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The Circles of Lester, Evans, Parry, and Their Generalizations

Paul Yiu

Abstract. Beginning with the famous Lester circle containing the circumcenter, nine-point center and the two Fermat points of a triangle, we survey a number of interesting circles in triangle geometry.

1. Introduction

This paper treats a number of interesting circles discovered by June Lester, Lawrence Evans, and Cyril Parry. We prove their existence and establish their equations. Lester [12] has discovered that the Fermat points, the circumcenter, and the nine-point center are concyclic. We call this the first Lester circle, and study it in §§3 – 6. Lester also conjectured in [12] the existence of a circle through the symmedian point, the Feuerbach point, the Clawson point, and the homothetic center of the orthic and the intangent triangles. This conjecture is validated in §15. Evans, during the preparation of his papers in Forum Geometricorum, has communicated two conjectures on circles through two perspectors $V_{\pm}$ which has since borne his name. In §9 we study in detail the first Evans circle in relation to the excentral circle. The second one is established in §14. In [9], a great number of circles have been reported relating to the Parry point, a point on the circumcircle. These circles are studied in §§10 – 12.

CONTENTS

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This paper is an extended version of a presentation with the same title at the Invited Paper Session: Classical Euclidean Geometry in MathFest, July 31–August 2, 2008 Madison, Wisconsin, USA. Thanks are due to Nikolaos Dergiades for many improvements of the paper, especially on the proof of Theorem 29.

2. Preliminaries

We refer to [15] for the standard notations of triangle geometry. Given a triangle ABC, with sidelengths a, b, c, the circumcircle is represented in homogeneous barycentric coordinates by the equation

$$a^2yz + b^2zx + c^2xy = 0.$$

The equation of a general circle $\mathcal{C}$ is of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z) \cdot L(x, y, z) = 0 \tag{1}$$

where $L(x, y, z)$ is a linear form, and the line $L(x, y, z) = 0$ is the radical axis of the circle $\mathcal{C}$ and the circumcircle.

2.1. *Intersection of a circle with the circumcircle* . The intersections of the circle $\mathcal{C}$ with the circumcircle can certainly be determined by solving the equations

$$\begin{aligned} a^2yz + b^2zx + c^2xy =& \ 0, \\ L(x, y, z) =& \ 0 \end{aligned}$$

simultaneously. Here is an interesting special case where these intersections can be easily identified. We say that a triangle center function $f(a,b,c)$ *represents an infinite point* if

$$f(a,b,c) + f(b,c,a) + f(c,a,b) = 0. \tag{2}$$

Proposition 1. *If a circle $\mathcal{C}$ is represented by an equation* (1) *in which*

$$L(x,y,z) = F(a,b,c) \cdot \sum_{\text{cyclic}} b^2c^2 \cdot f(a,b,c) \cdot g(a,b,c)x, \tag{3}$$

where $F(a,b,c)$ is symmetric in a, b, c, and $f(a,b,c)$, $g(a,b,c)$ are triangle center functions representing infinite points, then the circle intersects the circumcircle at the points

$$Q_f := \left(\frac{a^2}{f(a,b,c)} : \frac{b^2}{f(b,c,a)} : \frac{c^2}{f(c,a,b)}\right)$$

and

$$Q_g := \left(\frac{a^2}{g(a,b,c)} : \frac{b^2}{g(b,c,a)} : \frac{c^2}{g(c,a,b)}\right).$$

Proof. The line $L(x,y,z) = 0$ clearly contains the point Q_f, which by (2) is the isogonal conjugate of an infinite point, and so lies on the circumcircle. It is therefore a common point of the circumcircle and $\mathcal{C}$. The same reasoning applies to the point Q_g. □

For an application, see Remark after Proposition 11.

2.2. *Construction of circle equation.* Suppose we know the equation of a circle through two points Q_1 and Q_2, in the form of (1), and the equation of the line Q_1Q_2, in the form $L'(x,y,z) = 0$. To determine the equation of the circle through Q_1, Q_2 and a third point $Q = (x_0,\ y_0,\ z_0)$ not on the line Q_1Q_2, it is enough to find t such that

$$a^2y_0z_0 + b^2z_0x_0 + c^2x_0y_0 + (x_0+y_0+z_0)(L(x_0,y_0,z_0) + t \cdot L'(x_0,y_0,z_0)) = 0.$$

With this value of t, the equation

$$a^2yz + b^2zx + c^2xy + (x+y+z)(L(x,y,z) + t \cdot L'(x,y,z)) = 0$$

represents the circle Q_1Q_2Q. For an application of this method, see §6.3 (11) and Proposition 11.

2.3. *Some common triangle center functions.* We list some frequently occurring homogeneous functions associated with the coordinates of triangle centers or coefficients in equations of lines and circles. An asterisk indicates that the function represents an infinite point.

