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Conics Tangent at the Vertices to Two Sides of a Triangle

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Abstract. We study conics tangent to two sides of a given triangle at two vertices, and construct two interesting triads of such conics, one consisting of parabolas, the other rectangular hyperbolas. We also construct a new triangle center, the barycentric cube root of X_{25}, which is the homothetic center of the orthic and tangential triangles.

1. Introduction

In the plane of a given triangle ABC, a conic is the locus of points with homogeneous barycentric coordinates $x : y : z$ satisfying a second degree equation of the form

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2hxy = 0, \tag{1}$$

where $f,\ g,\ h,\ p,\ q,\ r$ are real coefficients (see [4]). In matrix form, (1) can be rewritten as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

If we have a circumconic, *i.e.*, a conic that passes through the vertices A, B, C, then from (1), $f = g = h = 0$. Since the equation becomes $pyz + qzx + rxy = 0$, the circumconic is the isogonal (or isotomic) conjugate of a line.

If the conic passes through B and C, we have $g = h = 0$. The conic has equation

$$fx^2 + 2pyz + 2qzx + 2rxy = 0. \tag{2}$$

2. Conic tangent to two sides of ABC

Consider a conic given by (2) which is tangent at B, C to the sides AB, AC. We call this an A-conic. Since the line AB has equation $z = 0$, the conic (2) intersects AB at $(x : y : 0)$ where $x(fx + 2ry) = 0$. The conic and the line are tangent at B only if $r = 0$. Similarly the conic is tangent to AC at C only if $q = 0$. The A-conic has equation

$$fx^2 + 2pyz = 0. \tag{3}$$

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We shall also consider the analogous notions of B- and C-conics. These are respectively conics with equations

$$gy^2 + 2qzx = 0, \quad hz^2 + 2rxy = 0. \tag{4}$$

If triangle ABC is scalene, none of these conics is a circle.

Lemma 1. *Let BC be a chord of a conic with center O, and M the midpoint of BC. If the tangents to the conic at B and C intersect at A, then the points A, M, O are collinear.*

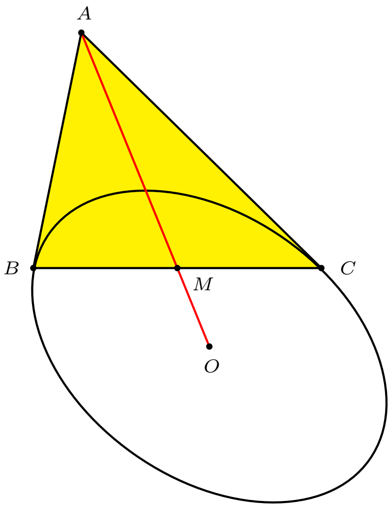

Figure 1.

Proof. The harmonic conjugate S of the point M relative to B, C is the point at infinity of the line BC. The center O is the midpoint of every chord passing through O. Hence, the polar of O is the line at infinity. The polar of A is the line BC (see Figure 1). Hence, the polar of S is a line that must pass through O, M and A. $\square$

From Lemma 1 we conclude that the center of an A-conic lies on the A-median. In the case of a parabola, the axis is parallel to the median of ABC since the center is a point at infinity. In general, the center of the conic (1) is the point

$$\left(\begin{vmatrix} 1 & r & q \\ 1 & g & p \\ 1 & p & h \end{vmatrix} : \begin{vmatrix} f & 1 & q \\ r & 1 & p \\ q & 1 & h \end{vmatrix} : \begin{vmatrix} f & r & 1 \\ r & g & 1 \\ q & p & 1 \end{vmatrix}\right). \tag{5}$$

Hence the centers of the A-, B-, C-conics given in (3) and (4) are the points $(p : f : f)$, $(g : q : g)$, and $h : h : r)$ respectively.

We investigate two interesting triads when the three conics (i) are parabolas, (ii) rectangular hyperbolas, and (iii) all pass through a given point $P = (u : v : w)$, and shall close with a generalization.

3. A triad of parabolas

The A-conic (3) is a parabola if the center $(p : f : f)$ is on the line at infinity. Hence, $p + f + f = 0$, and the A-parabola has equation $x^2 - 4yz = 0$. We label this parabola $\mathscr{P}_a$. Similarly, the B- and C-parabolas are $\mathscr{P}_b :\ y^2 - 4zx = 0$ and $\mathscr{P}_c :\ z^2 - 4xy = 0$ respectively. These are also known as the Artzt parabolas (see Figure 2).

