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Characterizations of Bicentric Quadrilaterals

Martin Josefsson

Abstract. We will prove two conditions for a tangential quadrilateral to be cyclic. According to one of these, a tangential quadrilateral is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.

1. Introduction

A *bicentric quadrilateral* is a convex quadrilateral with both an incircle and a circumcircle. One characterization of these quadrilaterals is obtained by combining the most useful characterizations of tangential and cyclic quadrilaterals, that the consecutive sides a, b, c and d, and angles A, B, C and D satisfy

$$a + c = b + d,$$
$$A + C = B + D = \pi.$$

We review a few other characterizations of bicentric quadrilaterals before proving two possibly new ones.

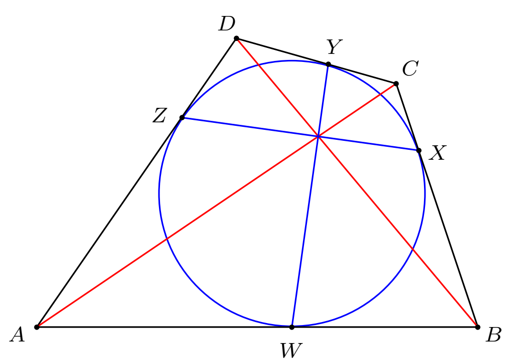

Figure 1. The tangency chords and diagonals

If the incircle in a tangential quadrilateral $ABCD$ is tangent to the sides AB, BC, CD and DA at W, X, Y and Z respectively, then the segments WY and XZ are called the *tangency chords* in [8, pp.188-189]. See Figure 1. In [4, 9, 13] it is proved that *a tangential quadrilateral is cyclic if and only if the tangency chords are perpendicular*.

Problem 10804 in the MONTHLY [14] states that a tangential quadrilateral is cyclic if and only if

$$\frac{AW}{WB} = \frac{DY}{YC}.$$

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Using the same notations, Hajja proved in [11] that a tangential quadrilateral is cyclic if and only if

$$\frac{AC}{BD} = \frac{AW + CY}{BX + DZ}.$$

If E, F, G and H are the midpoints of WX, XY, YZ and ZW respectively (see Figure 2), then *the tangential quadrilateral $ABCD$ is cyclic if and only if the quadrilateral $EFGH$ is a rectangle*. This characterization was Problem 6 on China Western Mathematical Olympiad 2003 [5, pp.182-183].

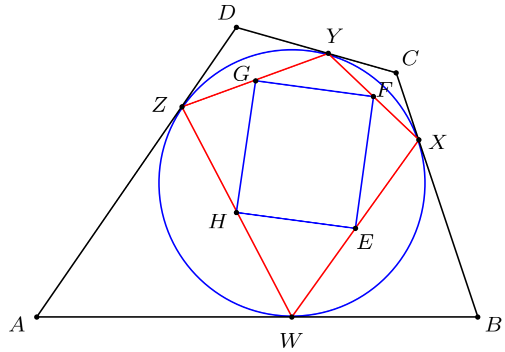

Figure 2. $ABCD$ is cyclic iff $EFGH$ is a rectangle

2. Two characterizations of right triangles

To prove one of the characterizations of bicentric quadrilaterals we will need the following characterization of right triangles. The direct part of the theorem is an easy exercise [1], but we have found no reference of the converse result.

Theorem 1. *In a non-isosceles triangle the median and altitude to one of the sides divide the opposite angle into three parts. This angle is a right one if and only if the angle between the median and the longer of the sides at the considered vertex is equal to the angle between the altitude and the shorter side at that vertex.*

Proof. We use notations as in Figure 3. If $C = \frac{\pi}{2}$, we shall prove that $\alpha = \beta$. Triangle AMC is isosceles [2] with $AM = CM$, so $A = \alpha$. Triangles ACB and CHB are similar, so $A = \beta$. Hence $\alpha = \beta$.

