)(b^2+c^2) + 2abc(b^2+bc+c^2) + 2(b+c)b^2c^2} \right.$$
$$\left. : \cdots : \cdots \right).$$

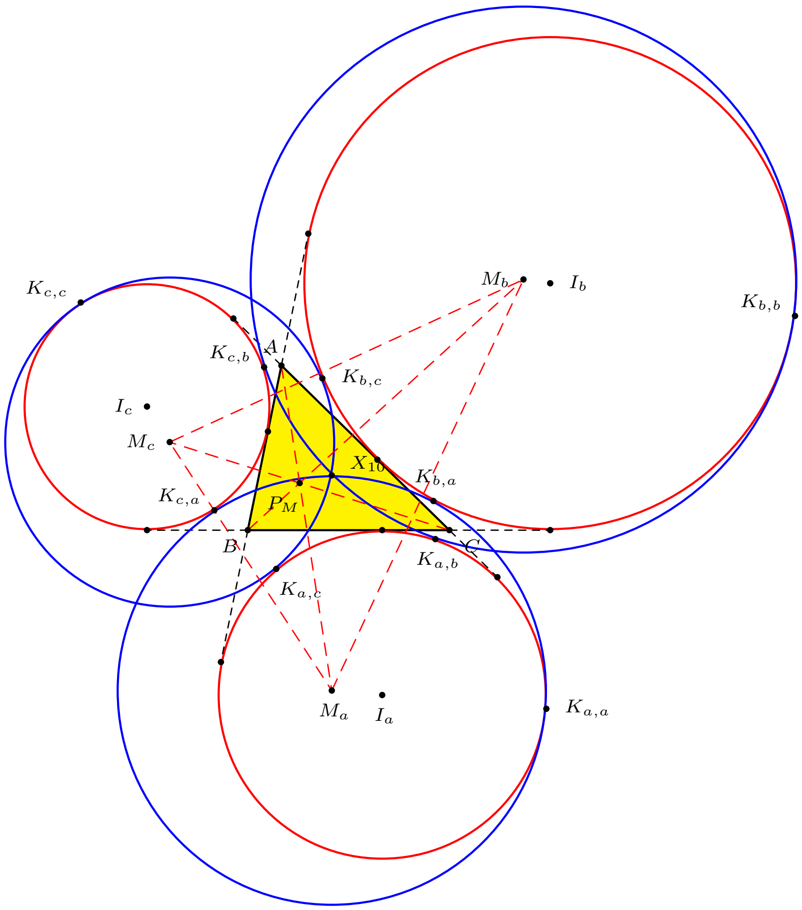

Figure 3.

Proof. The center of the circle $\mathcal{K}_a$ is the point

$$\begin{aligned} M_a = \ & -2a^4(b+c) - a^3(4b^2+4bc+3c^2) + a^2(b+c)(b^2+c^2) - (b+c-a)(b^2-c^2)^2 \\ : \ & -c^5 - c^4(a+b) + c^3(a-b)^2 + c^2(a+b)(a^2+b^2) + 2abc(a^2+ab+b^2) + 2a^2b^2(a+b) \\ : \ & -b^5 - b^4(c+a) + b^3(c-a)^2 + b^2(c+a)(c^2+a^2) + 2abc(c^2+ca+a^2) + 2c^2a^2(c+a). \end{aligned}$$

Similarly, the coordinates of M_b and M_c can be written down. From these, the perspectivity of $\mathbf{T}$ and $M_aM_bM_c$ follows, with the perspector given above. $\square$

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Boris Odehnal: Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria

E-mail address: `boris@geometrie.tuwien.ac.at`