3.1. *Construction.* A conic can be constructed with a dynamic software by locating 5 points on it. The A-parabola $\mathscr{P}_a$ clearly contains the vertices B, C, and the points $M_a = (2 : 1 : 1)$, $P_b = (4 : 1 : 4)$ and $P_c = (4 : 4 : 1)$. Clearly, M_a is the midpoint of the median AA_1, and P_b, P_c are points dividing the medians BB_1 and CC_1 in the ratio $BP_b : P_bB_1 = CP_c : P_cC_1 = 8 : 1$.

Similarly, if M_b, M_c are the midpoints of the medians BB_1 and CC_1, and P_a divides AA_1 in the ratio $AP_a : P_aA_1 = 8 : 1$, then the B-parabola $\mathscr{P}_b$ is the conic containing C, A, M_b, P_c, P_a, and the C-parabola $\mathscr{P}_c$ contains A, B, M_c, P_a, P_b.

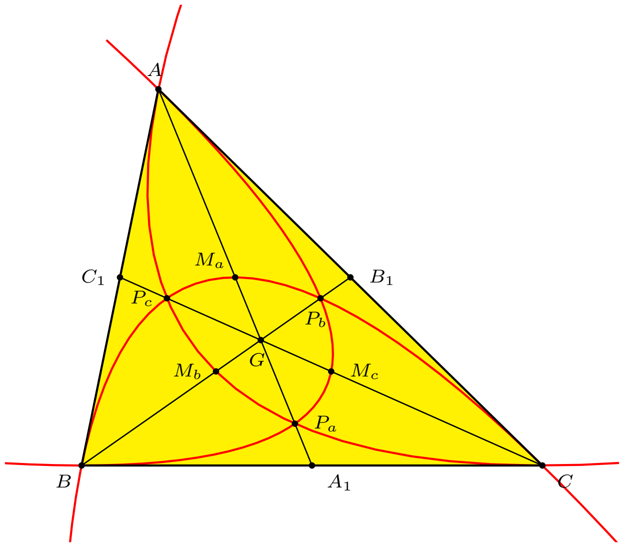

Figure 2.

Remarks. (1) P_a, P_b, P_c are respectively the centroids of triangles GBC, GCA, GAB.

(2) Since A_1 is the midpoint of BC and M_aA_1 is parallel to the axis of $\mathscr{P}_a$, by Archimedes' celebrated quadrature, the area of the parabola triangle M_aBC is $\frac{4}{3} \cdot \triangle M_aBC = \frac{2}{3} \cdot \triangle ABC$.

(3) The region bounded by the three Artzt parabolas has area $\frac{5}{27}$ of triangle ABC.

3.2. *Foci and directrices.* We identify the focus and directrix of the A-parabola $\mathscr{P}_a$. Note that the axis of the parabola is parallel to the median AA_1. Therefore, the parallel through B to the median AA_1 is parallel to the axis, and its reflection in the tangent AB passes through the focus F_a. Similarly the reflection in AC of the parallel through C to the median AA_1 also passes through F_a. Hence the focus F_a is constructible (in the Euclidean sense).

Let D, E be the orthogonal projections of the focus F_a on the sides AB, AC, and D', E' the reflections of F_a with respect to AB, AC. It is known that the line DE is the tangent to the A-parabola $\mathscr{P}_a$ at its vertex and the line $D'E'$ is the directrix of the parabola (see Figure 3).

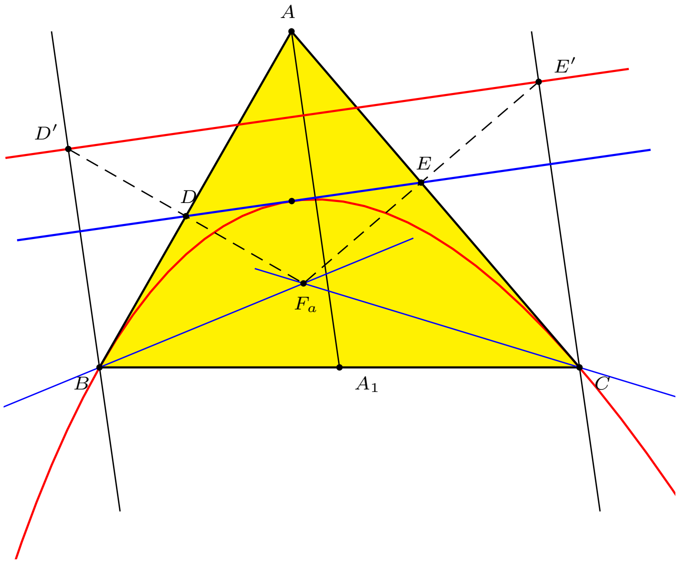

Figure 3.

