Forum Geometricorum
Volume 10 (2010) 79–91.

FORUM GEOM
ISSN 1534-1178

Orthic Quadrilaterals of a Convex Quadrilateral

Maria Flavia Mammana, Biagio Micale, and Mario Pennisi

Abstract. We introduce the orthic quadrilaterals of a convex quadrilateral, based on the notion of valtitudes. These orthic quadrilaterals have properties analogous to those of the orthic triangle of a triangle.

1. Orthic quadrilaterals

The orthic triangle of a triangle $\mathbf{T}$ is the triangle determined by the feet of the altitudes of $\mathbf{T}$. The orthic triangle has several and interesting properties (see [2, 4]). In particular, it is the triangle of minimal perimeter inscribed in a given acute-angled triangle (Fagnano's problem). It is possible to define an analogous notion for quadrilaterals, that is based on the valtitudes of quadrilaterals [6, p.20]. In this case, though, given any quadrilateral we obtain a family of "orthic quadrilaterals". Precisely, let $A_1A_2A_3A_4$ be a convex quadrilateral, which from now on we will denote by $\mathbf{Q}$. We call v-parallelogram of $\mathbf{Q}$ any parallelogram inscribed in $\mathbf{Q}$ and having the sides parallel to the diagonals of $\mathbf{Q}$. We denote by $\mathbf{V}$ a v-parallelogram of $\mathbf{Q}$ with vertices V_i, $i = 1, 2, 3, 4$, on the side A_iA_{i+1} (with indices taken modulo 4).

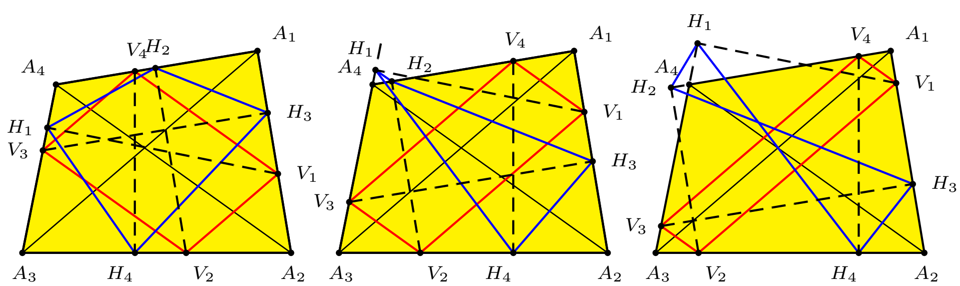

Figure 1.

The v-parallelograms of $\mathbf{Q}$ can be constructed as follows. Fix an arbitrary point V_1 on the segment A_1A_2. Draw from V_1 the parallel to the diagonal A_1A_3 and let V_2 be the intersection point of this line with the side A_2A_3. Draw from V_2 the parallel to the diagonal A_2A_4 and let V_3 be the intersection point of this line with the side A_3A_4. Finally, draw from V_3 the parallel to the diagonal A_1A_3 and let V_4 be the intersection point of this line with the side A_4A_1. The quadrilateral $V_1V_2V_3V_4$ is a v-parallelogram ([6, p.19]). By moving V_1 on the segment A_1A_2, we obtain all possible v-parallelograms of $\mathbf{Q}$. The v-parallelogram $M_1M_2M_3M_4$, with M_i the midpoint of the segment A_iA_{i+1}, is the Varignon's parallelogram of $\mathbf{Q}$.

Publication Date: June 15, 2010. Communicating Editor: Paul Yiu.

Given a v-parallelogram $\mathbf{V}$ of $\mathbf{Q}$, let H_i be the foot of the perpendicular from V_i to the line $A_{i+2}A_{i+3}$. We say that $H_1H_2H_3H_4$ is an orthic quadrilateral of $\mathbf{Q}$, and denote it by $\mathbf{Q}_{\rm o}$. Note that $\mathbf{Q}_{\rm o}$ may be convex, concave or self-crossing (see Figure 1). The lines V_iH_i are called the valtitudes of $\mathbf{Q}$ with respect to $\mathbf{V}$.

The orthic quadrilateral relative to the Varignon's parallelogram ($V_i = M_i$) will be called principal orthic quadrilateral of $\mathbf{Q}$ and will be denoted by $\mathbf{Q}_{\rm po}$. The line M_iH_i is the maltitude of $\mathbf{Q}$ on the side $A_{i+2}A_{i+3}$ (see Figure 2).

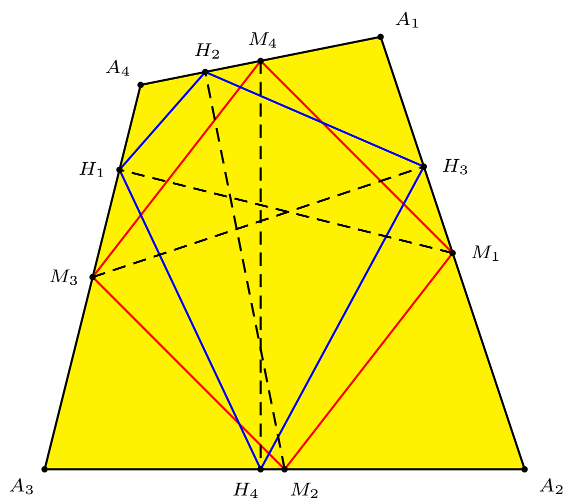

Figure 2.

The study of the orthic quadrilaterals, and in particular of the principal one, allows us to find some properties that are analogous to those of the orthic triangle. In §2 we study the orthic quadrilaterals of an orthogonal quadrilateral, in §3 we consider the case of cyclic and orthodiagonal quadrilaterals. In §4 we find some particular properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral. Finally, in §5 we introduce the notion of orthic axis of an orthodiagonal quadrilateral.