Conversely, if $\alpha = \beta$, we shall prove that $C = \frac{\pi}{2}$. By the exterior angle theorem, angle $CMB = A + \alpha$, so in triangle MCH we have

$$A + \alpha + \gamma = \frac{\pi}{2}. \tag{1}$$

[1]A similar problem also including the angle bisector can be found in [1, pp.46-49] and [12, p.32].

[2]The midpoint of the hypotenuse is the circumcenter.

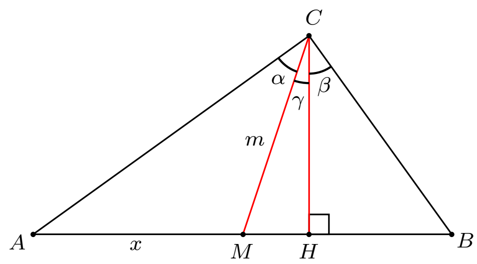

Figure 3. Median and altitude in a triangle

Let $x = AM = BM$ and $m = CM$. Using the law of sines in triangles CAM and CMB,

$$\frac{\sin \alpha}{x} = \frac{\sin A}{m} \quad \Leftrightarrow \quad \frac{x}{m} = \frac{\sin \alpha}{\sin A}$$

and with $\alpha = \beta$,

$$\frac{\sin(\alpha + \gamma)}{x} = \frac{\sin B}{m} \quad \Rightarrow \quad \frac{x}{m} = \frac{\sin\left(\frac{\pi}{2} - A\right)}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos A}{\cos \alpha}$$

since $B + \alpha = \frac{\pi}{2}$ in triangle BCH. Combining the last two equations, we get

$$\frac{\sin \alpha}{\sin A} = \frac{\cos A}{\cos \alpha} \quad \Leftrightarrow \quad \sin 2\alpha = \sin 2A.$$

This equation has the two solutions $2\alpha = 2A$ and $2\alpha = \pi - 2A$, hence $\alpha = A$ or $\alpha = \frac{\pi}{2} - A$. The second solution combined with $B + \alpha = \frac{\pi}{2}$ gives $A = B$, which is impossible since the triangle is not isosceles by the assumption in the theorem. Thus $\alpha = A$ is the only valid solution. Hence

$$C = \alpha + \gamma + \beta = A + \gamma + \alpha = \frac{\pi}{2}$$

according to (1), completing the proof. □

Corollary 2. *Let CM, CD and CH be a median, an angle bisector and an altitude respectively in triangle ABC. The angle C is a right angle if and only if CD bisects angle HCM.*

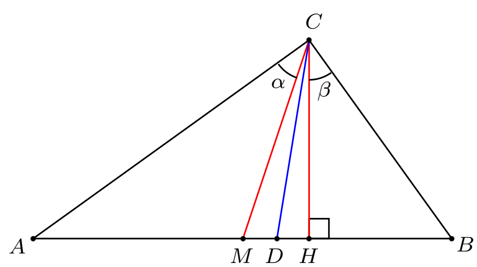

Figure 4. Median, angle bisector and altitude in a triangle

Proof. Since CD is an angle bisector in triangle ABC, we have (see Figure 4)

$$\alpha + \angle MCD = \angle HCD + \beta. \tag{2}$$

Using Theorem 1 and (2), we get

$$C = \frac{\pi}{2} \quad \Leftrightarrow \quad \alpha = \beta \quad \Leftrightarrow \quad \angle MCD = \angle HCD.$$

□

3. Corollaries of Pascal's theorem and Brocard's theorem

Pascal's theorem states that *if a hexagon is inscribed in a circle and the three pairs of opposite sides are extended until they meet, then the three points of intersection are collinear*. A proof is given in [6, pp.74-75]. Pascal's theorem is also true in degenerate cases.

In [7, p.15], the following theorem is called Brocard's theorem: *if the extensions of opposite sides in a cyclic quadrilateral intersect at J and K, and the diagonals intersect at P, then the circumcenter O of the quadrilateral is also the orthocenter in triangle JKP* (see Figure 5). An elementary proof of this theorem can be found at [16].

