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Iterates of Brocardian Points and Lines

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Abstract. We establish some interesting results on Brocardians in relation to the Steiner ellipses of a triangle.

1. Notations

Let ABC be a triangle with vertices A, B, C and sidelines a, b, c. For the representation of points and lines we use barycentric coordinates, and write

$P = (u : v : w)$ for a point,
$p = [u : v : w] := ux + vy + wz = 0$ for a line.

The point $P = (u : v : w)$ and the line $p = [u : v : w]$ are said to be dual, and we write $P = \star p$ and $p = \star P$. The trilinear polar (or simply tripolar) of a point $(u : v : w)$ is the line $\left[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right]$. The trilinear pole (or simply tripole) of a line $[u : v : w]$ is the point $\left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right)$. The conjugate [1] of a point $P = (u : v : w)$ is the point $P^\bullet = \left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right)$, and the conjugate of a line $p = [u : v : w]$ is the line $p^\bullet = \left[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right]$.

2. Brocardians of a point

Let P be a point (not on the sidelines of ABC and different from the centroid G) with cevian traces P_a, P_b, P_c. The parallels of b through P_a, c through P_b, and a through P_c intersect the sidelines c, a, b respectively in the point P_{ab}, P_{bc}, P_{ca}. These points are the traces of a point $P_\rightarrow$, called the forward (or right) Brocardian of P. Similarly, the parallels of c through P_a, a through P_b, and b through P_c intersect the sidelines b, c, a respectively in the point P_{ac}, P_{ba}, P_{cb}, the traces of a point $P_\leftarrow$, the backward (or left) Brocardian of P (see Figure 1). In barycentric coordinates,

$$P_\rightarrow = \left(\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right), \qquad P_\leftarrow = \left(\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right).$$

We say that a point $P = (u_1u_2 : v_1v_2 : w_1w_2)$ is the barycentric product of the points $P_1 = (u_1 : v_1 : w_1)$ and $P_2 = (u_2 : v_2 : w_2)$. A point P is obviously the barycentric product of its two Brocardians.

For $P = K$, the symmedian point, the Brocardian points are the Brocard points of the reference triangle ABC.

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[1] In this paper, the term “conjugate” always means isotomic conjugate.

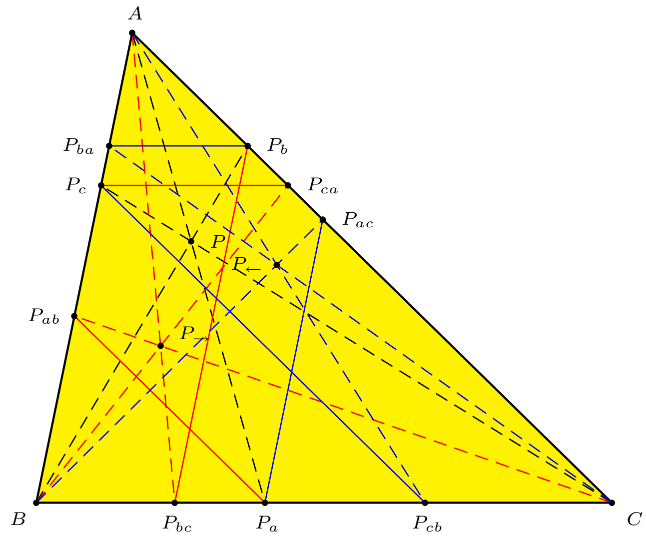

Figure 1.

3. Brocardians of a line

Let p be a line not parallel to the sidelines, intersecting a, b, c respectively at X_a, X_b, X_c. The parallels of c through X_a, a through X_b, and b through X_c intersect b, c, a respectively at Y_b, Y_c, Y_a. These three points are collinear in a line $p_{\rightarrow}$ which we call the right Brocardian line of p. Likewise, the parallels of b through X_a, c through X_b, and a through X_c intersect c, a, b respectively at Z_c, Z_a, Z_b. These points are on a line $p_{\leftarrow}$, the left Brocardian line of p (see Figure 2).

