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Three Maximal Cyclic Quadrangles

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Abstract. We study three problems concerning the determination of the maximal cyclic quadrangle under certain restrictions on its diagonals: (a) assuming that one of the diagonals as well as the intersection point of the two diagonals are fixed, (b) assuming that the location of the intersection point of the two diagonals is fixed, and (c) assuming that the intersection point of the diagonals is fixed and the diagonals intersect under a constant angle.

1. Introduction

Consider a convex cyclic quadrangle $ABCD$ and assume that one of its diagonals AC is a diameter of its circumcircle. Assume also that the intersection point E of the two diagonals is a fixed point and that the second diagonal BD turns about that point. For which position of the turning diagonal does the resulting quadrangle have maximum area? Figure 1 displays such a case of maximal quadrangle indicating that the solution is asymmetric, whereas changing the position of E on AC towards the center we do obtain symmetric maximal quadrangles (kites).

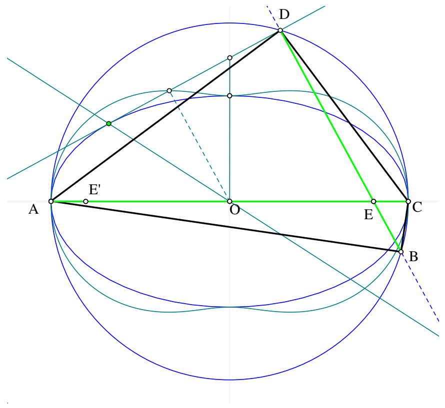

Figure 1. Maximal for fixed AC and E

Figure 1 also displays a nice quartic curve which controls this behavior. §2 below explores the geometric structure of the problem and establishes its connection to the aforementioned quartic. §3 solves the problem of finding the maximal cyclic quadrangles in the two cases:
(a) one diagonal AC and the intersection E of the diagonals are held fixed, and
(b) only the intersection E is held fixed.

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§4 handles the problem of finding the maximal cyclic quadrangle under the condition that its diagonals pass through a given point and intersect there at a fixed angle.

2. The Hippopede of Proclus

The basic configuration in this section is suggested by Figure 2. In this a chord AC turns about a fixed point E lying inside the circle c. The segment OG drawn from the center of the circle parallel and equal to this chord has its other end-point G moving on a curve called the *Hippopede of Proclus* [7, p. 88], [8, p. 35], [3] or the *Lemniscate of Booth* [2, vol.II, p. 163], [5, p.127].

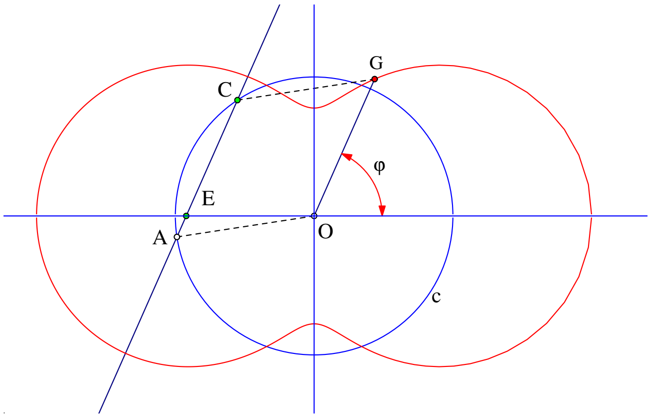

Figure 2. Hippopede of Proclus

This curve is a *quartic* described by the equation

$$(x^2 + y^2)^2 - 4R^2(x^2 + (1 - s^2)y^2) = 0.$$

Here $s \in (0, 1)$ defines the distance $OE = sR$ in terms of the radius R of the circle. A compendium of the various aspects of this interesting curve can be seen in [3]. Regarding the maximum quadrangle the following fact results immediately.

Proposition 1. *Let $ABCD$ be a cyclic quadrangle, E be the intersection point of its diagonals and $k(E)$ the corresponding hippopede. Let the positions of the diagonal AC and E be fixed and consider the other diagonal BD turning about E. Further, let G be the corresponding point on the hippopede, such that OG is parallel and equal and equally oriented to BD. Then the quadrangle attains its maximum area when the tangent t_G at G is parallel to the fixed diagonal AC and at maximum distance from the center O* (see Figure 3).

The proof follows trivially from the formula $|AC| \cdot |BD| \sin\theta$ expressing the area of the quadrangle in terms of the diagonals and their angle. Since AC is fixed the problem reduces to maximizing the product $|BD| \sin\theta$, which is the length of the projection of OG on the axis orthogonal to AC. The claim follows at once.