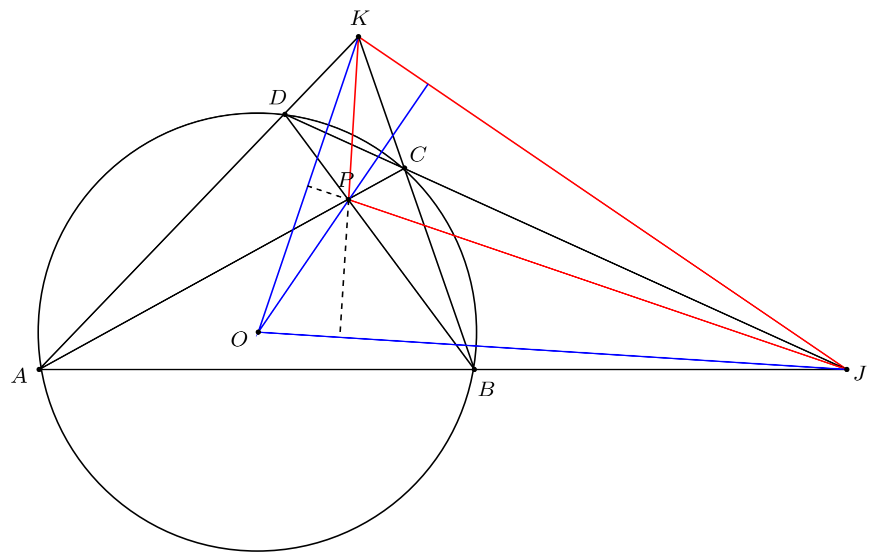

Figure 5. Brocard's theorem

To prove our second characterization of bicentric quadrilaterals we will need two corollaries of these theorems that are quite well known. The first is a special case of Pascal's theorem in a quadrilateral. If the incircle in a tangential quadrilateral $ABCD$ is tangent to the sides AB, BC, CD and DA at W, X, Y and Z respectively, then in [9] Yetti [3] calls the quadrilateral $WXYZ$ the *contact quadrilateral*.

[3]Yetti is the username of an American physicist at the website *Art of Problem Solving* [3].

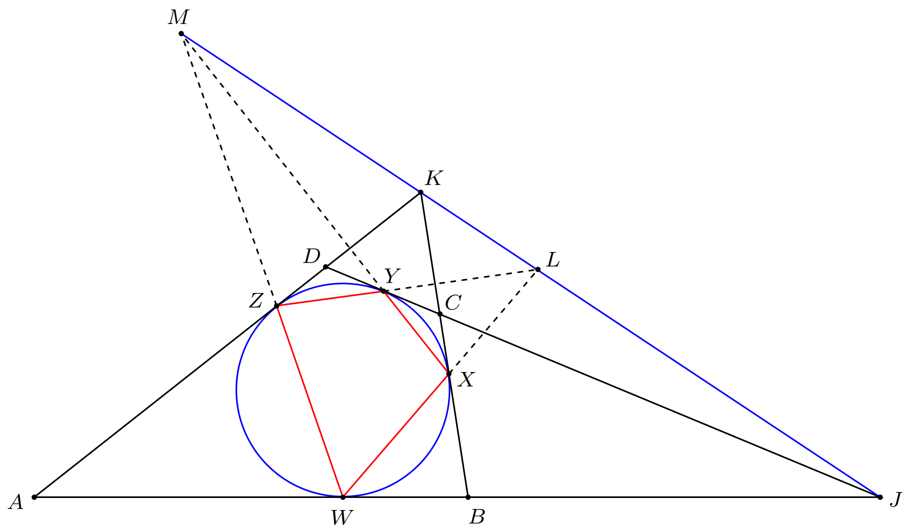

Figure 6. Pascal's theorem in a tangential quadrilateral

Corollary 3. *If the extensions of opposite sides in a tangential quadrilateral intersect at J and K, and the extensions of opposite sides in its contact quadrilateral intersect at L and M, then the four points J, L, K and M are collinear.*

Proof. Consider the degenerate cyclic hexagon $WWXYYZ$, where W and Y are double vertices. The extensions of the sides at these vertices are the tangents at W and Y, see Figure 6. According to Pascal's theorem, the points J, L and M are collinear.

