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On the Euler Reflection Point

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Abstract. The Euler reflection point E of a triangle is known in literature as the common point of the reflections of its Euler line OH in each of its sidelines, where O and H are the circumcenter and the orthocenter of the triangle, respectively. In this note we prove that E lies on six circles associated with the triangles of Napoleon.

1. Introduction

The Euler reflection point E of a triangle ABC is the concurrency point of the reflections of the Euler line in the sidelines of the triangle. The existence of E is justified by the following more general result.

Theorem 1 (S. N. Collings)**.** *Let ρ be a line in the plane of a triangle ABC. Its reflections in the sidelines BC, CA, AB are concurrent if and only if ρ passes through the orthocenter H of ABC. In this case, their point of concurrency lies on the circumcircle.*

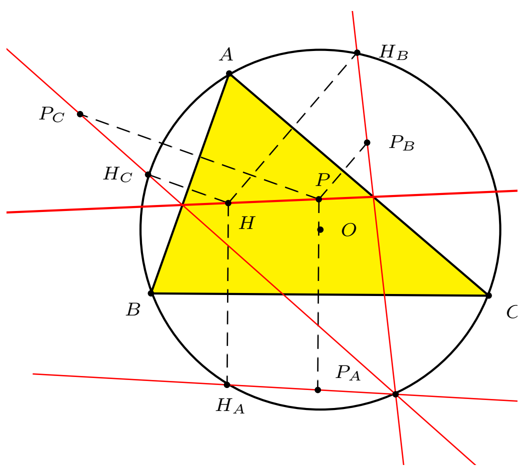

Figure 1

Synthetic proofs of Theorem 1 can be found in [1] and [2]. Known as X_{110} in Kimberling's list of triangle centers, the Euler reflection point is also the focus of the Kiepert parabola (see [8]) whose directrix is the line containing the reflections of E in the three sidelines.

Before proceeding to our main theorem, we give two preliminary results.

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Lemma 2 (J. Rigby)**.** *The three lines joining the vertices of a given triangle ABC with the circumcenter of the triangle formed by the other two vertices of ABC and the circumcenter O are concurrent at the isogonal conjugate of the nine-point center.*

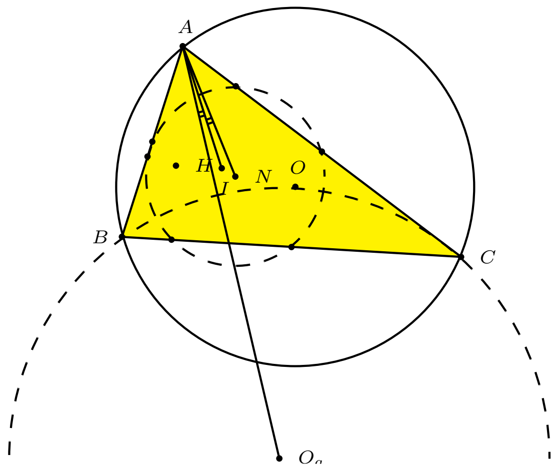

Figure 2.

The common point of these lines is also known as the Kosnita point of triangle ABC. For a synthetic proof of this result, see [7]. For further references, see [3] and [5].

Lemma 3. *The three lines joining the vertices of a given triangle ABC with the reflections of the circumcenter O into the opposite sidelines are concurrent at the nine-point center of triangle ABC.*

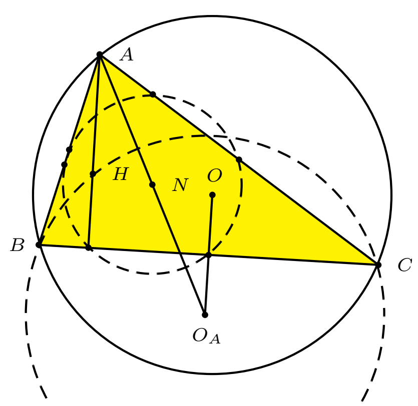

Figure 3.

This is a simple consequence of the fact that the reflection O_A of O into the sideline BC is the circumcenter of triangle BHC, where H is the orthocenter of

ABC. In this case, according to the definition of the nine-point circle, the circumcircle of BHC is the homothetic image of the nine-point circle under $\mathsf{h}(A, 2)$. See also [4].