If $p = [u : v : w]$, then the representation of its Brocardian lines (p-Brocardians)

$$p_{\rightarrow} = \left[\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right] \quad \text{and} \quad p_{\leftarrow} = \left[\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right]$$

can be derived by an easy calculation.

Here is an interesting connection between Brocardian points, Brocardian lines, and their trilinear elements.

Proposition 1. (a) *The tripolar of the right* (*left*) *Brocardian point of P is the right* (*left*) *Brocardian line of the tripolar of P.*

(b) *The tripole of the right* (*left*) *Brocardian line of p is the right* (*left*) *Brocardian point of the tripole of p.*

Proof. $\star P_{\rightarrow} = p_{\rightarrow}$ and $\star p_{\leftarrow} = P_{\leftarrow}$. □

We say that the line $p = [u_1u_2 : v_1v_2 : w_1w_2]$ is the barycentric product of the lines $p_1 = [u_1 : v_1 : w_1]$ and $p_2 = [u_2 : v_2 : w_2]$. Hence, a line is the barycentric product of its Brocardian lines.

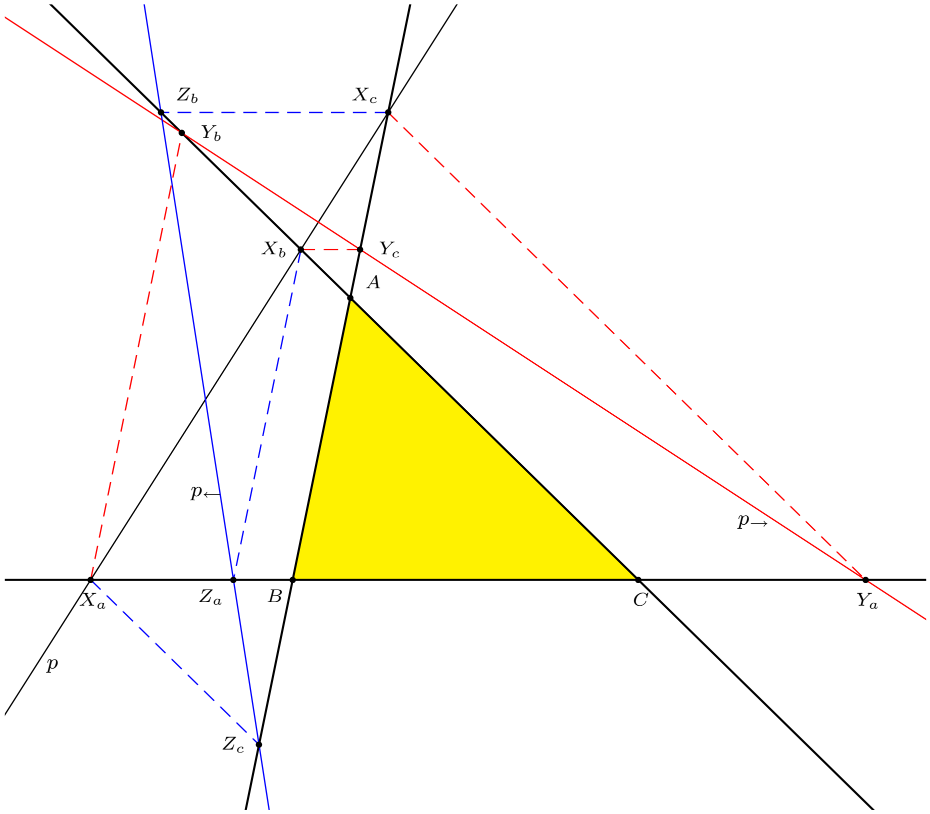

Figure 2.