Quartic forms

	$f_{4,1} := a^2(b^2+c^2)-(b^2-c^2)^2$
	$f_{4,2} := a^2(b^2+c^2)-(b^4+c^4)$
	$f_{4,3} := a^4-(b^4-b^2c^2+c^4)$
	$f_{4,4} := (b^2+c^2-a^2)^2-b^2c^2$
*	$f_{4,5} := 2a^4-a^2(b^2+c^2)-(b^2-c^2)^2$
	$f_{4,6} := 2a^4-3a^2(b^2+c^2)+(b^2-c^2)^2$
*	$f_{4,7} := 2a^4-2a^2(b^2+c^2)-(b^4-4b^2c^2+c^4)$

Sextic forms

	$f_{6,1} := a^6-3a^4(b^2+c^2)+a^2(3b^4-b^2c^2+3c^4)-(b^2+c^2)(b^2-c^2)^2$
*	$f_{6,2} := 2a^6-2a^4(b^2+c^2)+a^2(b^4+c^4)-(b^2+c^2)(b^2-c^2)^2$
*	$f_{6,3} := 2a^6-6a^4(b^2+c^2)+9a^2(b^4+c^4)-(b^2+c^2)^3$

Octic forms

	$f_{8,1} :=$	$a^8-2a^6(b^2+c^2)+a^4b^2c^2+a^2(b^2+c^2)(2b^4-b^2c^2+2c^4)$ $-(b^8-2b^6c^2+6b^4c^4-2b^2c^6+c^8)$
*	$f_{8,2} :=$	$2a^8-2a^6(b^2+c^2)-a^4(3b^4-8b^2c^2+3c^4)$ $+4a^2(b^2+c^2)(b^2-c^2)^2-(b^2-c^2)^2(b^4+4b^2c^2+c^4)$
*	$f_{8,3} :=$	$2a^8-5a^6(b^2+c^2)+a^4(3b^4+8b^2c^2+3c^4)$ $+a^2(b^2+c^2)(b^4-5b^2c^2+c^4)-(b^2-c^2)^4$
	$f_{8,4} :=$	$3a^8-8a^6(b^2+c^2)+a^4(8b^4+7b^2c^2+8c^4)$ $-a^2(b^2+c^2)(4b^4-3b^2c^2+4c^4)+(b^4-c^4)^2$

3. The first Lester circle

Theorem 2 (Lester)**.** *The Fermat points, the circumcenter, and the nine-point center of a triangle are concyclic.*

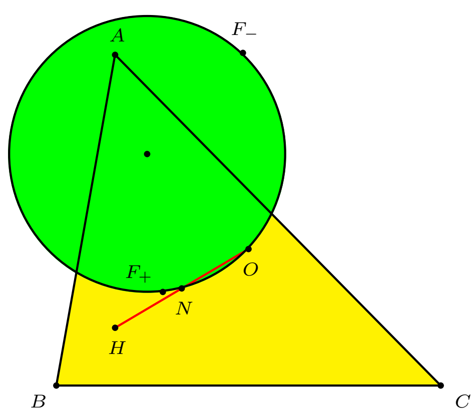

Figure 1. The first Lester circle through O, N and the Fermat points

Our starting point is a simple observation that the line joining the Fermat points intersects the Euler line at the midpoint of the orthocenter H and the centroid G.

Clearing denominators in the homogeneous barycentric coordinates of the Fermat point

$$F_+ = \left(\frac{1}{\sqrt{3}S_A + S}, \frac{1}{\sqrt{3}S_B + S}, \frac{1}{\sqrt{3}S_C + S}\right),$$

we rewrite it in the form

$$F_+ = (3S_{BC}+S^2, 3S_{CA}+S^2, 3S_{AB}+S^2)+\sqrt{3}S(S_B+S_C, S_C+S_A, S_A+S_B).$$

This expression shows that F_+ is a point of the line joining the symmedian point

$$K = (S_B + S_C, S_C + S_A, S_A + S_B)$$

to the point

$$\begin{aligned} M &= (3S_{BC} + S^2, 3S_{CA} + S^2, 3S_{AB} + S^2) \\ &= 3(S_{BC}, S_{CA}, S_{AB}) + S^2(1, 1, 1). \end{aligned}$$

Note that M is the midpoint of the segment HG, where $H = (S_{BC}, S_{CA}, S_{AB})$ is the orthocenter and $G = (1, 1, 1)$ is the centroid. It is the center of the orthocentroidal circle with HG as diameter. Indeed, F_+ divides MK in the ratio

$$MF_+ : F_+K = 2\sqrt{3}S(S_A + S_B + S_C) : 6S^2 = (S_A + S_B + S_C) : \sqrt{3}S.$$

