Conics Tangent at the Vertices to Two Sides of a Triangle

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Abstract. We study conics tangent to two sides of a given triangle at two vertices, and construct two interesting triads of such conics, one consisting of parabolas, the other rectangular hyperbolas. We also construct a new triangle center, the barycentric cube root of $X_{25}$, which is the homothetic center of the orthic and tangential triangles.

1. Introduction

In the plane of a given triangle $ABC$, a conic is the locus of points with homogeneous barycentric coordinates $x : y : z$ satisfying a second degree equation of the form

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2hxy = 0,$$

where $f$, $g$, $h$, $p$, $q$, $r$ are real coefficients (see [4]). In matrix form, (1) can be rewritten as

$$(x\ y\ z) \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

If we have a circumconic, i.e., a conic that passes through the vertices $A$, $B$, $C$, then from (1), $f = g = h = 0$. Since the equation becomes $pyz + qzx + rxy = 0$, the circumconic is the isogonal (or isotomic) conjugate of a line.

If the conic passes through $B$ and $C$, we have $g = h = 0$. The conic has equation

$$fx^2 + 2pzy + 2qzx + 2rxy = 0.$$  \hspace{1cm} (2)

2. Conic tangent to two sides of $ABC$

Consider a conic given by (2) which is tangent at $B$, $C$ to the sides $AB$, $AC$. We call this an $A$-conic. Since the line $AB$ has equation $z = 0$, the conic (2) intersects $AB$ at $(x : y : 0)$ where $x(fx + 2ry) = 0$. The conic and the line are tangent at $B$ only if $r = 0$. Similarly the conic is tangent to $AC$ at $C$ only if $q = 0$. The $A$-conic has equation

$$fx^2 + 2pzy = 0.$$  \hspace{1cm} (3)
We shall also consider the analogous notions of $B$- and $C$-conics. These are respectively conics with equations

\[ gy^2 + 2qzx = 0, \quad hz^2 + 2rxy = 0. \tag{4} \]

If triangle $ABC$ is scalene, none of these conics is a circle.

**Lemma 1.** Let $BC$ be a chord of a conic with center $O$, and $M$ the midpoint of $BC$. If the tangents to the conic at $B$ and $C$ intersect at $A$, then the points $A$, $M$, $O$ are collinear.

**Proof.** The harmonic conjugate $S$ of the point $M$ relative to $B$, $C$ is the point at infinity of the line $BC$. The center $O$ is the midpoint of every chord passing through $O$. Hence, the polar of $O$ is the line at infinity. The polar of $A$ is the line $BC$ (see Figure 1). Hence, the polar of $S$ is a line that must pass through $O$, $M$ and $A$.

From Lemma 1 we conclude that the center of an $A$-conic lies on the $A$-median. In the case of a parabola, the axis is parallel to the median of $ABC$ since the center is a point at infinity. In general, the center of the conic (1) is the point

\[
\begin{pmatrix}
1 & r & q & f & 1 & q & f & r & 1 \\
1 & g & p & r & 1 & p & r & g & 1 \\
1 & p & h & q & 1 & h & q & p & 1
\end{pmatrix}.
\tag{5}
\]

Hence the centers of the $A$-, $B$-, $C$-conics given in (3) and (4) are the points $(p : f : f)$, $(g : q : g)$, and $(h : h : r)$ respectively.

We investigate two interesting triads when the three conics (i) are parabolas, (ii) rectangular hyperbolas, and (iii) all pass through a given point $P = (u : v : w)$, and shall close with a generalization.
3. A triad of parabolas

The $A$-conic (3) is a parabola if the center $(p : f : f)$ is on the line at infinity. Hence, $p + f + f = 0$, and the $A$-parabola has equation $x^2 - 4yz = 0$. We label this parabola $\mathcal{P}_a$. Similarly, the $B$- and $C$-parabolas are $\mathcal{P}_b : y^2 - 4zx = 0$ and $\mathcal{P}_c : z^2 - 4xy = 0$ respectively. These are also known as the Artzt parabolas (see Figure 2).