4. Iterates of Brocardian points

Now we examine on what happens when we repeat the Brocardian operations. First of all,

$$P_{\rightarrow\rightarrow} = (v : w : u), \quad P_{\rightarrow\leftarrow} = P_{\leftarrow\rightarrow} = P, \quad P_{\leftarrow\leftarrow} = (w : u : v).$$

With respect to (isotomic) conjugation,

$$(P^{\bullet})_{\rightarrow} = (P_{\rightarrow})^{\bullet} = P_{\leftarrow\leftarrow}, \qquad (P^{\bullet})_{\leftarrow} = (P_{\leftarrow})^{\bullet} = P_{\rightarrow\rightarrow}.$$

More generally, for a positive integer, let $P_{n\rightarrow}$ denote the n-th iterate of $P_{\rightarrow}$ and $P_{n\leftarrow}$ the n-th iterate of $P_{\leftarrow}$. These operations form a cycle of period 6 (see Figure 3). The "neighbors" of each point are its Brocardians, and the "antipode" its conjugate, *i.e.*,

$$P_{3\rightarrow} = P_{3\leftarrow} = P^{\bullet}.$$

For example, consider the case of the symmedian point $P = K = X_6$ (in Kimberling's notation [1, 2]). The conjugate of X_6 is the third Brocard point X_{76}, the Brocardians of X_6 are the bicentric pair $P(1)$, $U(1)$, and the Brocardians of X_{76} are the bicentric pair $P(11)$, $U(11)$, which are also the conjugates of the Brocard points.

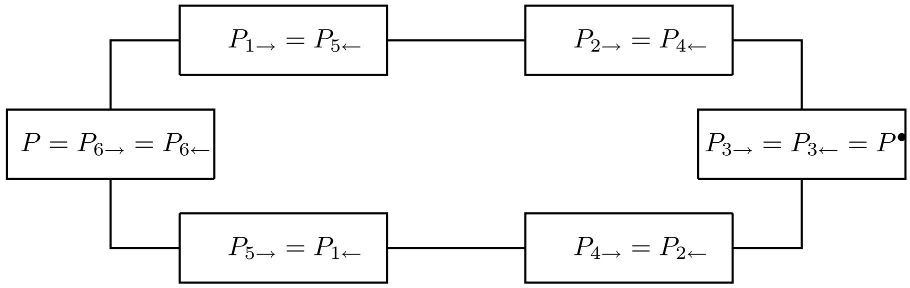

Figure 3.

The 6-cycle of Brocardians can be divided into two 3-cycles by selecting alternate points. We call $(P,\ P_{2\rightarrow},\ P_{2\leftarrow})$ the P-Brocardian triple, and $(P^{\bullet},\ P_{\leftarrow},\ P_{\rightarrow})$ the conjugate P-Brocardian triple (or simply the $P^{\bullet}$-Brocardian triple). For each point in such a triple, the remaining two points are the Brocardians of its conjugate.

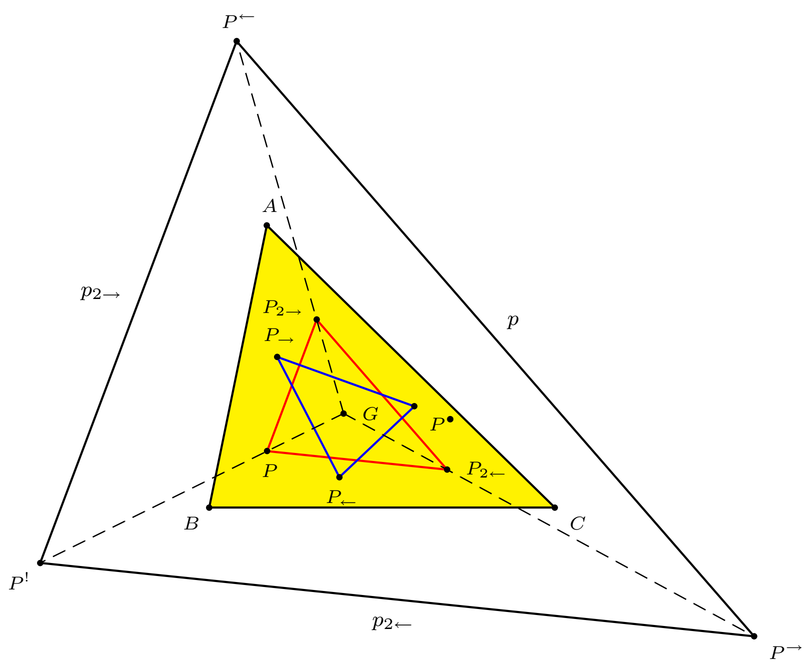

Figure 4.

