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Generalized Fibonacci Circle Chains

Giovanni Lucca

Abstract. We consider a particular circle chain where, each circle belonging to it, is tangent to the two previous ones and to a common straight line. We give a simple formula for calculating the limit point of the chain in terms of the radii of the first two circles and of the golden ratio.

1. Introduction

By taking a cue from [4] we define a generalized Fibonacci circle chain as follows. Let $\mathcal{L}$ be a line and $\mathcal{C}_1$ and $\mathcal{C}_2$ two circles of radii a and b respectively, tangent to each other and both tangent to $\mathcal{L}$. The generalized Fibonacci circles chain is the sequence of circles $\mathcal{C}_3, \mathcal{C}_4, \dots$, where $\mathcal{C}_n$ is tangent to $\mathcal{C}_{n-1}$, $\mathcal{C}_{n-2}$ and $\mathcal{L}$ (see Figure 1). Denote by r_n the radius of $\mathcal{C}_n$, with $r_1 = a$ and $r_2 = b$.

Let X_n be the point of tangency of the circle $\mathcal{C}_n$ with the line $\mathcal{L}$, with coordinates $(x_n,\ 0)$. Assuming $x_1 < x_2$, we have $x_3 < x_2$, $x_4 > x_3$, and more generally, $x_{n+1} - x_n > 0$ or < 0 according as n is odd or even.

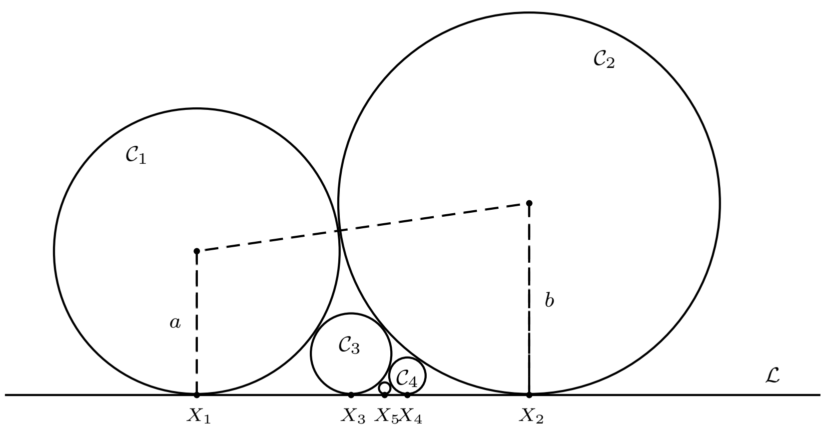

Figure 1.

In Figure 1, a simple application of the Pythagorean theorem shows that

$$(x_2 - x_1)^2 = (a + b)^2 - (a - b)^2 = 4ab,$$

so that $x_2 - x_1 = 2\sqrt{ab}$. Applying the same relation to three consecutive circles with points of tangency X_n, X_{n+1} and X_{n+2} with the line $\mathcal{L}$, we have

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$$\begin{aligned} x_{n+1} - x_n &= (-1)^{n-1} \cdot 2\sqrt{r_{n+1} r_n}, \\ x_{n+1} - x_{n+2} &= (-1)^{n-1} \cdot 2\sqrt{r_{n+2} r_{n+1}}, \\ x_{n+2} - x_n &= (-1)^{n-1} \cdot 2\sqrt{r_{n+2} r_n}. \end{aligned} \tag{1}$$

Since $x_{n+1} - x_n = (x_{n+2} - x_n) + (x_{n+1} - x_{n+2})$, we have

$$2\sqrt{r_{n+1} r_n} = 2\sqrt{r_{n+2} r_n} + 2\sqrt{r_{n+2} r_{n+1}},$$

and

$$\frac{1}{\sqrt{r_{n+2}}} = \frac{1}{\sqrt{r_{n+1}}} + \frac{1}{\sqrt{r_n}}. \tag{2}$$