Theorem 2. *The foci F_a, F_b, F_c of the three parabolas are the vertices of the second Brocard triangle of ABC and hence are lying on the Brocard circle. The triangles ABC and $F_aF_bF_c$ are perspective at the Lemoine point K.*

Proof. With reference to Figure 3, in triangle ADE, the median AA_1 is parallel to the axis of the A-parabola $\mathscr{P}_a$. Hence, AA_1 is an altitude of triangle ADE. The line AF_a is a diameter of the circumcircle of ADE and is the isogonal conjugate to AA_1. Hence the line AF_a is a symmedian of ABC, and it passes through the Lemoine point K of triangle ABC.

Similarly, if F_b and F_c are the foci of the B- and C-parabolas respectively, then the lines BF_b and CF_c also pass through K, and the triangles ABC and $F_aF_bF_c$ are perspective at the Lemoine point K.

Since the reflection of BF_a in AB is parallel to the median AA_1, we have

$$\angle F_aBA = \angle D'BA = \angle BAA_1 = \angle F_aAC, \tag{6}$$

and the circle F_aAB is tangent to AC at A. Similarly,

$$\angle ACF_a = \angle ACE' = \angle A_1AC = \angle BAF_a, \tag{7}$$

and the circle F_aCA is tangent to AB at A. From (6) and (7), we conclude that the triangles F_aAB and F_aCA are similar, so that

$$\angle AF_aB = \angle CF_aA = \pi - A \quad \text{and} \quad \angle BF_aC = 2A. \tag{8}$$

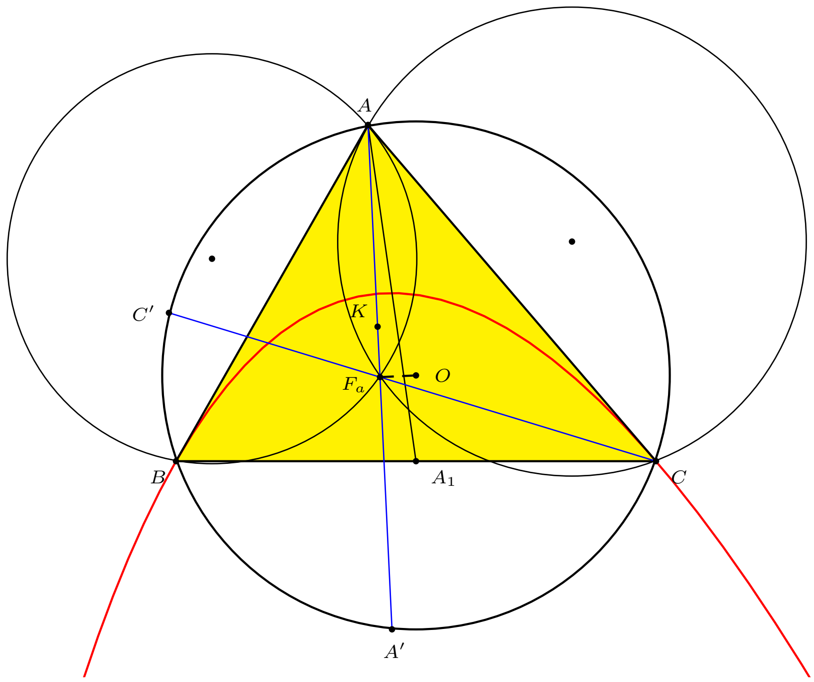

Figure 4.

If O is the circumcenter of ABC and the lines AF_a, CF_a meet the circumcircle again at the points A', C' respectively then from the equality of the arcs AC' and BA', the chords $AB = A'C'$. Since the triangles ABF_a and $A'C'F_a$ are similar to triangle CAF_a, they are congruent. Hence, F_a is the midpoint of AA', and is the orthogonal projection of the circumcenter O on the A-symmedian. As such, it is on the Brocard circle with diameter OK. Likewise, the foci of the B- and C-parabolas are the orthogonal projections of the point O on the B- and C-symmedians respectively. The three foci form the second Brocard triangle of ABC. □

Remarks. (1) Here is an alternative, analytic proof. In homogeneous barycentric coordinates, the equation of a circle is of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(Px + Qy + Rz) = 0,$$

where a, b, c are the lengths of the sides of ABC, and P, Q, R are the powers of A, B, C relative to the circle. The equation of the circle F_aAB tangent to AC at A is

$$a^2yz + b^2zx + c^2xy - b^2(x + y + z)z = 0, \tag{9}$$

and that of the circle F_aCA tangent to AB at A is

$$a^2yz + b^2zx + c^2xy - c^2(x+y+z)y = 0. \tag{10}$$

Solving these equations we find

$$F_a = (b^2 + c^2 - a^2 : b^2 : c^2).$$

It is easy to verify that this lies on the Brocard circle

$$a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2+b^2+c^2}(x+y+z)\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