3.1. Construction. A conic can be constructed with a dynamic software by locating 5 points on it. The $A$-parabola $\mathcal{P}_a$ clearly contains the vertices $B$, $C$, and the points $M_a = (2 : 1 : 1)$, $P_b = (4 : 1 : 4)$ and $P_c = (4 : 4 : 1)$. Clearly, $M_a$ is the midpoint of the median $AA_1$, and $P_b$, $P_c$ are points dividing the medians $BB_1$ and $CC_1$ in the ratio $BP_b : P_bB_1 = CP_c : P_cC_1 = 8 : 1$.

Similarly, if $M_b, M_c$ are the midpoints of the medians $BB_1$ and $CC_1$, and $P_a$ divides $AA_1$ in the ratio $AP_a : P_aA_1 = 8 : 1$, then the $B$-parabola $\mathcal{P}_b$ is the conic containing $C$, $A$, $M_b$, $P_c$, $P_a$, and the $C$-parabola $\mathcal{P}_c$ contains $A$, $B$, $M_c$, $P_a$, $P_b$.

![Figure 2](image_url)

Remarks. (1) $P_a$, $P_b$, $P_c$ are respectively the centroids of triangles $GBC$, $GCA$, $GAB$.

(2) Since $A_1$ is the midpoint of $BC$ and $M_aA_1$ is parallel to the axis of $\mathcal{P}_a$, by Archimedes’ celebrated quadration, the area of the parabola triangle $M_aBC$ is $\frac{4}{3} \cdot \Delta M_aBC = \frac{2}{3} \cdot \Delta ABC$.

(3) The region bounded by the three Artzt parabolas has area $\frac{5}{27}$ of triangle $ABC$.

3.2. Foci and directrices. We identify the focus and directrix of the $A$-parabola $\mathcal{P}_a$. Note that the axis of the parabola is parallel to the median $AA_1$. Therefore,
the parallel through $B$ to the median $AA_1$ is parallel to the axis, and its reflection in the tangent $AB$ passes through the focus $F_a$. Similarly the reflection in $AC$ of the parallel through $C$ to the median $AA_1$ also passes through $F_a$. Hence the focus $F_a$ is constructible (in the Euclidean sense).

Let $D, E$ be the orthogonal projections of the focus $F_a$ on the sides $AB, AC$, and $D', E'$ the reflections of $F_a$ with respect to $AB, AC$. It is known that the line $DE$ is the tangent to the $A$-parabola $\mathcal{P}_a$ at its vertex and the line $D'E'$ is the directrix of the parabola (see Figure 3).

**Figure 3.**

**Theorem 2.** The foci $F_a, F_b, F_c$ of the three parabolas are the vertices of the second Brocard triangle of $ABC$ and hence are lying on the Brocard circle. The triangles $ABC$ and $F_aF_bF_c$ are perspective at the Lemoine point $K$.

**Proof.** With reference to Figure 3, in triangle $ADE$, the median $AA_1$ is parallel to the axis of the $A$-parabola $\mathcal{P}_a$. Hence, $AA_1$ is an altitude of triangle $ADE$. The line $AF_a$ is a diameter of the circumcircle of $ADE$ and is the isogonal conjugate to $AA_1$. Hence the line $AF_a$ is a symmedian of $ABC$, and it passes through the Lemoine point $K$ of triangle $ABC$.

Similarly, if $F_b$ and $F_c$ are the foci of the $B$- and $C$-parabolas respectively, then the lines $BF_b$ and $CF_c$ also pass through $K$, and the triangles $ABC$ and $F_aF_bF_c$ are perspective at the Lemoine point $K$.

Since the reflection of $BF_a$ in $AB$ is parallel to the median $AA_1$, we have

$$\angle F_aBA = \angle D'BA = \angle BAA_1 = \angle F_aAC,$$  
and the circle $F_aAB$ is tangent to $AC$ at $A$. Similarly,

$$\angle ACF_a = \angle AEE' = \angle A_1AC = \angle BAF_a.$$
and the circle $F_a CA$ is tangent to $AB$ at $A$. From (6) and (7), we conclude that the triangles $F_a AB$ and $F_a CA$ are similar, so that

$$\angle AF_a B = \angle CF_a A = \pi - A \quad \text{and} \quad \angle BF_a C = 2A. \quad (8)$$