2. Orthic quadrilaterals of an orthodiagonal quadrilateral

We recall that the maltitudes of $\mathbf{Q}$ are concurrent if and only if $\mathbf{Q}$ is cyclic ([6]). If $\mathbf{Q}$ is cyclic, the point H of concurrence of the maltitudes is called anticenter of $\mathbf{Q}$ (see Figure 3). Moreover, if $\mathbf{Q}$ is cyclic and orthodiagonal, the anticenter is the common point to the diagonals of $\mathbf{Q}$ (Brahmagupta's theorem, [2, p.44]). In general, if $\mathbf{Q}$ is cyclic, with circumcenter O and centroid G, then H is the symmetric of O with respect to G, and the line containing the three points H, O and G is called Euler line of $\mathbf{Q}$.

The valtitudes of $\mathbf{Q}$ relative to a v-parallelogram may concur only if $\mathbf{Q}$ is cyclic or orthodiagonal [6]. Precisely, when $\mathbf{Q}$ is cyclic they concur if and only if they are the maltitudes of $\mathbf{Q}$. When $\mathbf{Q}$ is orthodiagonal there exists one and only one v-parallelogram of $\mathbf{Q}$ with concurrent valtitudes. In this case they concur in the

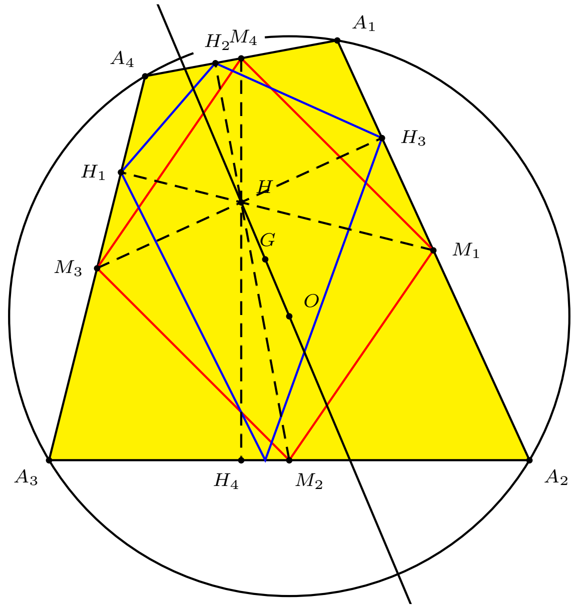

Figure 3.

point D common to the diagonals of $\mathbf{Q}$, and are perpendicular to the sides of $\mathbf{Q}$ through D.

Lemma 1. *If* $\mathbf{Q}$ *is orthodiagonal, the valtitudes* V_iH_i *and* $V_{i+1}H_{i+1}$ $(i = 1, 2, 3, 4)$ *with respect to a v-parallelogram* $\mathbf{V}$ *of* $\mathbf{Q}$ *meet on the diagonal* $A_{i+1}A_{i+3}$ *of* $\mathbf{Q}$.

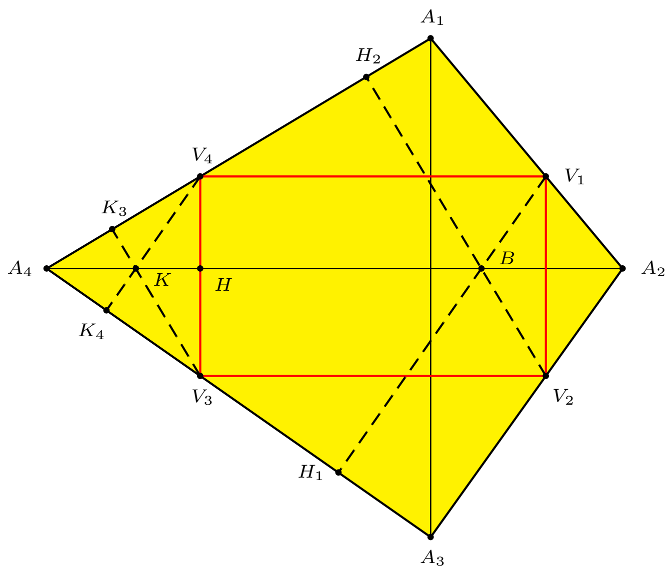

Figure 4.

Proof. Let $\mathbf{Q}$ be orthodiagonal and $\mathbf{V}$ a v-parallelogram of $\mathbf{Q}$. Let us prove that the valtitudes V_1H_1 and V_2H_2 meet on the line A_2A_4 (see Figure 4). The altitudes V_3K_3, V_4K_4, A_4H of triangle $V_3V_4A_4$ concur at a point K on the line A_2A_4. Let B be the common point to V_1H_1 and A_2A_4. We prove that B is on V_2H_2 as well. The quadrilateral V_1BKV_4 is a parallelogram, because its opposite sides are parallel. Thus, BK is equal and parallel to V_1V_4 and to V_2V_3, and the quadrilateral V_2V_3KB is a parallelogram because it has two opposite sides equal and parallel. It follows that V_2B is parallel to V_3K, and B lies on V_2H_2.