2. The Euler reflection point and the triangles of Napoleon

Let A_+, B_+, C_+, A_-, B_-, C_- be the apices of the outer and inner equilateral triangles erected on the sides BC, CA and AB of triangle ABC, respectively. Denote by N_A, N_B, N_C, N'_A, N'_B, N'_C the circumcenters of triangles BCA_+, CAB_+, ABC_+, BCA_-, CAB_-, ABC_-, respectively. The triangles $N_AN_BN_C$ and $N'_AN'_BN'_C$ are known as the two triangles of Napoleon (the outer and the inner).

Theorem 4. *The circumcircles of triangles AN_BN_C, BN_CN_A, CN_AN_B, $AN'_BN'_C$, $BN'_CN'_A$, $CN'_AN'_B$ are concurrent at the Euler reflection point E of triangle ABC.*

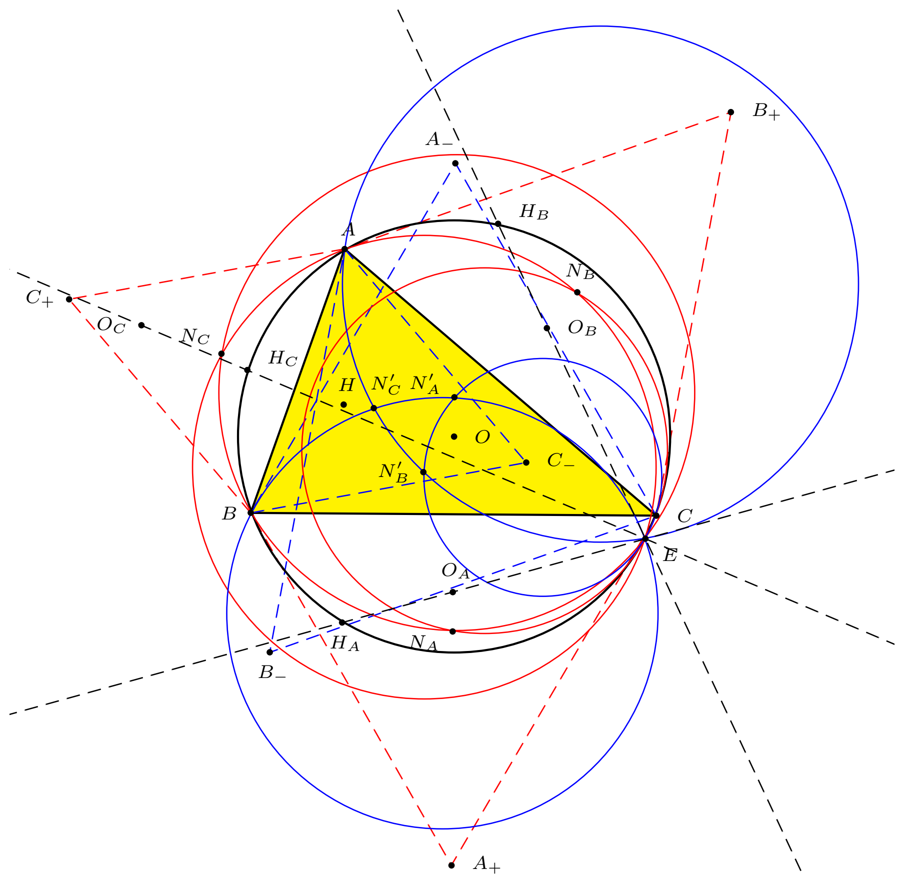

Figure 4

Proof. We shall show that each of these circles contains E. It is enough to consider the circle AB_BN_C.

Denote by O_B, O_C the reflections of the circumcenter O into the sidelines CA and AB, respectively. The lines EO_B, EO_C are the reflections of the Euler line OH in the sidelines CA and AB, respectively. Computing directed angles, we have

$$\begin{aligned}(EO_C, EO_B) &= (EO_C, OH) + (OH, EO_B) \\ &= 2(AB, OH) + 2(OH, AC) \\ &= 2(AB, AC) \pmod{\pi}.\end{aligned}$$

On the other hand,

$$(AO_C, AO_B) = (AO_C, AO) + (AO, AO_B) = 2(AB, AO) + 2(AO, AC) = 2(AB, AC).$$

Therefore, the quadrilateral O_CAO_BE is cyclic. We show that the centers of the three circles O_BAO_C, ABC and AN_BN_C are collinear. Since they all contain A, it follows that they are coaxial with two common points. Since E lies on the first two circles, it must also lie on the the third circle AN_BN_C. □