In general the construction of the point G cannot be effected through elementary operations by ruler and compasses. This is because the position of G, which

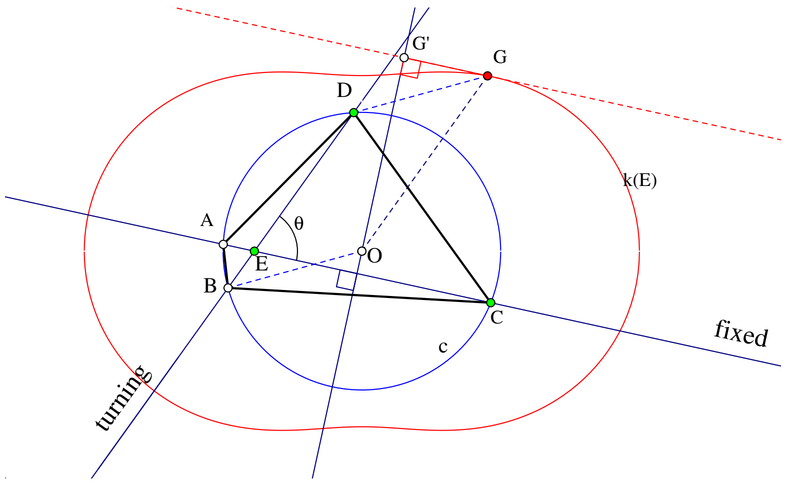

Figure 3. Maximal quadrangle for fixed AC and E

determines the *width* of the hippopede in a certain direction, amounts to a cubic equation. In fact, it can be easily seen that the parameterization of the curve in polar coordinates is given by:

$$(x,y) = 2R\sqrt{1-s^2\sin^2\phi}\cdot(\cos\phi,\sin\phi).$$

Then, the inner product $\langle\nabla f,\overline{e}_\alpha\rangle$ of the gradient of the function defining the curve with the direction $\overline{e}_\alpha = (\cos\alpha,\sin\alpha)$ of the fixed diagonal AC defining the width of the curve in the direction $\alpha+\frac{\pi}{2}$ is:

$$\begin{aligned}
\frac{1}{4}\langle\nabla f,\overline{e}_\alpha\rangle &= ((x^2+y^2)-2R^2)x\cos\alpha + ((x^2+y^2)-2R^2(1-s^2))y\sin\alpha \\
&= 4R^3\sqrt{1-s^2\sin^2\phi}\left(\cos^2\phi+(1-2s^2)\sin^2\phi\right)\cos\phi\cos\alpha \\
&\quad + 4R^3\sqrt{1-s^2\sin^2\phi}\left((1+s^2)\cos^2\phi+(1-s^2)\sin^2\phi\right)\sin\phi\sin\alpha \\
&= 0.
\end{aligned}$$

Assuming $x\neq 0$, *i.e.*, $\phi\neq\frac{\pi}{2}$, and also $\alpha\neq\frac{\pi}{2}$, simplifying, dividing by $\cos^3\phi\cos\alpha$, and setting $z=\tan\phi$, $a=\tan\alpha$, we get the cubic

$$a(1-s^2)z^3+(1-2s^2)z^2+a(1+s^2)z+1=0. \tag{1}$$

This, as expected, controls the position of the variable diagonal BD in dependence of the position (s) of E and the direction α of the fixed diagonal AC. In dependence of these parameters the previous cubic can have one, two or three real solutions. In fact, the cubic depends on the form of the hippopede which in turn depends only on s. It is easily seen that for $s\leq\frac{1}{\sqrt{2}}$ the hippopede is convex, whereas for $s>\frac{1}{\sqrt{2}}$ the curve is non-convex and has two real horizontal bitangents. In the latter case, depending on the orientation of the line AC (*i.e.*, on α), there may be one, two, or three tangents of the hippopede parallel to the fixed chord AC and the maximal quadrangle is determined by the one lying at maximum distance from the origin O (see Figure 4).