Analogously we can proceed for the other pairs of valtitudes. □

Theorem 2. *Let* $\mathbf{Q}$ *be orthodiagonal. Let* $\mathbf{V}$ *be a v-parallelogram of* $\mathbf{Q}$ *and* $\mathbf{Q}_0$ *be the orthic quadrilateral of* $\mathbf{Q}$ *relative to* $\mathbf{V}$. *The vertices of* $\mathbf{V}$ *and those of* $\mathbf{Q}_\mathrm{o}$ *lie on the same circle.*

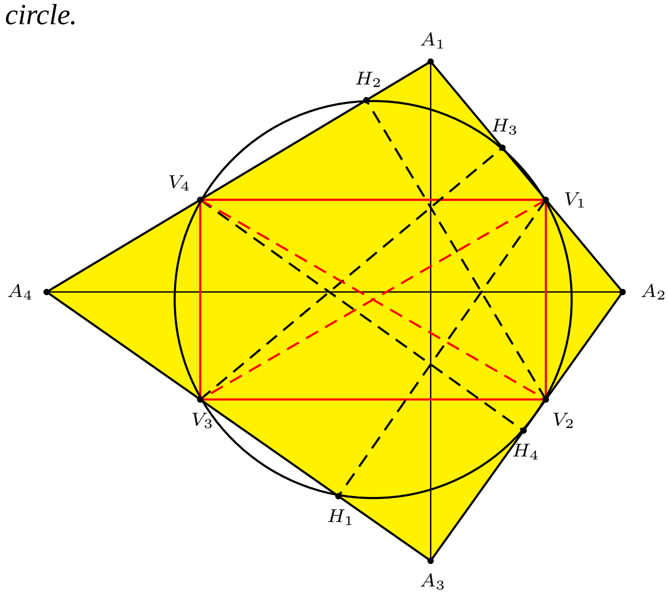

Figure 5.

Proof. In fact, since $\mathbf{Q}$ is orthodiagonal, $\mathbf{V}$ is a rectangle and it is inscribed in the circle $\mathcal{C}$ of diameter $V_1V_3 = V_2V_4$. The vertices of $\mathbf{Q}_\mathrm{o}$ lie on $\mathcal{C}$, because, for example, $\angle V_1H_1V_3$ is a right angle, and H_1 lie on $\mathcal{C}$ (see Figure 5). □

Note that if $\mathbf{V}$ is the Varignon's parallelogram, the center of the circle $\mathcal{C}$ is the centroid G of $\mathbf{Q}$. In this case $\mathcal{C}$ is known as the eight-point circle of $\mathbf{Q}$ (see [1, 3]).

Corollary 3. *If* $\mathbf{Q}$ *is orthodiagonal, then each orthic quadrilateral of* $\mathbf{Q}$, *in particular* $\mathbf{Q}_\mathrm{po}$, *is cyclic.*

3. Orthic quadrilaterals of a cyclic and orthodiagonal quadrilateral

The orthic quadrilaterals of $\mathbf{Q}$ may not be inscribed in $\mathbf{Q}$. In particular, $\mathbf{Q}_\mathrm{po}$ is inscribed in $\mathbf{Q}$ if and only if the angles formed by each side of $\mathbf{Q}$ with the lines joining its endpoints with the midpoint of the opposite side are acute. It follows that if $\mathbf{Q}$ is cyclic and orthodiagonal, then $\mathbf{Q}_\mathrm{po}$ is inscribed in $\mathbf{Q}$.

Theorem 4. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal and* $\mathbf{Q}_\mathrm{o}$ *is an orthic quadrilateral of* $\mathbf{Q}$ *that is inscribed in* Q*, the valtitudes that detect* $\mathbf{Q}_\mathrm{o}$ *are the internal angle bisectors of* $\mathbf{Q}_\mathrm{o}$.

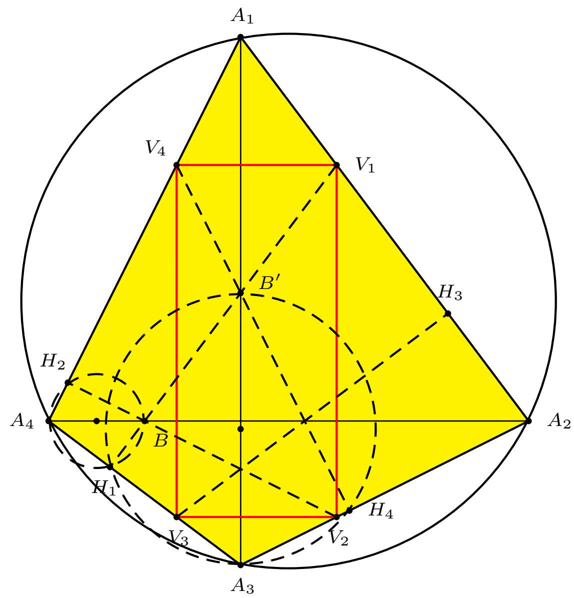

Figure 6.

Proof. We prove that the valtitude V_1H_1 is the bisector of $\angle H_2H_1H_4$ (see Figure 6).

Since $\mathbf{Q}$ is cyclic, we have

$$\angle A_1A_4A_2 = \angle A_1A_3A_2, \tag{1}$$

because they are subtended by the same arc A_1A_2. Let B be the common point to the valtitudes V_1H_1 and V_2H_2 and B' the common point to the valtitudes V_1H_1 and V_4H_4. The quadrilateral $BH_1A_4H_2$ is cyclic because the angles in H_1 and in H_2 are right angles; it follows that

$$\angle H_2H_1B = \angle H_2A_4B, \tag{2}$$

because they are subtended by the same arc H_2B. Analogously the quadrilateral $B'H_1A_3H_4$ is cyclic and

$$\angle B'H_1H_4 = \angle B'A_3H_4. \tag{3}$$

But, for Lemma 1, $\angle H_2A_4B = \angle A_1A_4A_2$ and $\angle B'A_3H_4 = \angle A_1A_3A_2$, then, from (1), (2), (3), it follows that $\angle H_2H_1V_1 = \angle V_1H_1H_4$. □

From Corollary 3 and Theorem 4 applied to the case of maltitudes, we obtain

Corollary 5. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal, then* $\mathbf{Q}_{\text{po}}$ *is bicentric and its centers are the centroid and the anticenter of* $\mathbf{Q}$ (see Figure 7).