(2) This computation would be more difficult if the above circles were not tangent at A to the sides AC and AB. So it is interesting to show another method. We can get the same result, as we know directed angles (defined modulo π) from

$$\theta = (F_aB, F_aC) = 2A, \quad \varphi = (F_aC, F_aA) = -A, \quad \psi = (F_aA, F_aB) = -A.$$

Making use of the formula given in [1], we obtain

$$\begin{aligned}
F_a = & \left(\frac{1}{\cot A - \cot\theta} : \frac{1}{\cot B - \cot\varphi} : \frac{1}{\cot C - \cot\psi}\right) \\
= & \left(\frac{1}{\cot A - \cot 2A} : \frac{1}{\cot B + \cot A} : \frac{1}{\cot C + \cot A}\right) \\
= & \left(\frac{\sin A \sin 2A}{\sin A} : \frac{\sin A \sin B}{\sin(A+B)} : \frac{\sin A \sin C}{\sin(A+C)}\right) \\
= & \ (2\cos A \sin B \sin C : \sin^2 B : \sin^2 C) \\
= & \ (b^2 + c^2 - a^2 : b^2 : c^2).
\end{aligned}$$

4. Rectangular A-, B-, C-hyperbolas

The A-conic $fx^2 + 2pyz = 0$ is a rectangular hyperbola if it contains two orthogonal points at infinity $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ where

$$x_1 + y_1 + z_1 = 0 \quad \text{and} \quad x_2 + y_2 + z_2 = 0.$$

and . Putting $x = sy$ and $z = -(x+y)$ we have $fs^2 - 2ps - 2p = 0$ with roots $s_1 + s_2 = \frac{2p}{f}$ and $s_1s_2 = -\frac{2p}{f}$. The two points are orthogonal if

$$S_Ax_1x_2 + S_By_1y_2 + S_Cz_1z_2 = 0.$$

From this, $S_Ax_1x_2 + S_By_1y_2 + S_C(x_1+y_1)(x_2+y_2) = 0$, $S_As_1s_2 + S_B + S_C(s_1+1)(s_2+1) = 0$, and

$$\frac{p}{f} = \frac{S_B + S_C}{2S_A} = \frac{a^2}{b^2+c^2-a^2}.$$

This gives the rectangular A-hyperbola

$$\mathscr{H}_a: \qquad (b^2 + c^2 - a^2)x^2 + 2a^2yz = 0,$$

with center

$$O_a = (a^2 : b^2 + c^2 - a^2 : b^2 + c^2 - a^2).$$

This point is the orthogonal projection of the orthocenter H of ABC on the A-median and lies on the orthocentroidal circle, *i.e.*, the circle with diameter GH. Similarly the centers O_b, O_c lie on the orthocentroidal circle as projections of H on the B-, C-medians. Triangle $O_aO_bO_c$ is similar to the triangle of the medians of ABC (see Figure 5).

The construction of the A-hyperbola can be done since we know five points of it: the points B, C, their reflections B', C' in O_a, and the othocenter of triangle $B'BC$.

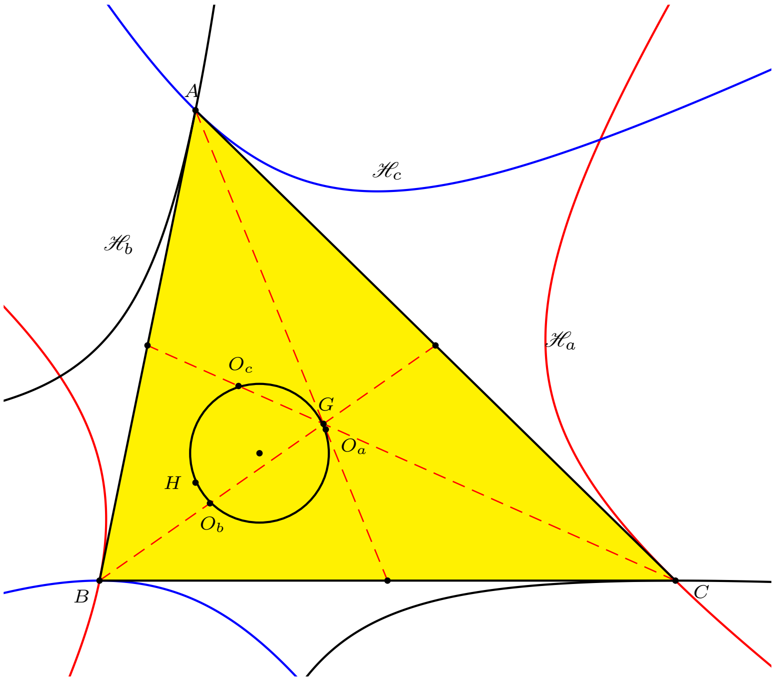

Figure 5.