If $O$ is the circumcenter of $ABC$ and the lines $AF_a, CF_a$ meet the circumcircle again at the points $A', C'$ respectively then from the equality of the arcs $AC'$ and $BA'$, the chords $AB = A'C'$. Since the triangles $ABF_a$ and $A'C'F_a$ are similar to triangle $CAF_a$, they are congruent. Hence, $F_a$ is the midpoint of $AA'$, and is the orthogonal projection of the circumcenter $O$ on the $A$-symmedian. As such, it is on the Brocard circle with diameter $OK$. Likewise, the foci of the $B$- and $C$-parabolas are the orthogonal projections of the point $O$ on the $B$- and $C$-symmedians respectively. The three foci form the second Brocard triangle of $ABC$. □

Remarks. (1) Here is an alternative, analytic proof. In homogeneous barycentric coordinates, the equation of a circle is of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(Px + Qy + Rz) = 0,$$

where $a, b, c$ are the lengths of the sides of $ABC$, and $P, Q, R$ are the powers of $A, B, C$ relative to the circle. The equation of the circle $F_a AB$ tangent to $AC$ at $A$ is

$$a^2yz + b^2zx + c^2xy - b^2(x + y + z)z = 0, \quad (9)$$
and that of the circle $F_a CA$ tangent to $AB$ at $A$ is
\[a^2 yz + b^2 zx + c^2 xy - c^2 (x + y + z) y = 0.\] (10)

Solving these equations we find
\[F_a = (b^2 + c^2 - a^2 : b^2 : c^2).\]

It is easy to verify that this lies on the Brocard circle
\[a^2 yz + b^2 zx + c^2 xy - \frac{a^2 b^2 c^2}{a^2 + b^2 + c^2} (x + y + z) \left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.\]

(2) This computation would be more difficult if the above circles were not tangent at $A$ to the sides $AC$ and $AB$. So it is interesting to show another method. We can get the same result, as we know directed angles (defined modulo $\pi$) from
\[\theta = (F_aB, F_aC) = 2A, \quad \varphi = (F_aC, F_aA) = -A, \quad \psi = (F_aA, F_aB) = -A.\]

Making use of the formula given in [1], we obtain
\[
F_a = \left(\frac{1}{\cot A - \cot \theta} : \frac{1}{\cot B - \cot \varphi} : \frac{1}{\cot C - \cot \psi}\right)
= \left(\frac{1}{\cot A - \cot 2A} : \frac{1}{\cot B + \cot A} : \frac{1}{\cot C + \cot A}\right)
= \left(\frac{\sin A \sin 2A}{\sin A} : \frac{\sin A \sin B}{\sin(A + B)} : \frac{\sin A \sin C}{\sin(A + C)}\right)
= \left(2 \cos A \sin B \sin C : \sin^2 B : \sin^2 C\right)
= \left(b^2 + c^2 - a^2 : b^2 : c^2\right).
\]

4. Rectangular $A$-, $B$-, $C$-hyperbolas

The $A$-conic $fx^2 + 2pyz = 0$ is a rectangular hyperbola if it contains two orthogonal points at infinity $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ where
\[x_1 + y_1 + z_1 = 0 \quad \text{and} \quad x_2 + y_2 + z_2 = 0.\]

and. Putting $x = sy$ and $z = -(x + y)$ we have $fs^2 - 2ps - 2p = 0$ with roots $s_1 + s_2 = \frac{2p}{f}$ and $s_1 s_2 = -\frac{2p}{f}$. The two points are orthogonal if
\[S_A x_1 x_2 + S_B y_1 y_2 + S_C z_1 z_2 = 0.\]

From this, $S_A x_1 x_2 + S_B y_1 y_2 + S_C (x_1 + y_1) (x_2 + y_2) = 0$, $S_A s_1 s_2 + S_B + S_C (s_1 + 1) (s_2 + 1) = 0$, and
\[\frac{p}{f} = \frac{S_B + S_C}{2S_A} = \frac{a^2}{b^2 + c^2 - a^2}.
\]

This gives the rectangular $A$-hyperbola
\[\mathcal{H}_a : \quad (b^2 + c^2 - a^2) x^2 + 2a^2 yz = 0,
\]

with center
\[O_a = (a^2 : b^2 + c^2 - a^2 : b^2 - c^2 - a^2).\]
This point is the orthogonal projection of the orthocenter $H$ of $ABC$ on the $A$-median and lies on the orthocentroidal circle, i.e., the circle with diameter $GH$. Similarly the centers $O_b$, $O_c$ lie on the orthocentroidal circle as projections of $H$ on the $B$-, $C$-medians. Triangle $O_aO_bO_c$ is similar to the triangle of the medians of $ABC$ (see Figure 5).