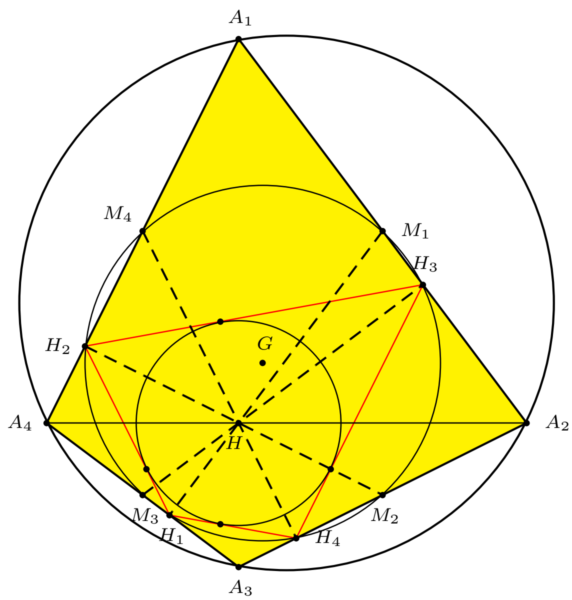

Figure 7.

Theorem 6. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal, the bimedians of* $\mathbf{Q}$ *are the axes of the diagonals of* $\mathbf{Q}_{\text{po}}$.

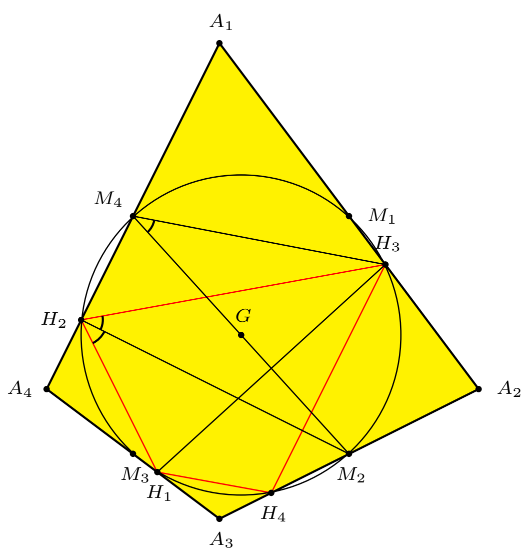

Figure 8.

Proof. It is enough to consider the eight-point circle of $\mathbf{Q}$ and prove that the bimedian M_2M_4 is the axis of the diagonal H_1H_3 of $\mathbf{Q}_{\rm po}$ (see Figure 8). Note that $\angle H_3M_4M_2 = \angle H_3H_2M_2$, because they are subtended by the same arc H_3M_2. Moreover, $\angle H_3H_2M_2 = \angle H_1H_2M_2$, because H_2M_2 bisects $\angle H_1H_2H_3$ (Theorem 4). It follows that $\angle H_3M_4M_2 = \angle H_1H_2M_2$. Then M_2 is the midpoint of the arc H_1H_3 and M_2M_4 is the axis of H_1H_3. □

Note that M_2 and M_4 are the midpoints of the two arcs with endpoints H_1, H_3, and M_1, M_3 are the midpoints of the two arcs with endpoints H_2, H_4.

Theorem 7. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal, the orthic quadrilaterals of* $\mathbf{Q}$ *inscribed in* $\mathbf{Q}$ *have the same perimeter. Moreover, they have the minimum perimeter of any quadrilateral inscribed in* $\mathbf{Q}$*.*

Proof. Let $\mathbf{Q}$ be cyclic and orthodiagonal and let $\mathbf{Q}_{\rm o}$ be any orthic quadrilateral of $\mathbf{Q}$ inscribed in $\mathbf{Q}$ (see Figure 9). Let $\overline{\mathbf{Q}}$ be any quadrilateral inscribed in $\mathbf{Q}$, different from $\mathbf{Q}_{\rm o}$. In the figure, $\mathbf{Q}_{\rm o}$ is the red quadrilateral and $\overline{\mathbf{Q}}$ is the blue quadrilateral.

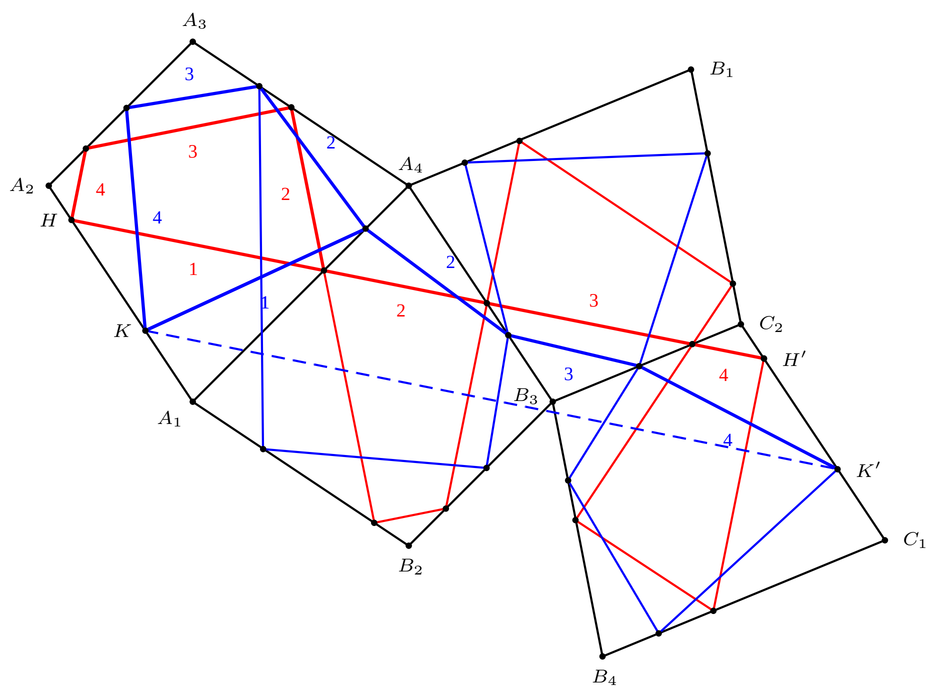

Figure 9.