5. Iterates of Brocardian lines

Analogous to the Brocardian operation for points one can iterate this process for lines. The results of two and three operations are the following:

$$p_{2\rightarrow} = [v : w : u], \quad p_{\rightarrow\leftarrow} = p_{\leftarrow\rightarrow} = p, \quad p_{2\leftarrow} = [w : u : v],$$
$$p_{3\rightarrow} = \ p_{3\leftarrow} = \left[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right] = p^{\bullet}.$$

The lines $p_{2\rightarrow}$ and $p_{2\leftarrow}$ are the tripolars of $P_{\leftarrow}$ and $P_{\rightarrow}$ (and the duals of $P_{2\rightarrow}$ and $P_{2\leftarrow}$) respectively. The line $p_{3\rightarrow}$ is the tripolar of P and the barycentric product of $p_{2\rightarrow}$ and $p_{2\leftarrow}$. It is obvious that the iterates of a Brocardian operation for lines have the same structure as those of a Brocardian operation for points. This is not surprising because the iterates of Brocardian lines are the duals of the iterates of Brocardian points.

We call the line triple $\{p,\ p_{2\rightarrow},\ p_{2\leftarrow})$ a p-Brocardian triple. It is the dual of the P-Brocardian triple and has the same centroid as the reference triangle. The triangles formed by the P-Brocardian triple and p-triple are homothetic at G, *i.e.*, they are similar and their corresponding sides are parallel. (The vertices of the p-triple in Figure 4 are defined in the next section.

6. Brocardian points on a line

There are geometric causes to complete the above structure of six points and their duals. For instance, it is desirable to solve following problem: Given a line $p = [u : v : w]$ and its dual P, does there exist a point X with Brocardians $X_{\rightarrow}$ and $X_{\leftarrow}$ lying on p? How can such a point be constructed?

Consider the lines $p_{2\rightarrow}$ and $p_{2\leftarrow}$. They generate three new points:

$$P^{!} := \ p_{2\rightarrow} \cap p_{2\leftarrow} = (u^2 - vw : v^2 - wu : w^2 - uv),$$
$$P^{\rightarrow} := \ p \cap p_{2\rightarrow} = (w^2 - uv : u^2 - vw : v^2 - wu),$$
$$P^{\leftarrow} := \ p \cap p_{2\leftarrow} = (v^2 - wu : w^2 - uv : u^2 - vw).$$

The point $P^{!}$ is also called the Steiner inverse of P (see [3]). It is interesting that $P^{\rightarrow}$ and $P^{\leftarrow}$ are the Steiner inverses of $P_{2\leftarrow}$ and $P_{2\rightarrow}$ respectively. The point with its Brocardians $P^{\rightarrow}$ and $P^{\leftarrow}$ is the conjugate of $P^{!}$:

$$P^{!\bullet} = \left(\frac{1}{u^2 - vw} : \frac{1}{v^2 - wu} : \frac{1}{w^2 - uv}\right).$$

The line containing $P_{2\rightarrow}$ and $P_{2\leftarrow}$ has tripole $P^{!\bullet}$. For $P = K$, the symmedian point, we have $P^{!} = X_{385}$ and $P^{!\bullet} = X_{1916}$.

7. Brocardian lines through a point

Given a point P not on the sidelines and different from the centroid G, are there two lines through P which are the Brocardian lines of a third line? This is easy to answer by making use of duality. The Brocardian lines are the duals of the points $P^{\rightarrow}$ and $P^{\leftarrow}$, and the third line is the dual of the point $P^{!\bullet}$.

8. Generation of further Brocardian triples

There are many possibilities to create new Brocardian triples from a given one. We consider a few of these.