Next consider the degenerate cyclic hexagon $WXXYZZ$. In the same way the points M, K and L are collinear. This proves that the four points J, L, K and M are collinear, since M and L are on both lines, so these lines coincide. □

Corollary 4. *If the extensions of opposite sides in a tangential quadrilateral intersect at J and K, and the diagonals intersect at P, then JK is perpendicular to the extension of IP where I is the incenter.*

Proof. The contact quadrilateral $WXYZ$ is a cyclic quadrilateral with circumcenter I, see Figure 7. It is well known that the point of intersection of WY and XZ is also the point of intersection of the diagonals in the tangential quadrilateral $ABCD$, see [10, 15, 17]. If the extensions of opposite sides in the contact quadrilateral $WXYZ$ intersect at L and M, then by Brocard's theorem $ML \perp IP$. According to Corollary 3, ML and JK are the same line. Hence $JK \perp IP$. □

4. Two characterizations of bicentric quadrilaterals

Many problems on quadrilaterals in text books and on problem solving web sites are formulated as implications of the form: if the quadrilateral is a special type (like a bicentric quadrilateral), then you should prove it has some property. How

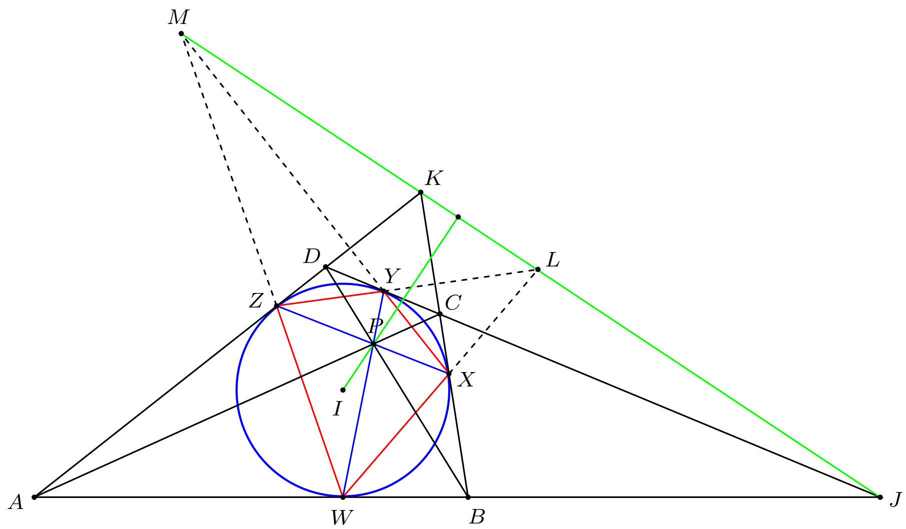

Figure 7. Perpendicular lines JK and IP

about the converse statement? Sometimes it is concidered, but far from always. The two characterizations we will prove here was found when considering if the converse statement of two such problems are also true. The first is a rather easy one and it would surprise us if it hasn't been published before; however we have been unable to find a reference for it. Besides, it will be used in the proof of the second characterization.