5. Triad of conics passing through a given point

Let $P = (u : v : w)$ be a given point. We denote the A-conic through P by $(\mathcal{A}_P)$; similarly for $(\mathcal{B}_P)$ and $(\mathcal{C}_P)$. These conics have equations

$$vwx^2 - u^2yz = 0, \quad wuy^2 - v^2zx = 0, \quad uvz^2 - w^2xy = 0.$$

Let XYZ be the cevian triangle of P and $X'Y'Z'$ be the trillinear polar of P (see Figure 6). It is known that the point X' is the intersection of YZ with BC, and is the harmonic conjugate of X relative to the points B, C; similarly for Y' and Z' are the harmonic conjugates of Y and Z relative to C, A and A, B. These points have coordinates

$$X' = (0 : -v : w), \quad Y' = (u : 0 : -w), \quad Z' = (-u : v : 0),$$

and they lie on the trilinear polar

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

Now, the polar of A relative to the conic $(\mathcal{A}_P)$ is the line BC. Hence, the polar of X' passes through A. Since X is harmonic conjugate of X' relative to B, C, the line AX is the polar of X' and the line $X'P$ is tangent to the conic $(\mathcal{A}_P)$ at P.

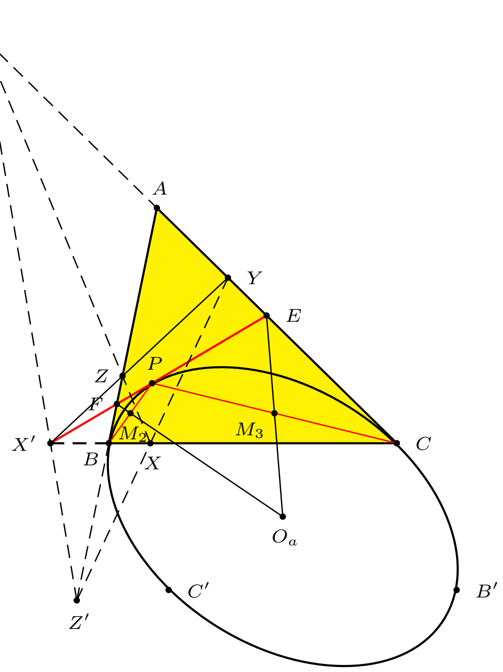

Figure 6.

Remark. The polar of an arbitrary point or the tangent of a conic at the point $P = (u : v : w)$ is the line given by

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

Hence the tangent of $(\mathcal{A}_P)$ at P is the line

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2vw & 0 & 0 \\ 0 & 0 & -u^2 \\ 0 & -u^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0,$$

or $2vwx - wuy - uvz = 0$. This meets BC at the point $X' = (0 : -v : w)$ as shown before.

5.1. *Construction of the conic* $(\mathcal{A}_P)$. We draw the line YZ to meet BC at X'. Then $X'P$ is tangent to the conic at P. Let this meet AC and AB at E and F respectively. Let M_2 and M_3 be the midpoints of PB and PC. Since AB, AC and EF are tangents to $(\mathcal{A}_P)$ at the points B, C, P respectively, the lines EM_3 and FM_2 meet at the center O_a of the conic $(\mathcal{A}_P)$. If B', C' are the symmetric points of B, C relative to O_a, then $(\mathcal{A}_P)$ is the conic passing through the five points B, P, C, B', C'.

The conics $(\mathcal{B}_P)$ and $(\mathcal{C}_P)$ can be constructed in a similar way.

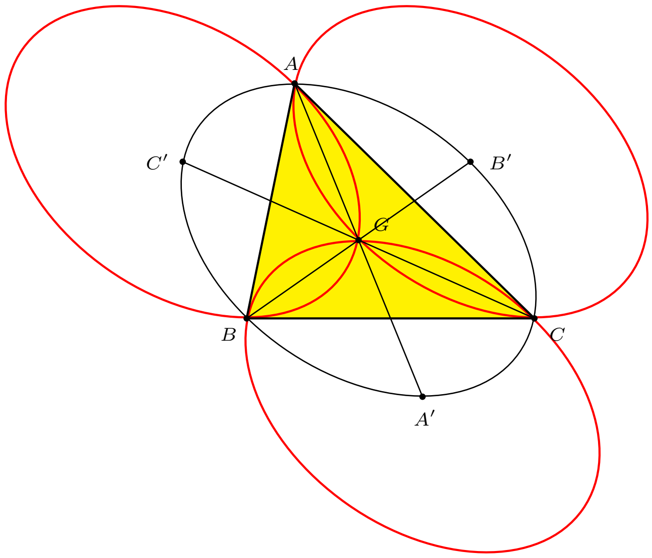

Figure 7.