8.1. The midpoints of each pair in a P-Brocardian triple form a new Brocardian triple with coordinates

$$(v+w:w+u:u+v), \quad (w+u:u+v:v+w), \quad (u+v:v+w:w+u).$$

8.2. The P-Brocardian and $P^{\bullet}$-Brocardian triples have an interesting property: they are triply perspective:

$PP^{\bullet}$, $P_{2\rightarrow}P_{\rightarrow}$, $P_{2\leftarrow}P_{\leftarrow}$ are concurrent in $P_1 := \left(\frac{u^2-vw}{u} : \frac{v^2-wu}{v} : \frac{w^2-uv}{w}\right)$,

$PP_{\rightarrow}$, $P_{2\rightarrow}P_{\leftarrow}$, $P_{2\leftarrow}P^{\bullet}$ are concurrent in $P_2 := \left(\frac{v^2-wu}{v} : \frac{w^2-uv}{w} : \frac{u^2-vw}{u}\right)$,

$PP_{\leftarrow}$, $P_{2\rightarrow}P^{\bullet}$, $P_{2\leftarrow}P_{\rightarrow}$ are concurrent in $P_3 := \left(\frac{w^2-uv}{w} : \frac{u^2-vw}{u} : \frac{v^2-wu}{v}\right)$.

These intersections form a Brocardian triple.

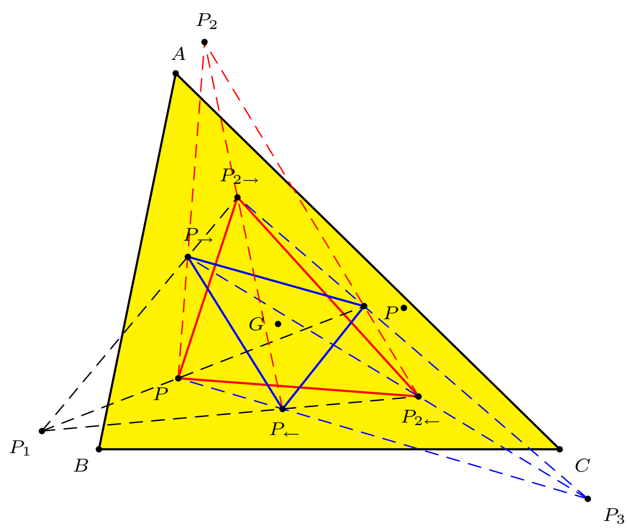

Figure 5.

8.3. Given a P-Brocardian triple and its dual, the lines p, $p_{2\rightarrow}$, $p_{2\leftarrow}$ of that dual intersect the infinite line at the points of another Brocardian triple:

$$\begin{aligned} Q^{\bullet} &:= (v-w:w-u:u-v), \\ Q^{\bullet}_{2\rightarrow} &= (w-u:u-v:v-w), \\ Q^{\bullet}_{2\leftarrow} &= (u-v:v-w:w-u). \end{aligned}$$

Likewise, their conjugates

$$Q = \left(\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right),$$
$$Q_{2\rightarrow} = \left(\frac{1}{w-u} : \frac{1}{u-v} : \frac{1}{v-w}\right),$$
$$Q_{2\leftarrow} = \left(\frac{1}{u-v} : \frac{1}{v-w} : \frac{1}{w-u}\right)$$

form a Brocardian triple with points lying on the Steiner circumellipse $yz + zx + xy = 0$. The connecting line of $Q_{2\rightarrow}$ and $Q_{2\leftarrow}$ is the dual of Q:

$$\star Q = \left[\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right]$$

and is tangent to the Steiner inellipse $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$ at

$$R := ((v-w)^2 : (w-u)^2 : (u-v)^2).$$

The point Q is the fourth intersection of the Steiner circumellipse with the P-circumconic. The midpoints of $QQ_{2\rightarrow}$ and $QQ_{2\leftarrow}$ are the points

$$R_{2\rightarrow} := ((u-v)^2 : (v-w)^2 : (w-u)^2),$$
$$R_{2\leftarrow} := ((w-u)^2 : (u-v)^2 : (v-w)^2)$$

lying on the Steiner inellipse, which, together with R, form a Brocardian triple.