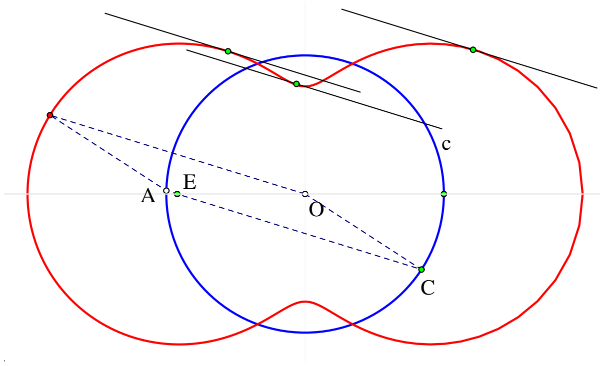

Figure 4. Meaning of roots of the cubic

This is also the reason (noticed in the introduction) for the diversity of behavior of the maximal quadrangle in the case AC is a diameter. In fact, for $a = 0$, (1) reduces to the quadratic

$$(1 - 2s^2)z^2 + 1 = 0,$$

which has real solutions $z = \pm 1/\sqrt{2s^2 - 1} \neq 0$ only in the case $s > \frac{1}{\sqrt{2}}$. The two corresponding solutions define an asymmetric quadrangle (see Figure 1) and its reflection on the x-axis. In the case $s \leq \frac{1}{\sqrt{2}}$ it is easily seen that only $\phi = \frac{\pi}{2}$ (*i.e.*, $x = 0$) is possible, resulting in symmetric maximal quadrangles (kites; see Figure 5).

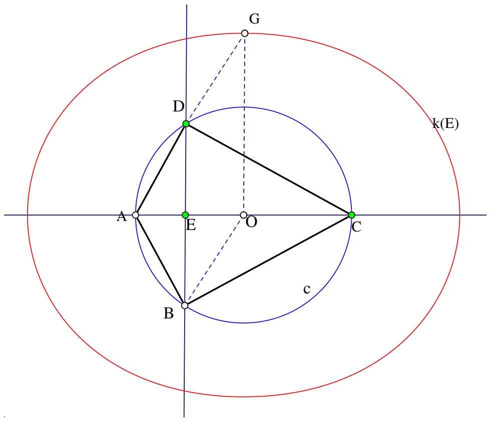

Figure 5. Kites for $s \leq \frac{1}{\sqrt{2}}$

3. The set of maximal quadrangles

Proposition 2 summarizes some well known facts about the various ways to generate the hippopede, as pedal of the ellipse with respect to its center (as does point I in Figure 6) [1, p. 148], but mainly, for us, as envelope of circles passing through the center of an ellipse with eccentricity s (with axes $(R, \sqrt{1-s^2}R)$) and having their centers at points of this ellipse, called the *deferent* of the hippopede [3].

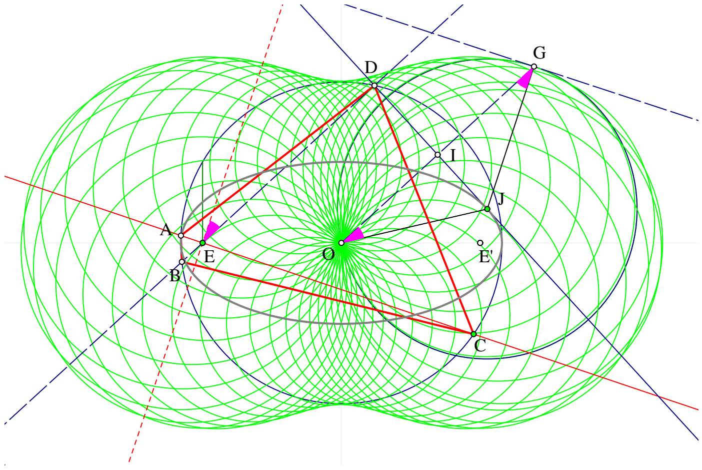

Figure 6. Hippopede as envelope

Proposition 2. *Let $c(O, R)$ be a circle of radius R centered at the origin and $E : |OE| = sR, s \in (0, 1)$ be a fixed point inside the circle.*
(1) *The ellipse $e(s)$ centered at O having one focus at E and eccentricity equal to s has c as its auxiliary circle and its pedal with respect to the center O is the homothetic with ratio $\frac{1}{2}$ of the hippopede.*
(2) *The hippopede is the envelope of all circles passing through O and having their centers on the ellipse $e(s)$.*
(3) *For each point J on the ellipse the circle $c(J, |OJ|)$ is tangent to the hippopede at a point G which is the reflection of O on the tangent t_J to the ellipse at J. The tangent t_G of the hippopede at G is the reflection of the tangent t_O of $c(O, |OJ|)$ with respect to t_J.*

