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Trilinear Distance Inequalities for the Symmedian Point, the Centroid, and Other Triangle Centers

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Abstract. Seven inequalities which appear to be new are derived using Hölder's inequality and the arithmetic-mean–geometric-mean inequality. In particular, bounds are found for power sums $x^q + y^q + z^q$, where x, y, z are the directed distances of a point to the sidelines of a triangle ABC, and the centroid maximizes the product xyz.

We begin with a *very* special point inside a triangle ABC and prove its well known extremal property; however, the proof is not the one often cited (*e.g.*, [2, p.75]). Instead, the proof given here depends on Hölder's inequality and extends to extremal properties of other special points. Then, a second classical inequality, namely the arithmetic-mean–geometric-mean inequality, is used to prove that yet another special point attains an extreme. These results motivate an open question stated at the end of the note.

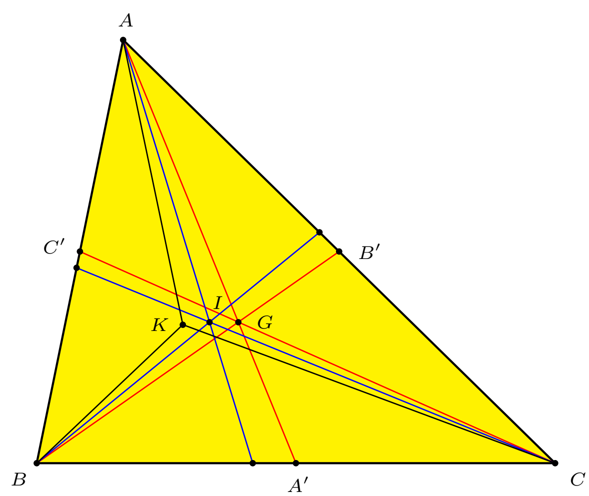

Figure 1.

Regarding the *very* special point, Honsberger [2] writes, "The symmedian point is one of the crown jewels of modern geometry." A construction of the symmedian point, K, of a triangle ABC, goes like this: let M_A be the A-median; *i.e.*, the line of the vertex A and the midpoint A' of segment BC. Let m_A be the A-symmedian; *i.e.*, the reflection of M_A in the bisector of angle A. Let m_B be the B-symmedian

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and m_C the C-symmedian. Then the three symmedians m_A, m_B, m_C concur in K, as shown in Figure 1, where you can also see the medians concurring in the centroid, G, and the bisectors concurring in the incenter, I.

In 1873, Emile Lemoine proved that if a point X inside a triangle ABC has distances x, y, z from the lines BC, CA, AB, then

$$\alpha^2 + \beta^2 + \gamma^2 \leq x^2 + y^2 + z^2, \tag{1}$$

where α, β, γ are the distances from K to those lines. That is, K minimizes $x^2 + y^2 + z^2$. Lemoine's proof [4] can be found by googling "minimum aura donc lieu pour le pied". Expositions in English can be found by googling the titles of John MacKay's historical articles ([6], [7]); these describe the proposal of "Yanto" about 1803, a proof in 1809 by Simon Lhuilier [5, pages 296-8], and rediscoveries. Hölder's inequality [1, pages 19 and 51])of 1889, generalizes Cauchy's inequality [1, pages 2 and 50] of 1821. (Perhaps rediscoverers of the minimal property of K recognized that it follows easily from Cauchy's inequality.)

We shall now generalize (1), starting with Hölder's inequality [1]: that if a, b, c, x, y, z are positive real numbers, then

$$ax + by + cz \leq (a^p + b^p + c^p)^{1/p}(x^q + y^q + z^q)^{1/q} \tag{2}$$

for all real p, q satisfying

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } q > 1,$$

with the inequality (2) reversed if $q < 1$ and $q \neq 0$. It is easy to check that Hölder's inequality is equivalent to

$$\frac{(ax + by + cz)^q}{[a^{q/(q-1)} + b^{q/(q-1)} + c^{q/(q-1)}]^{q-1}} \leq x^q + y^q + z^q \tag{3}$$

if $q > 1$ or $q < 0$, with (3) reversed if $0 < q < 1$.

Let ABC be a triangle with sidelengths $a = |BC|$, $b = |CA|$, $c = |AB|$ and area Δ.If U is any point inside ABC, then homogeneous trilinear coordinates (or simply trilinears) for U, written as $u : v : w$, are any triple of numbers proportional to the respective perpendicular distances x, y, z from U to the sidelines BC, CA, AB. Thus, there is a constant h such that $(x, y, z) = (hu, hv, hw)$. In fact,

$$h = \frac{2\Delta}{au + bv + cw},$$

since $\Delta = \frac{1}{2}(ax + by + cz)$ is the sum of areas $\frac{1}{2}ahu$, $\frac{1}{2}bhv$, $\frac{1}{2}chw$ of the triangles BCU, CAU, ABU. Writing $S(q)$ for $x^q + y^q + z^q$, we now recast (2), via (3), as

$$\frac{(2\Delta)^q}{[aa^{1/(q-1)} + bb^{1/(q-1)} + cc^{1/(q-1)}]^{q-1}} \begin{cases} \leq S(q) & \text{if } q < 0 \text{ or } q > 1, \\ \geq S(q) & \text{if } 0 < q < 1. \end{cases} \tag{4}$$

These two inequalities show that the point $a^{1/(q-1)} : b^{1/(q-1)} : c^{1/(q-1)}$ is an extreme point of $S(q)$ for $q \neq 1$. Taking $q = 2$ (which reduces Hölder's inequality to Cauchy's) gives (1). Other choices of q give inequalities not mentioned in [8]. In order to list several, we use the indexing of special points in the *Encyclopedia* *of Triangle Centers*[3], where K is indexed as X_6, and many of its properties, and properties of other special points to be mentioned here, are recorded.

