Abstract. The Euler reflection point $E$ of a triangle is known in literature as the common point of the reflections of its Euler line $OH$ in each of its sidelines, where $O$ and $H$ are the circumcenter and the orthocenter of the triangle, respectively. In this note we prove that $E$ lies on six circles associated with the triangles of Napoleon.

1. Introduction

The Euler reflection point $E$ of a triangle $ABC$ is the concurrency point of the reflections of the Euler line in the sidelines of the triangle. The existence of $E$ is justified by the following more general result.

**Theorem 1** (S. N. Collings). Let $\rho$ be a line in the plane of a triangle $ABC$. Its reflections in the sidelines $BC$, $CA$, $AB$ are concurrent if and only if $\rho$ passes through the orthocenter $H$ of $ABC$. In this case, their point of concurrency lies on the circumcircle.

![Figure 1](image)

Synthetic proofs of Theorem 1 can be found in [1] and [2]. Known as $X_{110}$ in Kimberling’s list of triangle centers, the Euler reflection point is also the focus of the Kiepert parabola (see [8]) whose directrix is the line containing the reflections of $E$ in the three sidelines.

Before proceeding to our main theorem, we give two preliminary results.

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Lemma 2 (J. Rigby). The three lines joining the vertices of a given triangle $ABC$ with the circumcenter of the triangle formed by the other two vertices of $ABC$ and the circumcenter $O$ are concurrent at the isogonal conjugate of the nine-point center.

![Figure 2.](image1)

The common point of these lines is also known as the Kosnita point of triangle $ABC$. For a synthetic proof of this result, see [7]. For further references, see [3] and [5].

Lemma 3. The three lines joining the vertices of a given triangle $ABC$ with the reflections of the circumcenter $O$ into the opposite sidelines are concurrent at the nine-point center of triangle $ABC$.

![Figure 3.](image2)

This is a simple consequence of the fact that the reflection $O_A$ of $O$ into the sideline $BC$ is the circumcenter of triangle $BHC$, where $H$ is the orthocenter of
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ABC. In this case, according to the definition of the nine-point circle, the circumcircle of $BHC$ is the homothetic image of the nine-point circle under $h(A, 2)$. See also [4].

2. The Euler reflection point and the triangles of Napoleon

Let $A_+, B_+, C_+, A_-, B_-, C_-$ be the apices of the outer and inner equilateral triangles erected on the sides $BC$, $CA$ and $AB$ of triangle $ABC$, respectively. Denote by $N_A$, $N_B$, $N_C$, $N'_A$, $N'_B$, $N'_C$ the circumcenters of triangles $BCA_+$, $CAB_+$, $ABC_+$, $BCA_-$, $CAB_-$, $ABC_-$, respectively. The triangles $N_AN_BN_C$ and $N'_AN'_BN'_C$ are known as the two triangles of Napoleon (the outer and the inner).

**Theorem 4.** The circumcircles of triangles $AN_BN_C$, $BN_CN_A$, $CN_AN_B$, $AN'_BN'_C$, $BN'_CN'_A$, $CN'_AN'_B$ are concurrent at the Euler reflection point $E$ of triangle $ABC$. 

![Figure 4](image-url)
Proof. We shall show that each of these circles contains $E$. It is enough to consider the circle $AB_BN_C$.

Denote by $O_B, O_C$ the reflections of the circumcenter $O$ into the sidelines $CA$ and $AB$, respectively. The lines $EO_B, EO_C$ are the reflections of the Euler line $OH$ in the sidelines $CA$ and $AB$, respectively. Computing directed angles, we have

\[
(EO_C, EO_B) = (EO_C, OH) + (OH, EO_B)
= 2(AB, OH) + 2(OH, AC)
= 2(AB, AC) \pmod{\pi}.
\]

On the other hand,

\[
(AO_C, AO_B) = (AO_C, AO) + (AO, AO_B) = 2(AB, AO) + 2(AO, AC) = 2(AB, AC).
\]

Therefore, the quadrilateral $OC_AO_BE$ is cyclic. We show that the centers of the three circles $O_BAO_C, ABC$ and $AN_BN_C$ are collinear. Since they all contain $A$, it follows that they are coaxial with two common points. Since $E$ lies on the first two circles, it must also lie on the third circle $AN_BN_C$. \hfill \Box

**Proposition 5.** Let $ABC$ be a triangle with circumcenter $O$ and orthocenter $H$. Consider the points $Y$ and $Z$ on the sides $CA$ and $AB$ respectively such that the directed angles $(AC, HY) = -\frac{\pi}{3}$ and $(AB, HZ) = \frac{\pi}{3}$. Let $U$ be the circumcenter of triangle $HYZ$.

(a) $A, U, H$ are collinear.