The construction of the $A$-hyperbola can be done since we know five points of it: the points $B$, $C$, their reflections $B'$, $C'$ in $O_a$, and the orthocenter of triangle $B'C'$. 

5. Triad of conics passing through a given point

Let $P = (u : v : w)$ be a given point. We denote the $A$-conic through $P$ by $(A_P)$; similarly for $(B_P)$ and $(C_P)$. These conics have equations 

$$vwx^2 - u^2yz = 0, \quad wuy^2 - v^2zx = 0, \quad uvz^2 - w^2xy = 0.$$ 

Let $XYZ$ be the cevian triangle of $P$ and $X'Y'Z'$ be the trilinear polar of $P$ (see Figure 6). It is known that the point $X'$ is the intersection of $YZ$ with $BC$, and is the harmonic conjugate of $X$ relative to the points $B$, $C$; similarly for $Y'$ and $Z'$ are the harmonic conjugates of $Y$ and $Z$ relative to $C$, $A$ and $A$, $B$. These points have coordinates

$$X' = (0 : -v : w), \quad Y' = (u : 0 : -v), \quad Z' = (-u : v : 0),$$
and they lie on the trilinear polar

\[ \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0. \]

Now, the polar of \( A \) relative to the conic \((A_P)\) is the line \( BC \). Hence, the polar of \( X' \) passes through \( A \). Since \( X \) is harmonic conjugate of \( X' \) relative to \( B, C \), the line \( AX \) is the polar of \( X' \) and the line \( X'P \) is tangent to the conic \((A_P)\) at \( P \).

**Figure 6.**

**Remark.** The polar of an arbitrary point or the tangent of a conic at the point \( P = (u : v : w) \) is the line given by

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
\end{pmatrix}
\begin{pmatrix}
  f & r & q \\
  r & g & p \\
  q & p & h \\
\end{pmatrix}
\begin{pmatrix}
  u \\
  v \\
  w \\
\end{pmatrix} = 0.
\]

Hence the tangent of \((A_P)\) at \( P \) is the line

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
\end{pmatrix}
\begin{pmatrix}
  2vw & 0 & 0 \\
  0 & 0 & -u^2 \\
  0 & -u^2 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  u \\
  v \\
  w \\
\end{pmatrix} = 0,
\]

or \( 2vwx - wuy - uvz = 0 \). This meets \( BC \) at the point \( X' = (0 : -v : w) \) as shown before.
5.1. Construction of the conic \((A_P)\). We draw the line \(YZ\) to meet \(BC\) at \(X'\). Then \(X'P\) is tangent to the conic at \(P\). Let this meet \(AC\) and \(AB\) at \(E\) and \(F\) respectively. Let \(M_2\) and \(M_3\) be the midpoints of \(PB\) and \(PC\). Since \(AB\), \(AC\) and \(EF\) are tangents to \((A_P)\) at the points \(B\), \(C\), \(P\) respectively, the lines \(EM_3\) and \(FM_2\) meet at the center \(O_a\) of the conic \((A_P)\). If \(B', C'\) are the symmetric points of \(B\), \(C\) relative to \(O_a\), then \((A_P)\) is the conic passing through the five points \(B\), \(P\), \(C\), \(B'\), \(C'\).

The conics \((B_P)\) and \((C_P)\) can be constructed in a similar way.

![Figure 7](image.png)

In the special case when \(P\) is the centroid \(G\) of triangle \(ABC\), \(X'\) is a point at infinity and the line \(YZ\) is parallel to \(BC\). The center \(O_a\) is the symmetric \(A'\) of \(A\) relative to \(G\). It is obvious that \((A_P)\) is the translation of the Steiner circumellipse (with center \(G\)), by the vector \(AG\); similarly for \((B_P)\), \((C_P)\) (see Figure 7).