Let us consider the reflection in the line A_1A_4, that transforms $A_1A_2A_3A_4$ in $A_1B_2B_3A_4$, the reflection in the line B_3A_4, that transforms $A_1B_2B_3A_4$ in $B_1C_2B_3A_4$, and the reflection in the line C_2B_3, that transforms $B_1C_2B_3A_4$ in

$C_1C_2B_3B_4$. Let H and K be the vertices of $\mathbf{Q}_o$ and $\overline{\mathbf{Q}}$ on the segment A_1A_2 respectively, and H' and K' the correspondent points of H and K in the product of the three reflections.

Let us consider the broken line $A_2A_1A_4B_3C_2C_1$. The angles formed by its sides, measured counterclockwise, are $\angle A_1$, $-\angle A_4$, $\angle A_3$, $-\angle A_2$. The sum of these angles is equal to zero, because $\mathbf{Q}$ is cyclic, then the final side C_1C_2 is parallel to A_1A_2. It follows that the segments HH' and KK' are congruent by translation.

For Theorem 4 the valtitudes of $\mathbf{Q}$ relative to $\mathbf{Q}_o$ are the internal angles bisectors of $\mathbf{Q}_o$, then with the three reflections in the lines A_1A_4, B_3A_4 and C_2B_3, the sides of $\mathbf{Q}_o$ will lie on the segment HH', whose length is then equal to the perimeter of $\mathbf{Q}_o$. But, the segment HH' is equal to the segment KK', that has the same endpoints of the broken line formed by the sides of $\overline{\mathbf{Q}}$. It follows that the perimeter of $\overline{\mathbf{Q}}$ is greater than or equal to the one of $\mathbf{Q}_o$, then the theorem is proved. □

4. Properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral

Let $\mathbf{Q}_o$ be an orthic quadrilateral of $\mathbf{Q}$ inscribed in $\mathbf{Q}$. Subtracting from $\mathbf{Q}$ the quadrilateral $\mathbf{Q}_o$ we produce the corner triangles $A_iH_{i+1}H_{i+2}$, $(i = 1, 2, 3, 4)$.

Lemma 8. *Let $\mathbf{Q}$ be cyclic and orthodiagonal and let $\mathbf{Q}_o$ be an orthic quadrilateral of $\mathbf{Q}$ inscribed in $\mathbf{Q}$. The triangle $A_iH_{i+1}H_{i+2}$ $(i = 1, 2, 3, 4)$ is similar to the triangle $A_iA_{i+1}A_{i+3}$.*

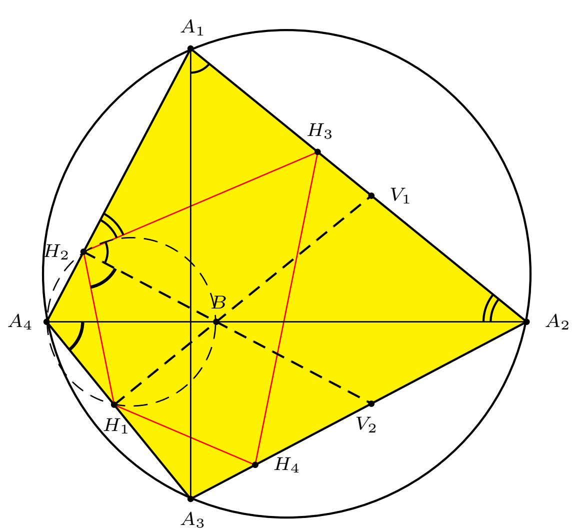

Figure 10.

Proof. Let us prove that the triangles $A_1H_2H_3$ and $A_1A_2A_4$ are similar. Then all we need to prove is that $\angle A_1H_2H_3 = \angle A_1A_2A_4$ (see Figure 10).

Let B be the common point to the valtitudes V_1H_1 and V_2H_2. Since the quadrilateral $A_4H_1BH_2$ is cyclic, it is $\angle BH_2H_1 = \angle BA_4H_1$. Moreover, $\angle BH_2H_3 = \angle BH_2H_1$, because the valtitude V_2H_2 bisects $\angle H_1H_2H_3$. We have $\angle A_3A_1A_2 = \angle A_2A_4A_3$, because $\mathbf{Q}$ is cyclic. Then $\angle A_3A_1A_2 = \angle BH_2H_3$. Since $\angle A_1H_2H_3 = 90° - \angle BH_2H_3$ and $\angle A_1A_2A_4 = 90° - \angle A_3A_1A_2$, because $\mathbf{Q}$ is orthodiagonal, it is $\angle A_1H_2H_3 = \angle A_1A_2A_4$. □

Suppose now that $\mathbf{Q}$ is cyclic and orthodiagonal. Let us find some properties that hold for the principal orthic quadrilateral $\mathbf{Q}_{\mathrm{po}}$, but not for any orthic quadrilateral of $\mathbf{Q}$.

Consider the quadrilateral $\mathbf{Q}'$ whose vertices are the points A'_i in which $\mathbf{Q}_{\mathrm{po}}$ is tangent to its incircle (Corollary 5) and the quadrilateral $\mathbf{Q}_{\mathrm{t}}$ whose sides are tangent to the circumcircle of $\mathbf{Q}$ at its vertices. We say that $\mathbf{Q}_{\mathrm{po}}$ is the tangential quadrilateral of $\mathbf{Q}'$ and $\mathbf{Q}_{\mathrm{t}}$ is the tangential quadrilateral of $\mathbf{Q}$.