Proposition 5. *Let ABC be a triangle with circumcenter O and orthocenter H. Consider the points Y and Z on the sides CA and AB respectively such that the directed angles $(AC, HY) = -\frac{\pi}{3}$ and $(AB, HZ) = \frac{\pi}{3}$. Let U be the circumcenter of triangle HYZ.*

(a) *A_-, U, H are collinear.*

(b) *A, U, O_A are collinear.*

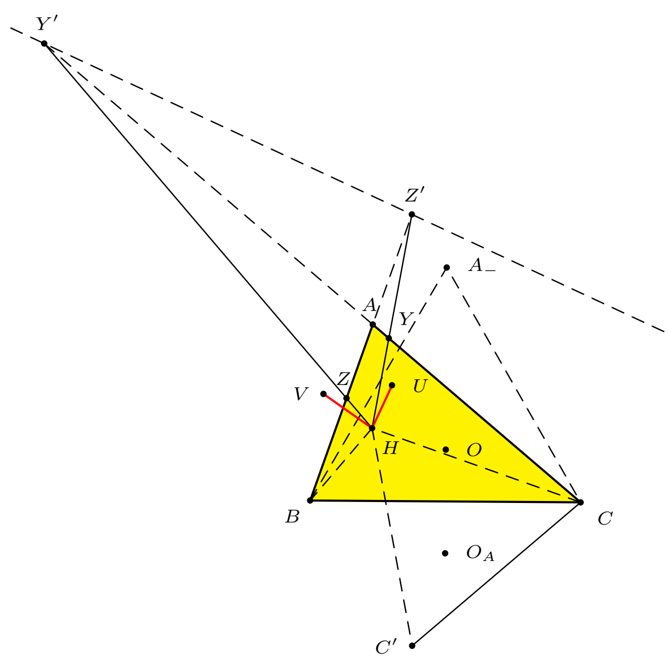

Figure 5

Proof. (a) Let V be the orthocenter of triangle HYZ, and denote by Y', Z' the intersections of the lines HZ with CA and HY with AB. Since the quadrilateral $YZY'Z'$ is cyclic, the lines $Y'Z'$, YZ are antiparallel. Since the lines HU, HV are isogonal conjugate with respect to the angle YHZ, it follows that the lines HU and $Y'Z'$ are perpendicular.

Let C' be the reflections of C in the line HY'. Triangle $HC'C$ is equilateral since

$$\begin{aligned}(HY', HC) &= (HY', CA) + (CA, HC)\\ &= (AB, AC) - \frac{\pi}{3} + \frac{\pi}{2} - (AB, AC)\\ &= \frac{\pi}{6}.\end{aligned}$$

Now, triangles $Y'HC$ and $Z'HB$ are similar since $\angle HY'C = \angle HZ'B$ and $\angle HCY' = \angle HBZ'$. Since $Y'HC'$ is the reflection of $Y'HC$ in HY', we conclude that triangles $Y'HC'$ and $Z'HB$ are similar. This means

$$\frac{HY'}{HZ'} = \frac{HC'}{HB} \quad \text{and} \quad \angle Y'HC' = \angle Z'HB,$$

and

$$\frac{HY'}{HC'} = \frac{HZ'}{HB} \quad \text{and} \quad \angle Z'HY' = \angle BHC'.$$

Hence, $Z'HY'$ and BHC' are directly similar. This implies that A_-H and $Y'Z'$ are perpendicular:

$$\begin{aligned}(Y'Z', A_-H) &= (Y'Z', BC') + (BC', A_-H)\\ &= (Z'H, BH) + (BC, A_-C)\\ &= \frac{\pi}{2}.\end{aligned}$$

Together with the perpendicularity of HU and $Y'Z'$, this yields the collinearity of A_-, U, and H.

(b) Note that the triangles $BC'C$ and A_-HC are congruent since $BC = A_-C$, $C'C = HC$, and $\angle BCC' = \angle A_-CH$. Applying the law of sines to triangle HYZ, we have

$$UH = \frac{YZ}{2\sin YHZ} = \frac{YZ}{2\sin\left(\frac{2\pi}{3} - A\right)}.$$

From the similarity of triangles $Z'HY'$ and BHC' and of HYZ and $HY'Z'$, we have

$$A_-H = BC' = Y'Z' \cdot \frac{BH}{Y'H} = YZ \cdot \frac{Y'H}{ZH} \cdot \frac{BH}{Y'H} = YZ \cdot \frac{\cos\frac{\pi}{6}}{|\cos A|} = YZ \cdot \frac{\sqrt{3}}{2|\cos A|}.$$