If we put $G_n = \frac{1}{\sqrt{r_n}}$, then (2) becomes a Fibonacci-like recursive relation

$$G_{n+2} = G_{n+1} + G_n, \qquad G_1 = \frac{1}{\sqrt{a}},\ G_2 = \frac{1}{\sqrt{b}}. \tag{3}$$

The solution of this recurrence relation can be expressed in terms of the Fibonacci numbers F_n according to a formula in [3]:

$$G_{n+2} = G_2 F_{n+1} + G_1 F_n = \frac{F_{n+1}}{\sqrt{b}} + \frac{F_n}{\sqrt{a}}. \tag{4}$$

Here, (F_n) is the Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + F_n, \qquad F_1 = F_2 = 1. \tag{5}$$

Relations (4) can be easily verified by induction. It is well known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi := \frac{\sqrt{5}+1}{2},$$

the golden ratio. In particular, when $a = b = 1$, (3) becomes the classical Fibonacci recurrence (5). Thus, the circles in the chain shown in Figure 1 can be regarded as generalized Fibonacci circles.

2. Limit point of the circle chain

We are interested in locating the limit point of the generalized Fibonacci circle chain. From (1), we have

$$x_{n+1} = x_n + (-1)^{n-1} \frac{2}{G_n G_{n+1}} = x_1 + 2\sum_{k=1}^{n} \frac{(-1)^{k-1}}{G_k G_{k+1}}. \tag{6}$$

The sum $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{G_k G_{k+1}} = \frac{\varphi}{G_1(G_1+\varphi G_2)}$ can be found as a particular case of formulas given in [1, 2]. We give a direct proof here. First of all, by induction, it is easy to establish

$$G_k G_{k+2} - G_{k+1}^2 = -(G_{k-1} G_{k+1} - G_k^2).$$

From this it follows that

$$\begin{aligned} G_kG_{k+2}-G_{k+1}^2 &= (-1)^{k-1}(G_1G_3-G_2^2) \\ &= (-1)^{k-1}\frac{b+\sqrt{ab}-a}{ab} \\ &= (-1)^{k-1}\frac{(\sqrt{b}+\varphi\sqrt{a})(\varphi\sqrt{b}-\sqrt{a})}{\varphi ab}. \end{aligned}$$

Using this, we rewrite (6) as

$$\begin{aligned} x_{n+1} &= x_1+\frac{2\varphi ab}{(\sqrt{b}+\varphi\sqrt{a})(\varphi\sqrt{b}-\sqrt{a})}\sum_{k=1}^{n}\frac{G_kG_{k+2}-G_{k+1}^2}{G_kG_{k+1}} \\ &= x_1+\frac{2\varphi ab}{(\sqrt{b}+\varphi\sqrt{a})(\varphi\sqrt{b}-\sqrt{a})}\sum_{k=1}^{n}\left(\frac{G_{k+2}}{G_{k+1}}-\frac{G_{k+1}}{G_k}\right) \\ &= x_1+\frac{2\varphi ab}{(\sqrt{b}+\varphi\sqrt{a})(\varphi\sqrt{b}-\sqrt{a})}\left(\frac{G_{n+2}}{G_{n+1}}-\frac{G_2}{G_1}\right). \end{aligned}$$

Since $\lim_{n\to\infty}\frac{G_{n+2}}{G_{n+1}}=\varphi$, which is easy to see from (4), we obtain the limit point

$$\begin{aligned} x_\infty &= x_1+\frac{2\varphi ab}{(\sqrt{b}+\varphi\sqrt{a})(\varphi\sqrt{b}-\sqrt{a})}\left(\varphi-\frac{\sqrt{a}}{\sqrt{b}}\right) \\ &= x_1+2\sqrt{ab}\cdot\frac{\sqrt{a}}{(\varphi-1)\sqrt{b}+\sqrt{a}}. \end{aligned}$$

Since $x_2=x_1+2\sqrt{ab}$, this limit point divides X_1X_2 in the ratio

$$x_\infty-x_1 : x_2-x_\infty=\sqrt{a}:(\varphi-1)\sqrt{b}=\varphi\sqrt{a}:\sqrt{b}.$$

References

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Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy
E-mail address: `vanni_lucca@inwind.it`