Using these facts and the symmetry of the figure we can eliminate the difficulties of selecting the furthest to O tangent of the hippopede and show that, by restricting the locations of the diagonal AC to the absolutely necessary needed to cover all possible quadrangles, the other diagonal BD of the corresponding maximal quadrangle depends continuously and in an invertible correspondence from AC. This among other things guarantees also the *existence* of maximal quadrangles under the restrictions we are considering. Denote by $P\alpha = ABCD$ the quadrangle with maximum area among all quadrangles inscribed in c with diagonals intersecting at E and having the diagonal AC fixed in the direction $\overline{e}_\alpha = (\cos\alpha, \sin\alpha)$. To obtain all such quadrangles it suffices to consider directions $\overline{e}_\alpha$ of the fixed diagonal AC which are restricted to one quadrant of the circle, for instance by selecting $\alpha \in \left[-\frac{\pi}{2}, 0\right]$. Then the polar angle ϕ of the corresponding point G on the hippopede, defining the other diagonal BD can be considered to lie in an interval $[0, \Phi]$ with $\Phi \le \frac{\pi}{2}$ (see Figure 7). With these conventions it is trivial to prove the following.

Proposition 3. *The correspondence $p : \alpha \mapsto \phi$ mapping each $\alpha \in \left[-\frac{\pi}{2}, 0\right]$ to the polar angle $\phi \in [0, \Phi]$ such that the tangent t_G of the hippopede at its point $G\phi$ is parallel to $\overline{e}_\phi$ and at maximum distance from O is a differentiable and invertible one.*

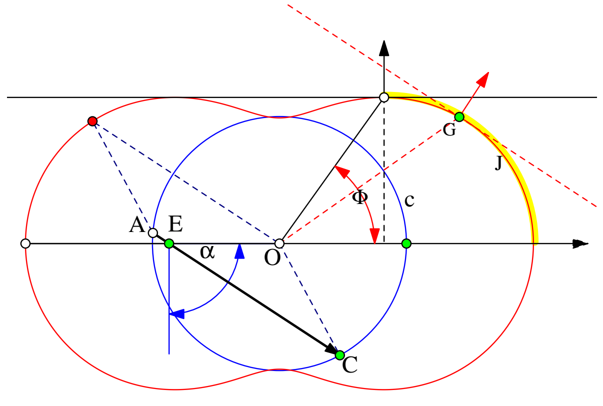

Figure 7. Domain Φ

This is obvious from the geometry of the figure. The analytic proof follows from the angle relations of the triangle OJG in Figure 6. In fact, if $(x, y) = (a\cos\theta, b\sin\theta)$ is a parametrization of the ellipse, then the direction of BD is that of the normal of the ellipse at (x, y), which is $\phi = \arccos\left(\frac{x/a^2}{\sqrt{x^2/a^4 + y^2/b^4}}\right)$. Setting $x = a\cos\theta = R\cos\theta$ and $y = b\sin\theta = R\sqrt{1-s^2}\sin\theta$, we obtain the function $\phi = g(\theta) = \arccos\left(\frac{1}{\sqrt{1+\frac{1}{1-s^2}\tan^2\theta}}\right)$, increasing for $\theta \in \left[0, \frac{\pi}{2}\right]$ (see Figure 8).

On the other hand, by the aforementioned relations, we obtain $\alpha = h(\theta) = 2\phi - \theta + \frac{\pi}{2}$ with its first root at Θ, such that $\alpha = h(\Theta) = 2\Phi - \Theta + \frac{\pi}{2} = 0$. It follows easily that in $[0, \Theta]$ the function $\alpha = h(\theta)$ is also increasing. The relation between the two angles is given by $\phi = p(\alpha) = g(h^{-1}(\alpha))$, which proves the claim.

In the following we restrict ourselves to quadrangles $P\alpha$ with $\alpha \in \left[0, \frac{\pi}{2}\right]$ and denote the set of all these maximal quadrangles by $\mathcal{P}$.

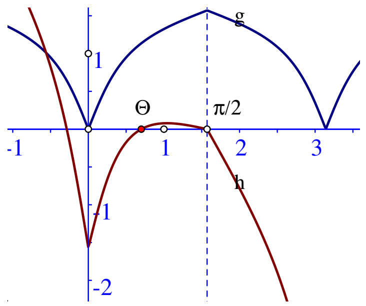

Figure 8. Composing the function $\phi(\alpha)$

Proposition 4. *For each point E there is a unique quadrangle $P\alpha \in \mathcal{P}$ having the diagonals symmetric with respect to the diameter OE hence equal, so that $P\alpha$ in this case is an isosceles trapezium. This quadrangle is also the maximal one in area among all quadrangles whose diagonals pass through E.*