Theorem 9. *If $\mathbf{Q}$ is cyclic and orthodiagonal, the quadrilaterals $\mathbf{Q}'$ and $\mathbf{Q}$ and the quadrilaterals $\mathbf{Q}_{\mathrm{po}}$ and $\mathbf{Q}_{\mathrm{t}}$ are correspondent in a homothetic transformation whose center lies on the Euler line of $\mathbf{Q}$.*

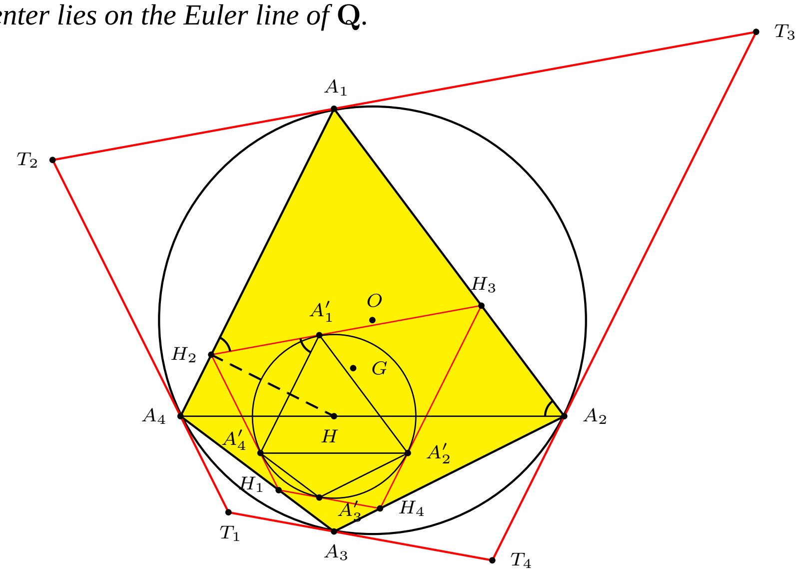

Figure 11.

Proof. It suffices to prove that the quadrilaterals $\mathbf{Q}'$ and $\mathbf{Q}$ are homothetic (see Figure 11).

Let us start proving that the sides of $\mathbf{Q}$ are parallel to the sides of $\mathbf{Q}'$, for example that A_1A_4 is parallel to $A'_1A'_4$. In fact, the maltitude HH_2 is perpendicular to A_1A_4; moreover, it bisects $\angle A'_1H_2H_4$, then it is perpendicular to $A'_1A'_4$ also, thus A_1A_4 and $A'_1A'_4$ are parallel. It follows, in particular, that the angles of $\mathbf{Q}$ are equal to those of $\mathbf{Q}'$, precisely $\angle A_i = \angle A'_i$.

Let us prove now that the sides of $\mathbf{Q}$ are proportional to the sides of $\mathbf{Q}'$. It is $\angle A_1H_2H_3 = \angle H_2A_1'A_4'$, because A_1A_4 and $A_1'A_4'$ are parallel, and $\angle H_2A'D' = \angle A'B'D'$, because they are subtended by the same arc $A'D'$, then $\angle AH_2H_3 = \angle A_1'A_2'A_4'$. It follows that the triangles $A_1H_2H_3$ and $A_1'A_2'A_4'$ are similar. But, for Lemma 8, $A_1H_2H_3$ is similar to $A_1A_2A_4$, then the triangles $A_1A_2A_4$ and $A_1'A_2'A_4'$ are similar. Analogously it is possible to prove that the triangles $A_3A_2A_4$ and $A_3'A_2'A_4'$ are similar. It follows that the sides of $\mathbf{Q}$ are proportional to the sides of $\mathbf{Q}'$. Then it is proved that the quadrilaterals $\mathbf{Q}'$ and $\mathbf{Q}$ are homothetic. Finally, the homothetic transformation that transforms $\mathbf{Q}'$ in $\mathbf{Q}$ transforms the circumcenter H of $\mathbf{Q}'$ in the circumcenter O of $\mathbf{Q}$, then the center P of the homothetic transformation lies on the Euler line of $\mathbf{Q}$. □

It is known that given a circumscriptible quadrilateral and considered the quadrilateral whose vertices are the points of contact of the incircle with the sides, the diagonals of the two quadrilaterals intersect at the same point (see [5] and [7, p.156]). By applying this result to $\mathbf{Q}_{\mathrm{t}}$ and $\mathbf{Q}$ it follows that the diagonals of $\mathbf{Q}_{\mathrm{t}}$ are concurrent in H. Thus the common point to the diagonals of $\mathbf{Q}_{\mathrm{po}}$, N, lies on the Euler line. Moreover, $\mathbf{Q}_{\mathrm{t}}$ is cyclic, because $\mathbf{Q}_{\mathrm{po}}$ is cyclic, and its circumcenter T lies on the Euler line (see Figure 12).