In the special case when P is the centroid G of triangle ABC, X' is a point at infinity and the line YZ is parallel to BC. The center O_a is the symmetric A' of A relative to G. It is obvious that $(\mathcal{A}_P)$ is the translation of the Steiner circumellipse (with center G), by the vector AG; similarly for $(\mathcal{B}_P)$, $(\mathcal{C}_P)$ (see Figure 7).

Theorem 3. *For an arbitrary point Q the line $X'Q$ intersects the line AP at the point R, and we define the mapping $h(Q) = S$, where S is the harmonic conjugate of Q with respect to X' and R. The mapping h swaps the conics $(\mathcal{B}_P)$ and $(\mathcal{C}_P)$.*

Proof. The mapping h is involutive because $h(Q) = S$ if and only if $h(S) = Q$. Since the line AP is the polar of X' relative to the pair of lines AB, AC we have $h(A) = A$, $h(P) = P$, $h(C) = B$, so that $h(AB) = AC$, and $h(BC) = CB$. Hence, the conic $h((\mathcal{B}_P))$ is the one passing through A, P, B and tangent to the sides AC, BC at A, B respectively. This is clearly the conic $(\mathcal{C}_P)$. □

Therefore, a line passing through X' tangent to $(\mathcal{B}_P)$ is also tangent to $(\mathcal{C}_P)$. This means that the common tangents of the conics $(\mathcal{B}_P)$ and $(\mathcal{C}_P)$ intersect at X'.

Similarly we can define mappings with pivot points Y', Z' swapping $(\mathcal{C}_P)$, $(\mathcal{A}_P)$ and $(\mathcal{A}_P)$, $(\mathcal{B}_P)$.

Consider the line $X'P$ tangent to the conic $(\mathcal{A}_P)$ at P. It has equation

$$-2vwx + wuy + uvz = 0.$$

This line meets the conic $(\mathcal{B}_P)$ at the point $X_b = (u : -2v : 4w)$, and the conic $(\mathcal{C}_P)$ at the point $X_c = (u : 4v : -2w)$.

Similarly, the tangent $Y'P$ of $(\mathcal{B}_P)$ intersects $(\mathcal{C}_P)$ again at $Y_c = (4u : v : -2w)$ and $(\mathcal{A}_P)$ again at $Y_a = (-2u : v : 4w)$. The tangent $Z'P$ of $(\mathcal{C}_P)$ intersects $(\mathcal{A}_P)$ again at $Z_a = (-2u : 4v : w)$ and $(\mathcal{B}_P)$ again at $Z_b = (4u : -2v : w)$ respectively.

Theorem 4. *The line Z_bY_c is a common tangent of $(\mathcal{B}_P)$ and $(\mathcal{C}_P)$, so is X_cZ_a of $(\mathcal{C}_P)$ and $(\mathcal{A}_P)$, and Y_aX_b of $(\mathcal{A}_P)$ and $(\mathcal{B}_P)$.*

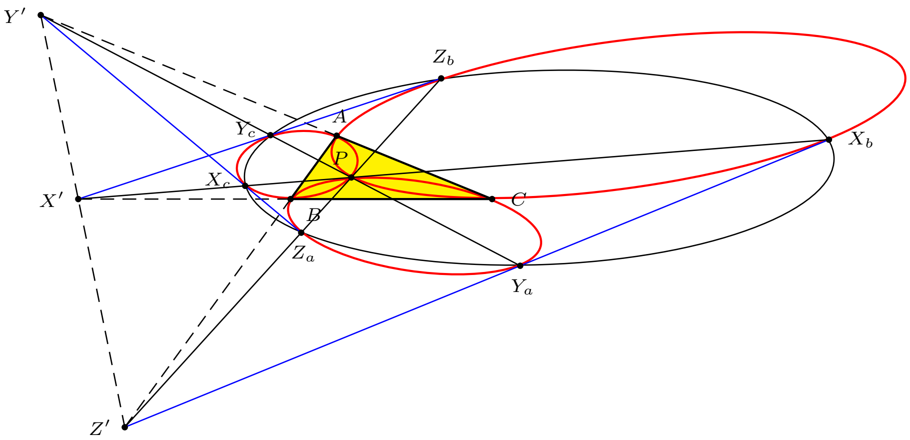

Figure 8.