Therefore,

$$\frac{UH}{A_-H} = \frac{|\cos A|}{\sqrt{3}\sin\left(\frac{2\pi}{3} - A\right)}.$$

Since $AH = 2R\cos A$, we have

$$AH+A\_O_A = 3R\cos A + a\sin\frac{\pi}{3} = R\cos A + \sqrt{3}R\sin A = 2\sqrt{3}R\sin\left(\frac{2\pi}{3} - A\right),$$

and

$$\frac{|AH|}{AH + A\_O_A} = \frac{|\cos A|}{\sqrt{3}\sin\left(\frac{2\pi}{3} - A\right)} = \frac{UH}{A\_H} = \frac{UH}{A\_U + UH}.$$

Since U, H and $A\_$ are collinear by (a), we have $\frac{AH}{A\_O_A} = \frac{UH}{A\_U}$. Combining this with the parallelism of the lines AH and A_O_A, we have the direct similarity of triangles AHU and O_AA_U. We now conclude that the angles HUA and A_UO' are equal. This, together with (a) above, implies the collinearity of the points A, U, O_A.

On the other hand, according to Lemma 3, the points A, N, O_A are collinear. Hence, the points A, U, N are collinear as well. □

According to Lemma 1, the lines AO_a and AN are isogonal conjugate with respect to angle BAC. Thus, by reflecting the figure in the internal bisector of angle BAC, and following Lemma 6, we obtain the following result.

Corollary 6. *Given a triangle ABC with circumcenter O, let Y, Z be points on the sides AC, AB satisfying $(AC, OY) = -\frac{\pi}{3}$ and $(AB, OZ) = \frac{\pi}{3}$. The circumcenters of triangles OYZ and BOC, and the vertex A are collinear.*

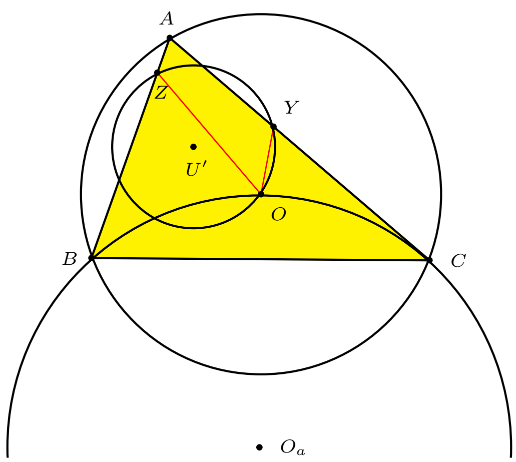

Figure 6

Now we complete the proof of Theorem 4. By applying Corollary 6 to triangle OO_BO_C with the points N_B, N_C lying on the sidelines OO_B and OO_C such that $(OO_B, AN_B) = -\frac{\pi}{3}$ and $(OO_C, AN_C) = \frac{\pi}{3}$, we conclude that the circumcenters of triangles AN_BN_C, O_BAO_C and ABC are collinear.

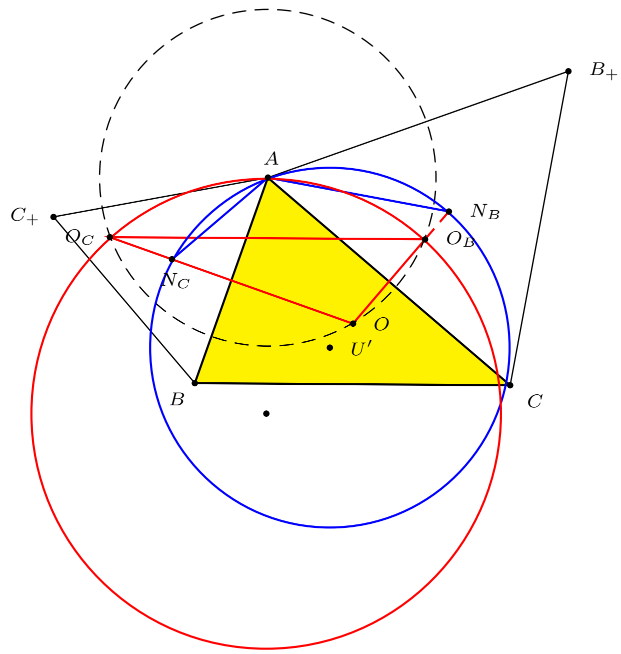

Figure 7

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