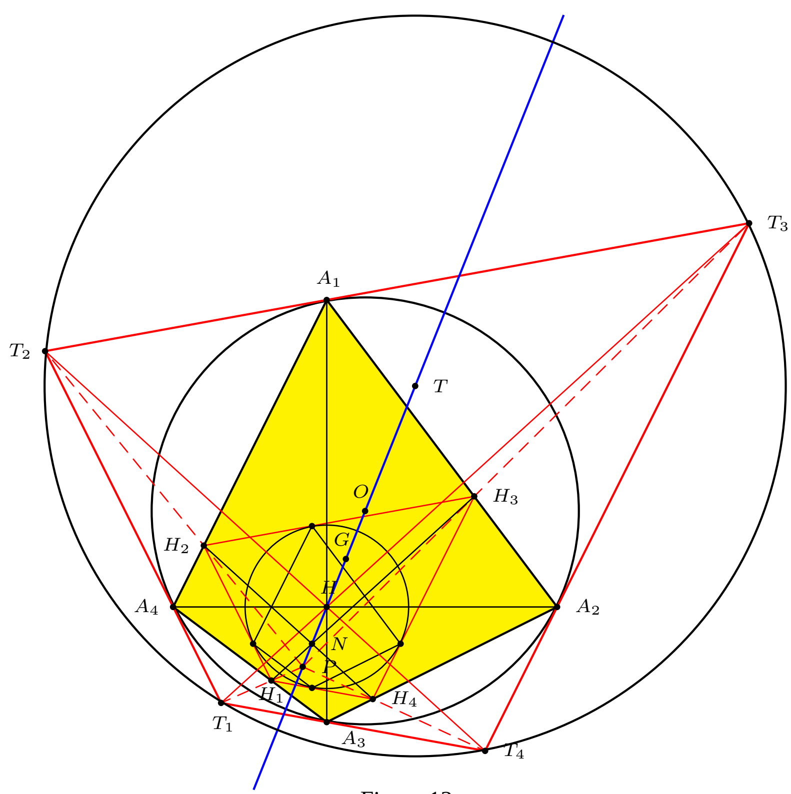

Figure 12.

Theorem 10. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal and* $\mathbf{Q}_{\mathrm{o}}$ *is an orthic quadrilateral of* $\mathbf{Q}$ *inscribed in* $\mathbf{Q}$*, the perimeter of* $\mathbf{Q}_{\mathrm{o}}$ *is twice the ratio between the area of* $\mathbf{Q}$ *and the radius of the circumcircle of* $\mathbf{Q}$*.*

Proof. In fact, from Theorem 7 all orthic quadrilaterals inscribed in $\mathbf{Q}$ have the same perimeter, then it suffices to prove the property for $\mathbf{Q}_{\mathrm{po}}$. The segments H_1H_2 and T_1T_2 are parallel, because $\mathbf{Q}$ and $\mathbf{Q}_{\mathrm{t}}$ are homothetic, then they both are perpendicular to OA_4, radius of the circumcircle of $\mathbf{Q}$ (see Figure 13). It follows that the area of the quadrilateral $OH_1A_4H_2$ is equal to $\frac{1}{2} \cdot OA_4 \cdot H_1H_2$. □

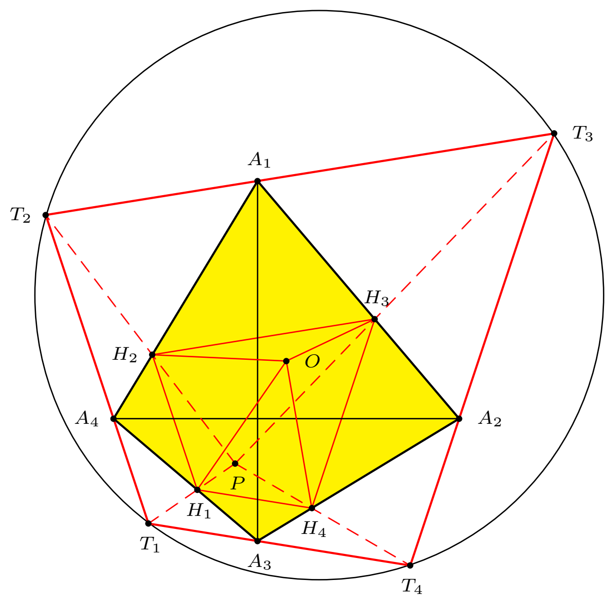

Figure 13.

Conjecture. If $\mathbf{Q}$ is cyclic and orthodiagonal, among all orthic quadrilaterals of $\mathbf{Q}$ inscribed in $\mathbf{Q}$ the one of maximum area is $\mathbf{Q}_{\mathrm{po}}$.

The conjecture, which we have been unable to prove, arises from several proofs that we made by using Cabri Géomètre.

5. Orthic axis of an orthodiagonal quadrilateral

Suppose that $\mathbf{Q}$ is not a parallelogram. If $\mathbf{Q}$ does not have parallel sides, let $\mathcal{R}$ be the straight line joining the common points of the lines containing opposite sides of $\mathbf{Q}$; if $\mathbf{Q}$ is a trapezium, let $\mathcal{R}$ be the line parallel to the basis of $\mathbf{Q}$ and passing through the common point of the lines containing the oblique sides of $\mathbf{Q}$. Let $\mathbf{Q}_{\mathrm{o}}$ be any orthic quadrilateral of $\mathbf{Q}$ and let S_i $(i = 1, 2, 3, 4)$ be the common point of the lines H_iH_{i+1} and V_iV_{i+1}, when these lines intersect (see Figure 14).

Theorem 11. *If* $\mathbf{Q}$ *is orthodiagonal and is not a square, for any orthic quadrilateral* $\mathbf{Q}_{\mathrm{o}}$ *of* $\mathbf{Q}$ *the points* S_1*,* S_2*,* S_3*,* S_4 *lie on a line* $\mathcal{R}$*.*

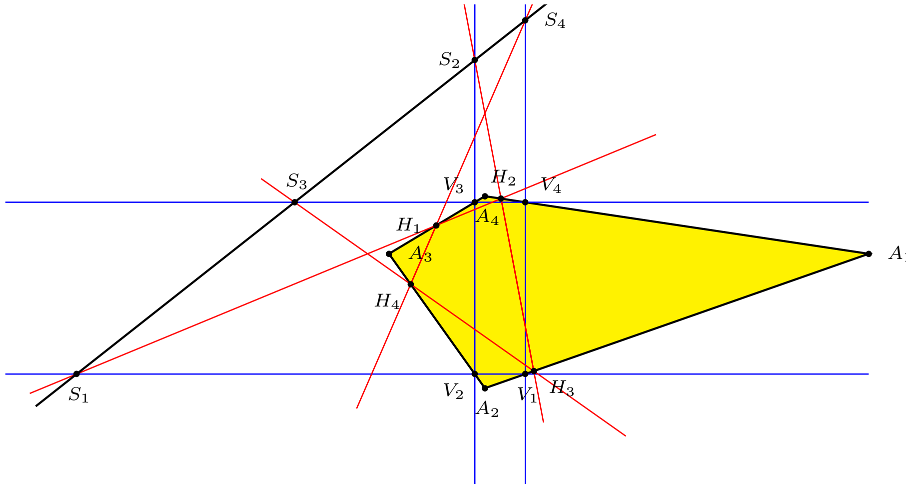

Figure 14.