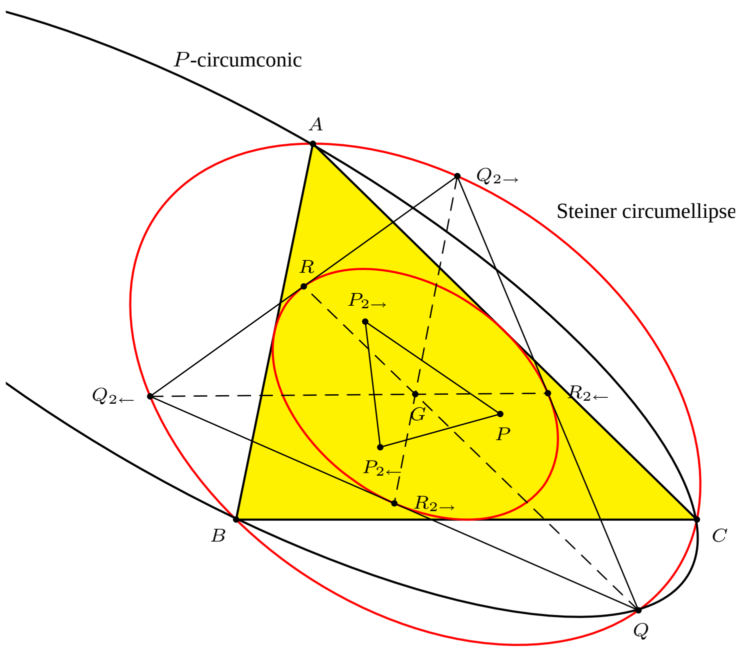

Figure 6.

9. Brocardians on a circumconic

Let $\mathscr{Q}_P$ be the circumconic $uyz + vzx + wxy = 0$. The triangle formed by the tangents of $\mathscr{Q}_P$ at the vertices is perspective with ABC at $P = (u : v : w)$. We call $\mathscr{Q}_P$ the P-circumconic of triangle ABC.

A natural question is the following: Are there a pair of points $X_\rightarrow$ and $X_\leftarrow$ on a given conic $\mathscr{Q}$ which are the Brocardians of a point X?

Let us begin with the special case of the Steiner circumellipse $\mathscr{Q} = \mathscr{Q}_G$: $xy + yz + zx = 0$ with perspector $G = (1 : 1 : 1)$. If $X = (x : y : z)$ is a point on $\mathscr{Q}_G$, its conjugate $X^\bullet$ is an infinite point and has Brocardians

$$X^\bullet_\rightarrow = (z : x : y), \qquad X^\bullet_\leftarrow = (y : z : x),$$

which also lie on $\mathscr{Q}_G$. These three points on the Steiner ellipse obviously form a Brocardian triple. The Brocardians of a point Y lie on the Steiner ellipse if and only if Y is an infinite point.

We can answer the above question for the Steiner circumellipse as follows: the set of points X constitutes the infinite line.

Now, for the case of $\mathscr{Q} = \mathscr{Q}_P$ with $P \neq G$, consider a variable point $X = (x : y : z)$ on $\mathscr{Q}_P$. It is easy to see that the Brocardians $X_\rightarrow$ and $X_\leftarrow$ lie on the tripolars of $P_\rightarrow$ and $P_\leftarrow$ respectively. The intersection of these lines is the Steiner inverse $P^!$ of P. Hence there must be a pair of points X_1 and X_2 on $\mathscr{Q}$ such that $(X_2)_\rightarrow = (X_1)_\leftarrow = P^!$. Then we have

$$P^!_\rightarrow = (X_1)_{\leftarrow\rightarrow} = X_1 \qquad P^!_\leftarrow = (X_2)_{\rightarrow\leftarrow} = X_2$$

with coordinates

$$P^!_\rightarrow = \left(\frac{1}{w^2 - uv} : \frac{1}{u^2 - vw} : \frac{1}{v^2 - wu}\right),$$
$$P^!_\leftarrow = \left(\frac{1}{v^2 - wu} : \frac{1}{w^2 - uv} : \frac{1}{u^2 - vw}\right).$$

Since the P-circumconic is the point-by-point conjugate of p, it is clear that these points are the conjugates of $P^\leftarrow$ and $P^\rightarrow$.