Theorem 5. *Let the extensions of opposite sides in a tangential quadrilateral intersect at J and K. If I is the incenter, then the quadrilateral is also cyclic if and only if JIK is a right angle.*

Proof. We use notations as in Figure 8, where G and H are the midpoints of the tangency chords WY and XZ respectively and P is the point of intersection of WY and XZ. In isosceles triangles WJY and XKZ, $IJ \perp WY$ and $IK \perp XZ$. Hence opposite angles IGP and IHP in quadrilateral $GIHP$ are right angles, so by the sum of angles in quadrilateral $GIHP$,

$$\angle JIK = \angle GIH = 2\pi - 2 \cdot \frac{\pi}{2} - \angle WPZ.$$

Hence we have

$$\angle JIK = \frac{\pi}{2} \quad \Leftrightarrow \quad \angle WPZ = \frac{\pi}{2} \quad \Leftrightarrow \quad WY \perp XZ$$

and according to [4, 9, 13] the tangency chords in a tangential quadrilateral are perpendicular if and only if it is cyclic[4]. □

Now we are ready for the main theorem in this paper, our second characterization of bicentric quadrilaterals. The direct part of the theorem was a problem

[4]This was also mentioned in the introduction to this paper.

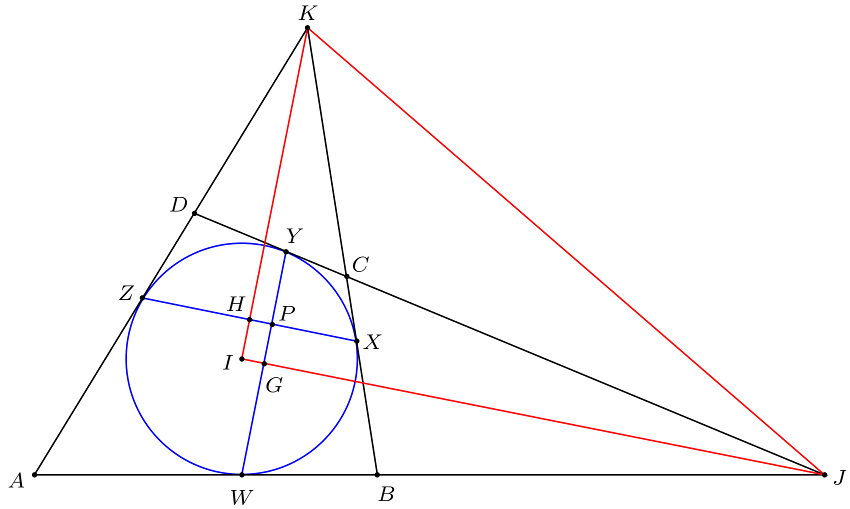

Figure 8. $ABCD$ is cyclic iff JIK is a right angle

studied at [2]. The *Newton line* [5] of a quadrilateral is the line defined by the midpoints of the two diagonals.

Theorem 6. *A tangential quadrilateral is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.*

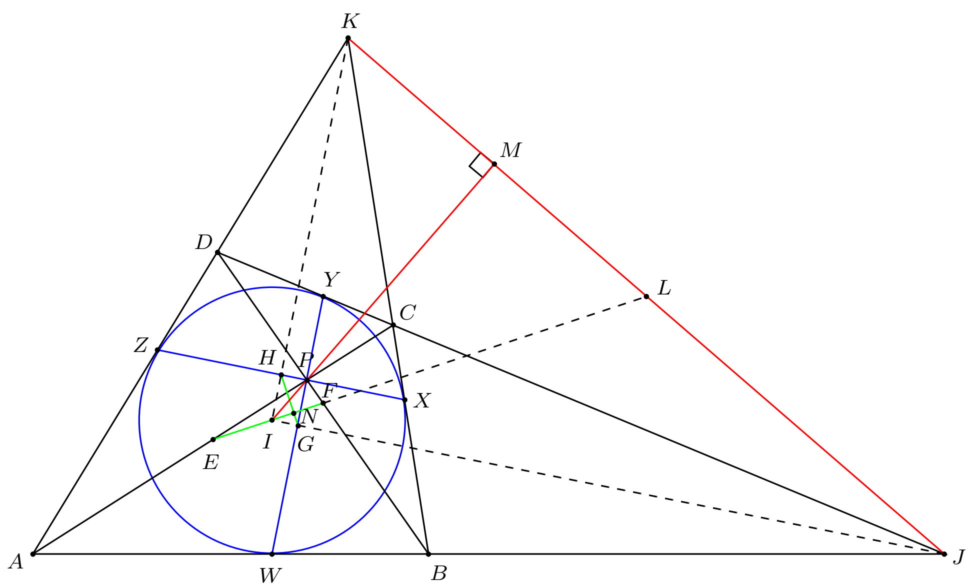

Figure 9. The Newton lines in $ABCD$ and $WXYZ$

[5]It is sometimes known as the Newton-Gauss line.