**Theorem 3.** For an arbitrary point \(Q\) the line \(X'Q\) intersects the line \(AP\) at the point \(R\), and we define the mapping \(h(Q) = S\), where \(S\) is the harmonic conjugate of \(Q\) with respect to \(X'\) and \(R\). The mapping \(h\) swaps the conics \((B_P)\) and \((C_P)\).

**Proof.** The mapping \(h\) is involutive because \(h(Q) = S\) if and only if \(h(S) = Q\). Since the line \(AP\) is the polar of \(X'\) relative to the pair of lines \(AB\), \(AC\) we have \(h(A) = A\), \(h(P) = P\), \(h(C) = B\) so that \(h(AB) = AC\), and \(h(BC) = CB\). Hence, the conic \(h((B_P))\) is the one passing through \(A\), \(P\), \(B\) and tangent to the sides \(AC\), \(BC\) at \(A\), \(B\) respectively. This is clearly the conic \((C_P)\). \(\square\)

Therefore, a line passing through \(X'\) tangent to \((B_P)\) is also tangent to \((C_P)\). This means that the common tangents of the conics \((B_P)\) and \((C_P)\) intersect at \(X'\).

Similarly we can define mappings with pivot points \(Y'\), \(Z'\) swapping \((C_P)\), \((A_P)\) and \((A_P), (B_P)\).
Consider the line $X'P$ tangent to the conic $(A_P)$ at $P$. It has equation
\[-2vwx + wuy + wzv = 0.\]
This line meets the conic $(B_P)$ at the point $X_b = (u : -2v : 4w)$, and the conic $(C_P)$ at the point $X_c = (u : 4v : -2w)$.

Similarly, the tangent $Y'P$ of $(B_P)$ intersects $(C_P)$ again at $Y_c = (4u : v : -2w)$ and $(A_P)$ again at $Y_a = (-2u : v : 4w)$. The tangent $Z'P$ of $(C_P)$ intersects $(A_P)$ again at $Z_a = (-2u : 4v : w)$ and $(B_P)$ again at $Z_b = (4u : -2v : w)$ respectively.

**Theorem 4.** The line $Z_bY_c$ is a common tangent of $(B_P)$ and $(C_P)$, so is $X_cZ_a$ of $(C_P)$ and $(A_P)$, and $Y_aX_b$ of $(A_P)$ and $(B_P)$.

**Proof.** We need only prove the case $Y_cZ_b$. The line has equation
\[vwx + 4wuy + 4uvz = 0.\]
This is tangent to $(B_P)$ at $Z_b$ and to $(C_P)$ at $Y_c$ as the following calculation confirms.
\[
\begin{pmatrix}
0 & 0 & v^2 \\
0 & -2wu & 0 \\
v^2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
4u \\
-2v \\
w
\end{pmatrix}
= 
\begin{pmatrix}
vw \\
4wu \\
4uv
\end{pmatrix}.
\]

**Theorem 5.** The six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ lie on a conic (see Figure 7).

**Proof.** It is easy to verify that the six points satisfy the equation of the conic
\[2v^2w^2x^2 + 2w^2u^2y^2 + 2w^2u^2z^2 + 7u^2uvwz + 7uv^2wzx + 7uw^2xy = 0,\] (11)
Conics tangent at the vertices to two sides of a triangle

which has center

\[(u(-11u + 7v + 7w) : v(7u - 11v + 7w) : w(7u + 7v - 11w)).\]

\[\square\]

Remark. The equation of the conic (11) can be rewritten as

\[u^2(2v^2 - 7uw + 2w^2)yz + v^2(2w^2 - 7wu + 2u^2)zx + w^2(2u^2 - 7uv + 2v^2)xy - 2(x + y + z)(v^2w^2x + w^2u^2y + u^2v^2z) = 0.\]

This is a circle if and only if

\[\frac{u^2(2v^2 - 7uw + 2w^2)}{a^2} = \frac{v^2(2w^2 - 7wu + 2u^2)}{b^2} = \frac{w^2(2u^2 - 7uv + 2v^2)}{c^2}.\]

Equivalently, the isotomic conjugate of \(P\), namely, \((\frac{1}{u} : \frac{1}{v} : \frac{1}{w})\) is the intersection of the three conics defined by

\[\frac{2y^2 - 7yz + 2z^2}{a^2} = \frac{2x^2 - 7zx + 2x^2}{b^2} = \frac{2x^2 - 7xy + 2y^2}{c^2}.\]