Now we want to list some possibilities to construct the points $P^!_\rightarrow$ and $P^!_\leftarrow$:

(1) Construct the Brocardians of $P^!$.
(2) Construct the conjugates of $P^\rightarrow$ and $P^\leftarrow$.
(3) Construct the tripoles of the lines $PP_{2\rightarrow}$ and $PP_{2\leftarrow}$.
(4) Reflect the Brocardians $P_\rightarrow$ and $P_\leftarrow$ in $R_{2\leftarrow}$ and $R_{2\rightarrow}$ respectively.
(5) The tripoles of the lines $PP_\rightarrow$ and $PP_\leftarrow$ are the points

$$Z_1 = \left(\frac{u}{u-w} : \frac{v}{v-u} : \frac{w}{w-v}\right) \quad \text{and} \quad Z_2 = \left(\frac{u}{u-v} : \frac{v}{v-w} : \frac{w}{w-u}\right).$$

These are the barycentric products of P and $Q_{2\rightarrow}$ and $Q_{2\leftarrow}$ respectively. They lie on the P-circumconic and are the intersections of the lines $QQ_{2\leftarrow}$ with $GP^!_\rightarrow$ and $QQ_{2\rightarrow}$ with $GP^!_\leftarrow$ respectively. The line Z_1Z_2 is tangent to the Steiner inellipse. Construct the intersections of the P-circumconic with the lines GZ_1 and GZ_2 to obtain the points $P^!_\rightarrow$ and $P^!_\leftarrow$.

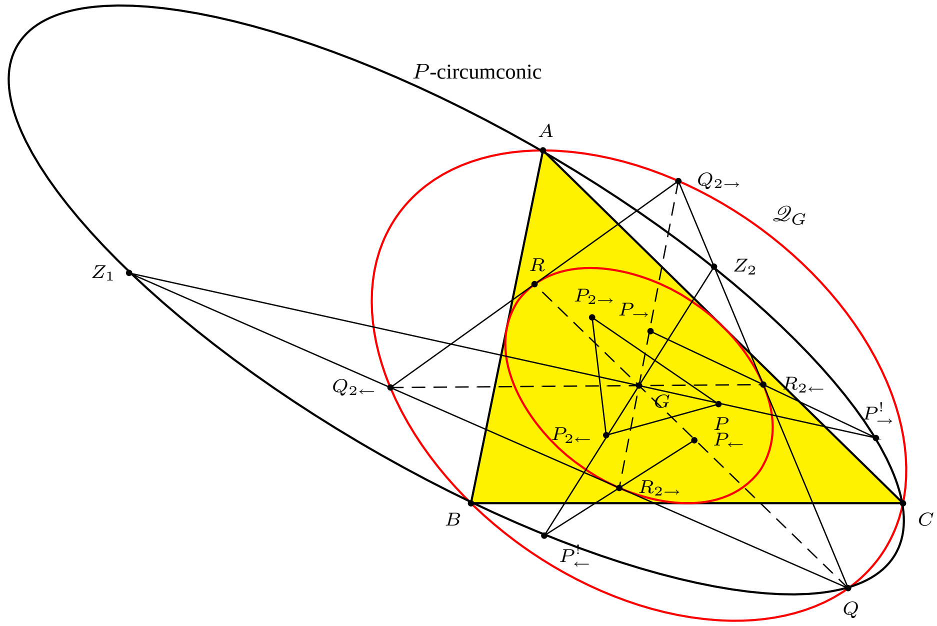

Figure 7.

10. Brocardians of a curve

Are there simple types of curves with the property that a point of one such curve has Brocardians on some simple curves?