Proof. We use notations as in Figure 9, where P is the point where both the diagonals and the tangency chords intersect (see [10, 15, 17]) and L is the midpoint of JK. If I is the incenter, then the points E, I, F and L are collinear on the Newton line, see Newton's theorem in [7, p.15] (this is proved in two different theorems in [1, p.42] [6] and [17, p.169]). Let M be the intersection of JK and the extension of IP. By Corollary 4 $IM \perp JK$. In isosceles triangles ZKX and WJY, $IK \perp ZX$ and $IJ \perp WY$.

Since it has two opposite right angles ($\angle IHP$ and $\angle IGP$), the quadrilateral $GIHP$ is cyclic, so $\angle HGI = \angle HPI$. From the sum of angles in a triangle, we have

$$\angle ING = \pi - (\angle GIF + \angle HGI) = \pi - (\angle JIL + \angle HPI)$$

where N is the intersection of EF and GH. Thus

$$\angle ING = \pi - \angle JIL - \left(\frac{\pi}{2} - \angle HIP\right) = \frac{\pi}{2} - \angle JIL + \angle KIM.$$

So far we have only used properties of tangential quadrilaterals, so

$$\angle ING = \frac{\pi}{2} - \angle JIL + \angle KIM$$

is valid in all tangential quadrilaterals where no pair of opposite sides are parallel[7]. Hence we have

$$EF \perp GH \quad \Leftrightarrow \quad \angle ING = \frac{\pi}{2} \quad \Leftrightarrow \quad \angle JIL = \angle KIM \quad \Leftrightarrow \quad \angle JIK = \frac{\pi}{2}$$

where the last equivalence is due to Theorem 1 and the fact that $IM \perp JK$ (Corollary 4). According to Theorem 5, $\angle JIK = \frac{\pi}{2}$ if and only if the tangential quadrilateral is also cyclic.

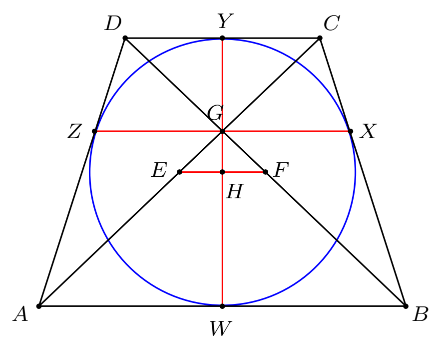

Figure 10. An isosceles tangential trapezoid

It remains to concider the case when at least one pair of opposite sides are parallel. Then the tangential quadrilateral is a trapezoid, so[8]

$$A + D = B + C \quad \Leftrightarrow \quad A - B = C - D.$$

[6] That the incenter I lies on the Newton line EF is actually a solved problem in this book.

[7] Otherwise at least one of the points J and K do not exist.

[8] We suppose without loss of generality that $AB \parallel CD$.

The trapezoid has a circumcircle if and only if

$$A + C = B + D \quad \Leftrightarrow \quad A - B = D - C.$$

Hence the quadrilateral is bicentric if and only if

$$C - D = D - C \quad \Leftrightarrow \quad C = D \quad \Leftrightarrow \quad A = B,$$

that is, the quadrilateral is bicentric if and only if it is an isosceles tangential trapezoid. In these $EF \perp GH$ (see Figure 10, where $EF \parallel AB$ and $GH \perp AB$) completing the proof. □

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Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: `martin.markaryd@hotmail.com`