Proof. Set up an orthogonal coordinate system whose axes are the diagonals of **Q**; then the vertices of **Q** have coordinates $A_1 = (a_1, 0)$, $A_2 = (0, a_2)$, $A_3 = (a_3, 0)$, $A_4 = (0, a_4)$. The equation of line $\mathcal{R}$ is

$$a_2a_4(a_1 + a_3)x + a_1a_3(a_2 + a_4)y - 2a_1a_2a_3a_4 = 0. \tag{4}$$

If **V** is a v-parallelogram of **Q**, with x-coordinate α for the vertex V_1, then

$$S_1 = \left(\frac{a_3(a_2(\alpha - a_1) + a_4(\alpha + a_1))}{a_4(a_1 + a_3)}, \frac{a_2(a_1 - \alpha)}{a_1}\right),$$
$$S_2 = \left(\frac{\alpha a_3}{a_1}, \frac{a_2a_4(2a_1^2 - \alpha(a_1 + a_3))}{a_1^2(a_2 + a_4)}\right),$$
$$S_3 = \left(\frac{a_3(a_2(\alpha + a_1) + a_4(\alpha - a_1))}{a_2(a_1 + a_3)}, \frac{a_4(a_1 - \alpha)}{a_1}\right),$$
$$S_4 = \left(\alpha, \frac{a_2a_4(2a_1a_3 - \alpha(a_1 + a_3))}{a_1a_3(a_2 + a_4)}\right).$$

It is not hard to verify that the coordinates of the points S_i satisfy (4). □

We call the line $\mathcal{R}$ the orthic axis of **Q**. It is possible to verify that if **Q** is cyclic and orthodiagonal, *i.e.*, $a_1a_3 = a_2a_4$, the orthic axis of **Q** is perpendicular to the Euler line of **Q** (see Figure 15). Moreover, it is known that in a cyclic quadrilateral **Q** without parallel sides the tangent lines to the circumcircle of **Q** in two opposite vertices meet on the line joining the common points of the lines containing the opposite sides of **Q** (see [2, p. 76]). It follows that if **Q** is cyclic and orthodiagonal and it has not parallel sides, the common points to the lines tangent to the circumcircle of **Q** in the opposite vertices of **Q** lie on the orthic axis of **Q**.

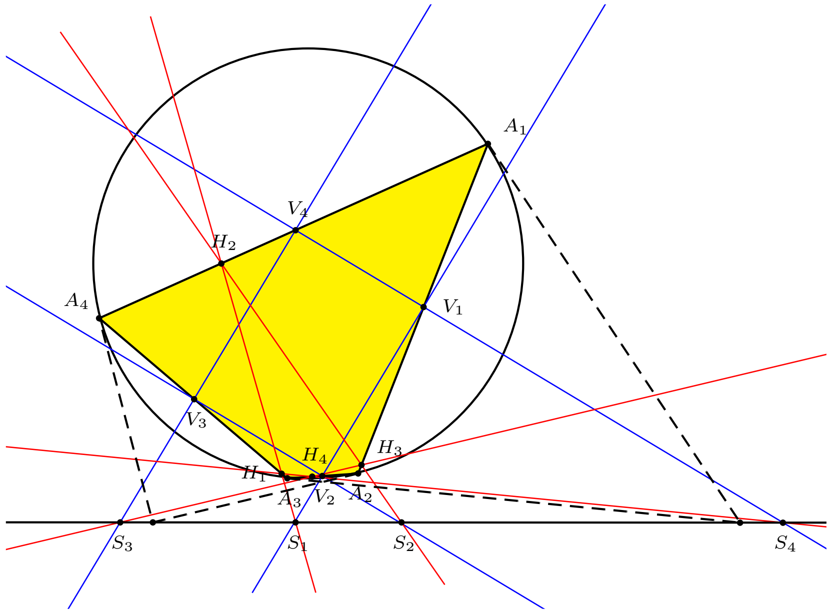

Figure 15.

References

[1] L. Brand, The eight-point circle and the nine-point circle, *Amer. Math. Monthly*, 51, (1944).

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, Math. Assoc. America, 1967.

[3] R. Honsberger, *Mathematical Gems II*, Math. Assoc. America, 1976.

[4] R. Honsberger, *Episodes in nineteenth and twentieth century Euclidean geometry*, Math. Assoc. America, 1995.

[5] J. Konhauser, Solution to Problem 199 (proposed by H. G. Dworschak), *Crux Math.*, 3 (1977) 113–114.

[6] B. Micale and M. Pennisi, On the altitudes of quadrilaterals, *Int. J. Math. Educ. Sci. Technol.*, 36 (2005) 817–828.

[7] P. Yiu, Notes on Euclidean Geometry, Florida Atlantic University Lecture Notes, 1998; available at `http://www.math.fau.edu/Yiu/EuclideanGeometryNotes.pdf`.

Maria Flavia Mammana: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125, Catania, Italy

E-mail address: `fmammana@dmi.unict.it`

Biagio Micale: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125, Catania, Italy

E-mail address: `micale@dmi.unict.it`

Mario Pennisi: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125, Catania, Italy

E-mail address: `pennisi@dmi.unict.it`