(b) $A, U, O_A$ are collinear.
Proof. (a) Let $V$ be the orthocenter of triangle $HYZ$, and denote by $Y'$, $Z'$ the intersections of the lines $HZ$ with $CA$ and $HY$ with $AB$. Since the quadrilateral $YZY'Z'$ is cyclic, the lines $Y'Z'$, $YZ$ are antiparallel. Since the lines $HU$, $HV$ are isogonal conjugate with respect to the angle $YHZ$, it follows that the lines $HU$ and $Y'Z'$ are perpendicular.

Let $C'$ be the reflections of $C$ in the line $HY'$. Triangle $HC'C$ is equilateral since

$$
(HY', HC) = (HY', CA) + (CA, HC)
= (AB, AC) - \frac{\pi}{3} + \frac{\pi}{2} - (AB, AC)
= \frac{\pi}{6}.
$$

Now, triangles $Y'HC$ and $Z'H$ are similar since $\angle HY'C = \angle HZ'B$ and $\angle HCY' = \angle HBZ'$. Since $Y'HC'$ is the reflection of $Y'HC$ in $HY'$, we conclude that triangles $Y'HC'$ and $Z'H$ are similar. This means

$$
\frac{HY'}{HZ'} = \frac{HC'}{HB} \quad \text{and} \quad \angle Y'HC' = \angle Z'H,
$$
and

$$
\frac{HY'}{HC'} = \frac{HZ'}{HB} \quad \text{and} \quad \angle Z'HY' = \angle BHC'.
$$
Hence, $Z'HY'$ and $BHC'$ are directly similar. This implies that $A_H$ and $Y'Z'$ are perpendicular:

$$
(Y'Z', A_H) = (Y'Z', BC') + (BC', A_H)
= (Z'H, BH) + (BC, A_C)
= \frac{\pi}{2}.
$$
Together with the perpendicularly of $HU$ and $Y'Z'$, this yields the collinearity of $A_H$, $U$, and $H$.

(b) Note that the triangles $BC'C$ and $A_HC$ are congruent since $BC = A_C$, $C'C = HC$, and $\angle BCC' = \angle A_CH$. Applying the law of sines to triangle $HYZ$, we have

$$
UH = \frac{YZ}{2 \sin YHZ} = \frac{YZ}{2 \sin \left(\frac{2\pi}{3} - A\right)}.
$$
From the similarity of triangles $Z'HY'$ and $BHC'$ and of $HYZ$ and $HY'Z'$, we have

$$
A_H = BC' = Y'Z' \cdot BH \quad \text{and} \quad \frac{Y'Z'}{Y'H} = \frac{BH}{Y'H} = YZ \cdot \frac{\cos \frac{\pi}{6}}{\cos A} = YZ \cdot \frac{\sqrt{3}}{2 |\cos A|}.
$$
Therefore,

$$
\frac{UH}{A_H} = \frac{|\cos A|}{\sqrt{3} \sin \left(\frac{2\pi}{3} - A\right)}.
$$
Since $AH = 2R \cos A$, we have

$$AH + A_OA = 3R \cos A + a \sin \frac{\pi}{3} = R \cos A + \sqrt{3}R \sin A = 2\sqrt{3}R \sin \left(\frac{2\pi}{3} - A\right),$$

and

$$\frac{|AH|}{AH + A_OA} = \frac{|\cos A|}{\sqrt{3} \sin \left(\frac{2\pi}{3} - A\right)} = \frac{UH}{A_H} = \frac{UH}{A_U + UH}.$$

Since $U$, $H$ and $A_O$ are collinear by (a), we have $\frac{AH}{A_OA} = \frac{UH}{A_U + UH}$. Combining this with the parallelism of the lines $AH$ and $A_OA$, we have the direct similarity of triangles $AHU$ and $A_OA$. We now conclude that the angles $HUA$ and $A_{U'O}$ are equal. This, together with (a) above, implies the collinearity of the points $A$, $U$, $O_A$.

On the other hand, according to Lemma 3, the points $A$, $N$, $O_A$ are collinear. Hence, the points $A$, $U$, $N$ are collinear as well. \hfill \Box

According to Lemma 1, the lines $AO_a$ and $AN$ are isogonal conjugate with respect to angle $BAC$. Thus, by reflecting the figure in the internal bisector of angle $BAC$, and following Lemma 6, we obtain the following result.

**Corollary 6.** Given a triangle $ABC$ with circumcenter $O$, let $Y$, $Z$ be points on the sides $AC$, $AB$ satisfying $(AC, OY) = -\frac{\pi}{3}$ and $(AB, OZ) = \frac{\pi}{3}$. The circumcenters of triangles $OYZ$ and $BOC$, and the vertex $A$ are collinear.

![Figure 6](image)

Now we complete the proof of Theorem 4. By applying Corollary 6 to triangle $OO_BO_C$ with the points $N_B$, $N_C$ lying on the sidelines $OO_B$ and $OO_C$ such that $(OO_B, AN_B) = -\frac{\pi}{3}$ and $(OO_C, AN_C) = \frac{\pi}{3}$, we conclude that the circumcenters of triangles $AN_BN_C$, $O_BO_C$ and $ABC$ are collinear.
References


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