5.2. The type of the three conics \((A_P), (B_P), (C_P)\). The type of a conic with given equation (1) can be determined by the quantity

\[d = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & f & r & q \\ 1 & r & g & p \\ 1 & q & p & h \end{vmatrix}\]

For the conic \(A_P : vwz^2 - u^2yz = 0\), this is

(i) a parabola if \(u^2 - 4vw = 0\) (and \(A_P = \mathcal{P}_a\)),

(ii) an ellipse if \(u^2 - vw < 0\), i.e., \(P\) lying inside \(\mathcal{P}_a\),

(iii) a hyperbola if \(u^2 - vw > 0\), i.e., \(P\) lying outside \(\mathcal{P}_a\).

Thus the conics in the triad \((A_P, B_P, C_P)\) are all ellipses if and only if \(P\) lies in the interior of the curvilinear triangle \(P_aP_bP_c\) bounded by arcs of the Artzt parabolas (see Figure 2).

Remarks. (1) It is impossible for all three conics \((A_P, B_P, C_P)\) to be parabolas.

(2) The various possibilities of conics of different types are given in the table below.

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<th>1</th>
<th>1</th>
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<td>1</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Hyperbolas</td>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that the conics in the triad \((A_P, B_P, C_P)\) cannot be all rectangular hyperbolas. This is because, as we have seen in §4 that the only rectangular \(A\)-hyperbola is \(\mathcal{H}_a\). The conic \(A_P\) is a rectangular hyperbola if and only if \(P\) lies on \(\mathcal{H}_a\), and in this case, \(A_P = \mathcal{H}_a\).
6. Generalization

Slightly modifying (3) and (4), we rewrite the equations of a triad of $A$-, $B$-, $C$-conics as

\[ x^2 + 2Pyz = 0, \quad y^2 + 2Qzx = 0, \quad z^2 + 2Rxy = 0. \]

Apart from the vertices, these conics intersect at

\[
Q_a = \left(-\left(\frac{1}{QR}\right)^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : 2R^{\frac{1}{3}}\right),
\]

\[
Q_b = \left(2P^{\frac{1}{3}} : -\left(\frac{1}{RP}\right)^{\frac{1}{3}} : 2R^{\frac{1}{3}}\right),
\]

\[
Q_c = \left(2P^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : -\left(\frac{1}{PQ}\right)^{\frac{1}{3}}\right),
\]

where, for a real number $x$, $x^{\frac{1}{3}}$ stands for the real cube root of $x$. Clearly, the triangles $Q_aQ_bQ_c$ and $ABC$ are perspective at

\[ Q = \left(P^{\frac{1}{3}} : Q^{\frac{1}{3}} : R^{\frac{1}{3}}\right). \]

Let $XYZ$ be the cevian triangle of $Q$ relative to $ABC$ and $X'$, $Y'$, $Z'$ the intersection of the sidelines with the trilinear polar of $Q$.

**Proposition 6.** For an arbitrary point $S$, the line $X'S$ meets the line $AQ$ at the point $T$, and we define the mapping $h(S) = U$, where $U$ is the harmonic conjugate of $S$ with respect to $X'$ and $T$. The mapping $h$ swaps the conics $B$- and $C$-conics.

Similarly we can define mappings with pivot points $Y'$, $Z'$ swapping the $C$- and $A$-conics, and the $A$- and $B$-conics.

<table>
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<th>intersects at</th>
<th>the point</th>
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<td>$Q_b$</td>
<td>$C_a$</td>
</tr>
</tbody>
</table>

From these data we deduce the following theorem.

**Theorem 7.** The line $Z_bY_c$ is a common tangent of the $B$- and $C$-conics; so is $X_cZ_a$ of the $C$- and $A$-conics, and $Y_aX_b$ of the $A$- and $B$-conics.
Figure 9 shows the case of the triad of rectangular hyperbolas ($\mathcal{H}_a$, $\mathcal{H}_b$, $\mathcal{H}_c$). The perspector $Q$ is the barycentric cube root of $X_{25}$ (the homothetic center of the orthic and tangential triangles). $Q$ does not appear in [3].

References


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