Proof. We need only prove the case Y_cZ_b. The line has equation

$$vwx + 4wuy + 4uvz = 0.$$

This is tangent to $(\mathcal{B}_P)$ at Z_b and to $(\mathcal{C}_P)$ at Y_c as the following calculation confirms.

$$\begin{pmatrix} 0 & 0 & v^2 \\ 0 & -2wu & 0 \\ v^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4u \\ -2v \\ w \end{pmatrix} = v \begin{pmatrix} vw \\ 4wu \\ 4uv \end{pmatrix},$$
$$\begin{pmatrix} 0 & w^2 & 0 \\ w^2 & 0 & 0 \\ 0 & 0 & -2uv \end{pmatrix} \begin{pmatrix} 4u \\ v \\ -2w \end{pmatrix} = w \begin{pmatrix} vw \\ 4wu \\ 4uv \end{pmatrix}.$$

□

Theorem 5. *The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a conic* (see Figure 7).

Proof. It is easy to verify that the six points satisfy the equation of the conic

$$2v^2w^2x^2 + 2w^2u^2y^2 + 2u^2v^2z^2 + 7u^2vwyz + 7uv^2wzx + 7uvw^2xy = 0, \quad (11)$$

which has center

$$(u(-11u+7v+7w):v(7u-11v+7w):w(7u+7v-11w)).$$

□

Remark. The equation of the conic (11) can be rewritten as

$$u^2(2v^2-7vw+2w^2)yz+v^2(2w^2-7wu+2u^2)zx+w^2(2u^2-7uv+2v^2)xy$$
$$-2(x+y+z)(v^2w^2x+w^2u^2y+u^2v^2z)=0.$$

This is a circle if and only if

$$\frac{u^2(2v^2-7vw+2w^2)}{a^2}=\frac{v^2(2w^2-7wu+2u^2)}{b^2}=\frac{w^2(2u^2-7uv+2v^2)}{c^2}.$$

Equivalently, the isotomic conjugate of P, namely, $\left(\frac{1}{u}:\frac{1}{v}:\frac{1}{w}\right)$ is the intersection of the three conics defined by

$$\frac{2y^2-7yz+2z^2}{a^2}=\frac{2x^2-7zx+2x^2}{b^2}=\frac{2x^2-7xy+2y^2}{c^2}.$$

5.2. *The type of the three conics* $(\mathcal{A}_P)$, $(\mathcal{B}_P)$, $(\mathcal{C}_P)$. The type of a conic with given equation (1) can be determined by the quantity

$$d=\begin{vmatrix}0&1&1&1\\1&f&r&q\\1&r&g&p\\1&q&p&h\end{vmatrix}$$

For the conic $\mathcal{A}_P:\ vwx^2-u^2yz=0$, this is
(i) a parabola if $u^2-4vw=0$ (and $\mathcal{A}_P=\mathscr{P}_a$),
(ii) an ellipse if $u^2-vw<0$, *i.e.*, P lying inside $\mathscr{P}_a$,
(iii) a hyperbola if $u^2-vw>0$, *i.e.*, P lying outside $\mathscr{P}_a$.

Thus the conics in the triad $(\mathcal{A}_P,\ \mathcal{B}_P,\ \mathcal{C}_P)$ are all ellipses if and only if P lies in the interior of the curvilinear triangle $P_aP_bP_c$ bounded by arcs of the Artzt parabolas (see Figure 2).

Remarks. (1) It is impossible for all three conics $(\mathcal{A}_P,\mathcal{B}_P,\mathcal{C}_P)$ to be parabolas.

(2) The various possibilities of conics of different types are given in the table below.

Ellipses	3	2	2	1	1	1		
Parabolas		1		2	1		1	
Hyperbolas			1		1	2	2	3

Note that the conics in the triad $(\mathcal{A}_P,\ \mathcal{B}_P,\ \mathcal{C}_P)$ cannot be all rectangular hyperbolas. This is because, as we have seen in §4 that the only rectangular A-hyperbola is $\mathscr{H}_a$. The conic $\mathcal{A}_P$ is a rectangular hyperbola if and only if P lies on $\mathscr{H}_a$, and in this case, $\mathcal{A}_P=\mathscr{H}_a$.

6. Generalization

Slightly modifying (3) and (4), we rewrite the equations of a triad of A-, B-, C-conics as

$$x^2 + 2Pyz = 0, \qquad y^2 + 2Qzx = 0, \qquad z^2 + 2Rxy = 0.$$

Apart from the vertices, these conics intersect at

$$\begin{aligned} Q_a &= \left(-\left(\frac{1}{QR}\right)^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : 2R^{\frac{1}{3}} \right), \\ Q_b &= \left(2P^{\frac{1}{3}} : -\left(\frac{1}{RP}\right)^{\frac{1}{3}} : 2R^{\frac{1}{3}} \right), \\ Q_c &= \left(2P^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : -\left(\frac{1}{PQ}\right)^{\frac{1}{3}} \right), \end{aligned}$$

where, for a real number x, $x^{\frac{1}{3}}$ stands for the real cube root of x. Clearly, the triangles $Q_aQ_bQ_c$ and ABC are perspective at

$$Q = \left(P^{\frac{1}{3}} : Q^{\frac{1}{3}} : R^{\frac{1}{3}}\right).$$

Let XYZ be the cevian triangle of Q relative to ABC and X', Y', Z' the intersection of the sidelines with the trilinear polar of Q.