The point $X_6 = a : b : c$ minimizes $S(2)$:

$$\frac{4\Delta^2}{a^2+b^2+c^2} \leq x^2+y^2+z^2$$

The point $X_{365} = a^{1/2} : b^{1/2} : c^{1/2}$ minimizes $S(3)$:

$$\frac{8\Delta^3}{(a^{3/2}+b^{3/2}+c^{3/2})^2} \leq x^3+y^3+z^3$$

The point $X_{31} = a^2 : b^2 : c^2$ minimizes $S(3/2)$:

$$\left(\frac{8\Delta^3}{a^3+b^3+c^3}\right)^{1/2} \leq x^{3/2}+y^{3/2}+z^{3/2}$$

The point $X_{32} = a^3 : b^3 : c^3$ minimizes $S(4/3)$:

$$\left(\frac{16\Delta^4}{a^4+b^4+c^4}\right)^{1/3} \leq x^{4/3}+y^{4/3}+z^{4/3}$$

The point $X_{75} = a^{-2} : b^{-2} : c^{-2}$ maximizes $S(1/2)$:

$$[2\Delta(a^{-1}+b^{-1}+c^{-1})]^{1/2} \geq x^{1/2}+y^{1/2}+z^{1/2}$$

The point $X_{76} = a^{-3} : b^{-3} : c^{-3}$ maximizes $S(2/3)$:

$$[4\Delta^2(a^{-2}+b^{-2}+c^{-2})]^{1/3} \geq x^{2/3}+y^{2/3}+z^{2/3}$$

The point $X_{366} = a^{-1/2} : b^{-1/2} : c^{-1/2}$ minimizes $S(-1)$:

$$\frac{(a^{1/2}+b^{1/2}+c^{1/2})^2}{2\Delta} \leq x^{-1}+y^{-1}+z^{-1}$$

In these examples, the extreme point is of the form $P(t) = a^t : b^t : c^t$. If ABC has distinct sidelengths then $P(t)$ traces the *power curve*, as in Figure 2. Taking limits as $t \to \infty$ and $t \to -\infty$ shows that this curve is tangent to the shortest and longest sides of ABC at the vertices opposite those sides. The incenter $I = P(0)$ lies on the power curve, and for each $t \neq 0$, the points $P(t)$ and $P(-t)$ are a pair of isogonal conjugates; *e.g.*, for $t = 1$, they are the symmedian point and the centroid.

Here are two more choices of q : the solutions of $1/(q-1) = q$, namely the golden ratio $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$. These are the only values of q for which the point $a^q : b^q : c^q$ minimizes $x^q+y^q+z^q$.

We turn now to a different sort of inequality for points inside ABC. As a special case of the arithmetic-mean–geometric-mean inequality [1],

$$27uvw \leq (u+v+w)^3$$

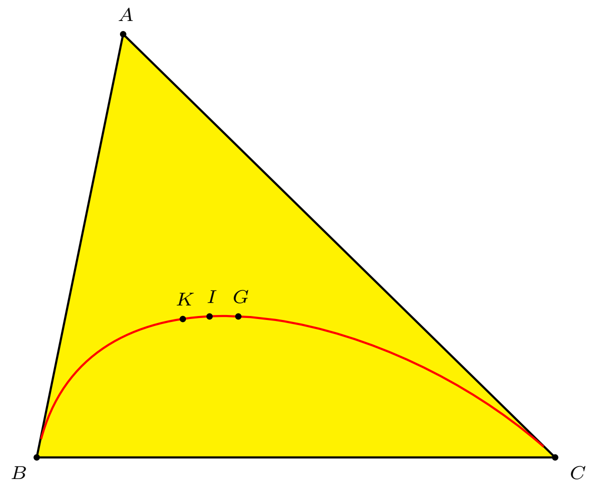

Figure 2.

for all positive u, v, w, so that if a, b, c, x, y, z are arbitrary positive real numbers, then

$$xyz \leq \frac{(ax+by+cz)^3}{27abc}. \tag{5}$$

The distances from the centroid ($X_2 = a^{-1} : b^{-1} : c^{-1}$, labeled G in Figures 1 and 2) to the sidelines BC, CA, AB are given by

$$\left(\frac{2\Delta}{3a}, \frac{2\Delta}{3b}, \frac{2\Delta}{3c}\right),$$

so that their product is $8\Delta^3/(27abc)$. Since $ax + by + cz = 2\Delta$, we may interpret (5) thus: *the centroid maximizes the product* xyz.

What other extrema are attained by special points inside a triangle? To seek others, it is easy and entertaining to devise computer-based searches – either visual searches with dynamic distance measurements using *Cabri* or *The Geometer's Sketchpad*, or algebraic using trilinears or other coordinates. Such searches prompt an intriguing problem: to determine conditions on a function in x, y, z for which an extreme value is attained inside ABC. If the function is homogeneous and symmetric in a, b, c, as in the examples considered above, then must such an extreme point be a triangle center? Is there a simple example in which the extreme point is not on the power curve?

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