In fact, by the previous discussion we see that the diametral position of AC does not deliver equal diagonals, except in the trivial case for $s = 0$ defining the square inscribed in the circle. Thus, we can assume $\phi \neq \frac{\pi}{2}$ and the proof follows by observing the role of $a = \tan\alpha$ in the cubic equation. The symmetry condition implies $z = \tan\phi = -a$, hence the biquadratic equation in a:

$$(1-s^2)a^4 + 4s^2a^2 - 1 = 0.$$

The only acceptable solution for a^2 producing two symmetric solutions, hence a unique quadrangle (see Figure 9), is

$$a^2 = \frac{1}{1-s^2}\left(\sqrt{4s^4 - s^2 + 1} - 2s^2\right).$$

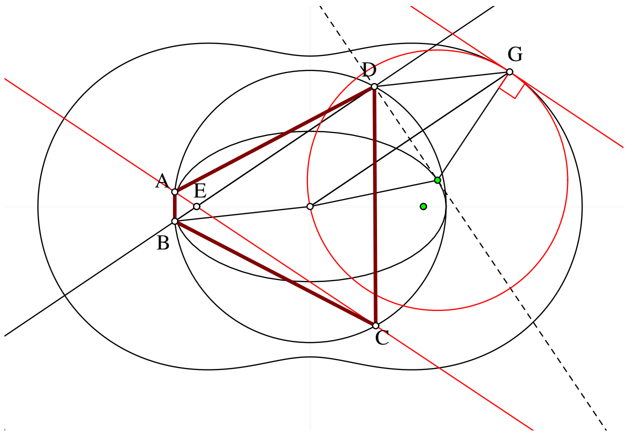

Figure 9. Maximal for diagonals through E

The last claim results again from the necessary cubic equation holding for AC as well as for BD in the case of the maximal quadrangle among all quadrangles with both diagonals varying but also passing through E. Taking the symmetric quadrangle with respect to the diameter EO we may assume that the cubic equation holds true two times with the roles of α and ϕ reversed. Subtracting the resulting cubic equations we obtain

$$\begin{aligned}&\big(a(1-s^2)z^3+(1-2s^2)z^2\big)-\big(z(1-s^2)a^3+(1-2s^2)a^2\big)\\ =\ &(1-s^2)az(z^2-a^2)+(1-2s^2)(z^2-a^2)\\ =\ &\big((1-s^2)az+(1-2s^2)\big)(z-a)(z+a)=0.\end{aligned}$$

It is readily seen that only $z+a=0$ is possible, and this proves the last claim.

4. Diagonals making a fixed angle

Here we turn to the problem of finding the quadrangle with maximum area among all quadrangles $ABCD$ inscribed in a circle $c(O,R)$ whose diagonals pass through the fixed point E and intersect there at a fixed angle α. By the discussion in §2 we know that the area of the quadrilateral in this case can be expressed by the product:

$$A=4R^2\sqrt{1-s^2\sin^2\phi}\sqrt{1-s^2\sin^2(\phi+\alpha)}\cdot\sin\alpha.$$

Essentially this amounts to maximizing the product

$$(1-s^2\sin^2\phi)(1-s^2\sin^2(\phi+\alpha)).$$

Using the identity $\sin^2\phi=\frac{1-\cos 2\phi}{2}$, we rewrite this product as

$$\left((1-\frac{s^2}{2})+\frac{s^2}{2}x\cos\alpha\right)^2-\left(\frac{s^4}{4}(1-x^2)\sin^2\alpha\right),$$

with $x=\cos(2\phi+\alpha)$. A further short calculation leads to the quadratic polynomial

$$\begin{aligned}&\frac{s^4}{4}\cdot x^2+\left(s^2(1-\frac{s^2}{2})\cos\alpha\right)x+\left(\frac{s^4}{4}\cos^2\alpha+1-s^2\right)\\ =\ &(1-s^2)+\frac{s^2}{2}\left(\frac{s^2}{2}x^2+(1-\frac{s^2}{2})x\cos\alpha+\frac{s^2}{2}\cos^2\alpha\right)\end{aligned}$$

The quadratic in x is easily seen to maximize among all values $|x|\le 1$ for $x=1$. Since $x=\cos(2\phi+\alpha)$, this corresponds to $2\phi+\alpha=2k\pi$ *i.e.*, $\phi=-\frac{\alpha}{2}+k\pi$. This establishes the following result.

Proposition 5. *The maximal in area quadrangle inscribed in a circle $c(O,R)$ and having its diagonals passing through a fixed point E and intersecting there under a constant angle α is the equilateral trapezium whose diagonals are inclined to the diameter OE by an angle of $\pm\frac{\alpha}{2}$.*

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