Here is one special case for lines.

Given a line $p = [u : v : w]$ with dual P, the circumconics $\mathscr{Q} = \mathscr{Q}_P$, $\mathscr{Q}_\rightarrow = \mathscr{Q}_{P_{2\rightarrow}}$ and $\mathscr{Q}_\leftarrow = \mathscr{Q}_{P_{2\leftarrow}}$, the following statements hold for X is a point on p (see Figure 8).

(1) The Brocardians $X_\rightarrow$ and $X_\leftarrow$ lie on the circumconics $\mathscr{Q}_\leftarrow : wyz + uzx + vxy = 0$ and $\mathscr{Q}_\rightarrow : vyz + wzx + uxy = 0$.
(2) The fourth intersection of $\mathscr{Q}_\rightarrow$ and $\mathscr{Q}_\leftarrow$ is the point $P^{!\bullet}$.
(3) The fourth intersections of the Steiner circumellipse with $\mathscr{Q}_\rightarrow$ and $\mathscr{Q}_\leftarrow$ are the points $Q_{2\rightarrow}$ and $Q_{2\leftarrow}$ respectively.
(4) The fourth intersections of $\mathscr{Q}$ with $\mathscr{Q}_\rightarrow$ and $\mathscr{Q}_\leftarrow$ are $P^!_\rightarrow$ and $P^!_\leftarrow$ respectively.

It is easy to show that the Brocardians $X_\rightarrow$ and $X_\leftarrow$ of all points X on a circumconic lie on the Brocardians $\ell_\rightarrow$ and $\ell_\leftarrow$ of a line ℓ.

References

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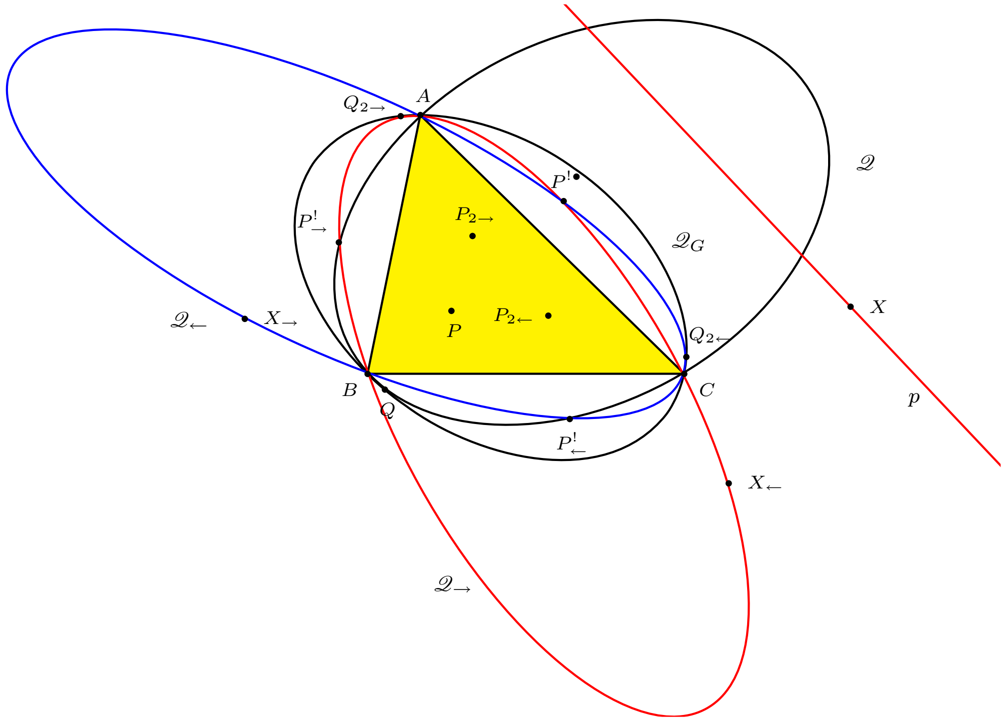

Figure 8.

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