Proposition 6. *For an arbitrary point S, the line $X'S$ meets the line AQ at the point T, and we define the mapping $h(S) = U$, where U is the harmonic conjugate of S with respect to X' and T. The mapping h swaps the conics B- and C-conics.*

Similarly we can define mappings with pivot points Y', Z' swapping the C- and A-conics, and the A- and B-conics.

The tangent to	at	intersects	at the point
$\mathcal{C}_a$	Q_b	$\mathcal{C}_c$	$X_c = \left(-(RP^2)^{\frac{1}{3}} : 2 : 2(R^2P)^{\frac{1}{3}}\right)$
$\mathcal{C}_a$	Q_c	$\mathcal{C}_b$	$X_b = \left(-(P^2Q)^{\frac{1}{3}} : 2(PQ^2)^{\frac{1}{3}} : 2\right)$
$\mathcal{C}_b$	Q_c	$\mathcal{C}_a$	$Y_a = \left(2(P^2Q)^{\frac{1}{3}} : -(PQ^2)^{\frac{1}{3}} : 2\right)$
$\mathcal{C}_b$	Q_a	$\mathcal{C}_c$	$Y_c = \left(2 : -(Q^2R)^{\frac{1}{3}} : 2(QR^2)^{\frac{1}{3}}\right)$
$\mathcal{C}_c$	Q_a	$\mathcal{C}_b$	$Z_b = \left(2 : 2(Q^2R)^{\frac{1}{3}} : -(QR^2)^{\frac{1}{3}}\right)$
$\mathcal{C}_c$	Q_b	$\mathcal{C}_a$	$Z_a = \left(2(RP^2)^{\frac{1}{3}} : 2 : -(R^2P)^{\frac{1}{3}}\right)$

From these data we deduce the following theorem.

Theorem 7. *The line Z_bY_c is a common tangent of the B- and C-conics; so is X_cZ_a of the C- and A-conics, and Y_aX_b of the A- and B-conics.*

Figure 9 shows the case of the triad of rectangular hyperbolas $(\mathscr{H}_a,\ \mathscr{H}_b,\ \mathscr{H}_c)$. The perspector Q is the barycentric cube root of X_{25} (the homothetic center of the orthic and tangential triangles). Q does not appear in [3].

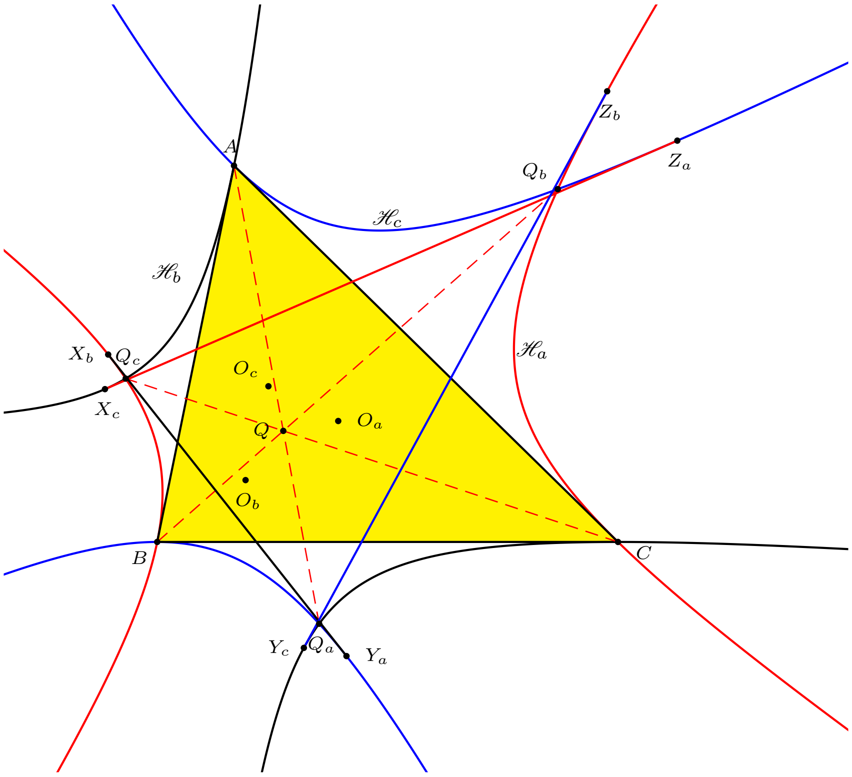

Figure 9.

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