With an obvious change in sign, we also have the negative Fermat point F_- dividing MK in the ratio

$$MF_- : F_-K = (S_A + S_B + S_C) : -\sqrt{3}S.$$

We have therefore established

Proposition 3. *The Fermat points divide MK harmonically.*

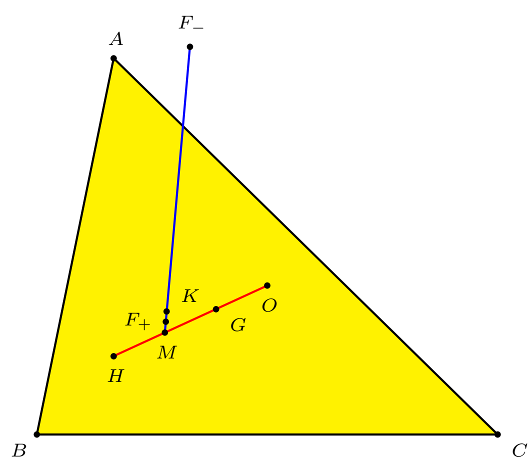

Figure 2. Intersection of Fermat line and Euler line

This simple observation suggests a proof of Lester's circle theorem by the intersection chords theorem.

Proposition 4. *The following statements are equivalent.*
(A) $MF_+ \cdot MF_- = MO \cdot MN$.
(B) *The circle F_+F_-G is tangent to the Euler line at G, i.e., $MF_+ \cdot MF_- = MG^2$.*
(C) *The circle F_+F_-H is tangent to the Euler line at H, i.e., $MF_+ \cdot MF_- = MH^2$.*
(D) *The Fermat points are inverse in the orthocentroidal circle.*

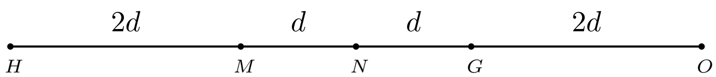

Figure 3. The Euler line

Proof. Since M is the midpoint of HG, the statements (B), (C), (D) are clearly equivalent. On the other hand, putting $OH = 6d$, we have

$$MO \cdot MN = (MH)^2 = (MG)^2 = 4d^2,$$

see Figure 3. This shows that (A), (B), (C) are equivalent. □

Note that (A) is Lester's circle theorem (Theorem 2). To complete its proof, it is enough to prove (D). We do this by a routine calculation.

Theorem 5. *The Fermat points are inverse in the orthocentroidal circle.*

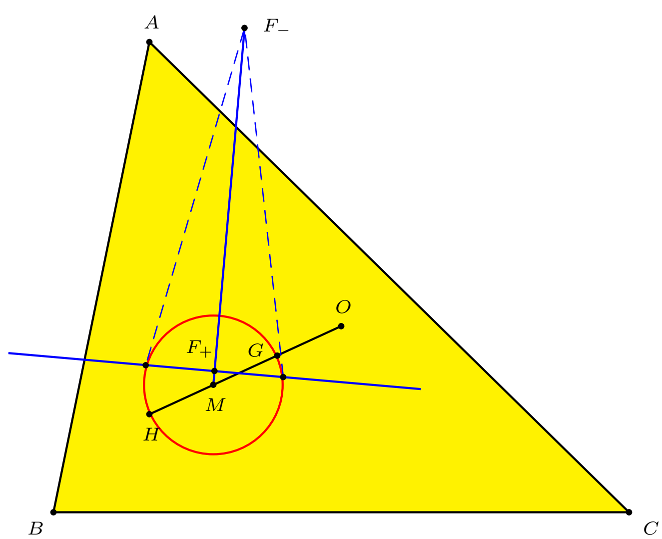

Figure 4. F_+ on the polar of F_- in the orthocentroidal circle

Proof. The equation of the orthocentroidal circle is

$$3(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(S_Ax + S_By + S_Cz) = 0,$$

equivalently, [1]

$$-2(S_Ax^2+S_By^2+S_Cz^2)+((S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy)=0.$$

This is represented by the matrix

$$\mathsf{M}=\begin{pmatrix}-4S_A & S_A+S_B & S_A+S_C\\ S_A+S_B & -4S_B & S_B+S_C\\ S_A+S_C & S_B+S_C & -4S_C\end{pmatrix}.$$

The coordinates of the Fermat points can be written as

$$F_+=X+Y \quad \text{and} \quad F_-=X-Y,$$

with

$$X=\left(3S_{BC}+S^2 \quad 3S_{CA}+S^2 \quad 3S_{AB}+S^2\right),$$
$$Y=\sqrt{3}S\left(S_B+S_C \quad S_C+S_A \quad S_A+S_B\right).$$

With these, we have

$$X\mathsf{M}X^t=Y\mathsf{M}Y^t=6S^2(S_A(S_B-S_C)^2+S_B(S_C-S_A)^2+S_C(S_A-S_B)^2),$$

and

$$F_+\mathsf{M}F_-^t=(X+Y)\mathsf{M}(X-Y)^t=X\mathsf{M}X^t-Y\mathsf{M}Y^t=0.$$

This shows that the Fermat points are inverse in the orthocentroidal circle. □

The proof of Theorem 2 is now complete, along with tangency of the Euler line with the two circles F_+F_-G and F_+F_-H (see Figure 5). We call the circle through O, N, and $F_\pm$ the first Lester circle.

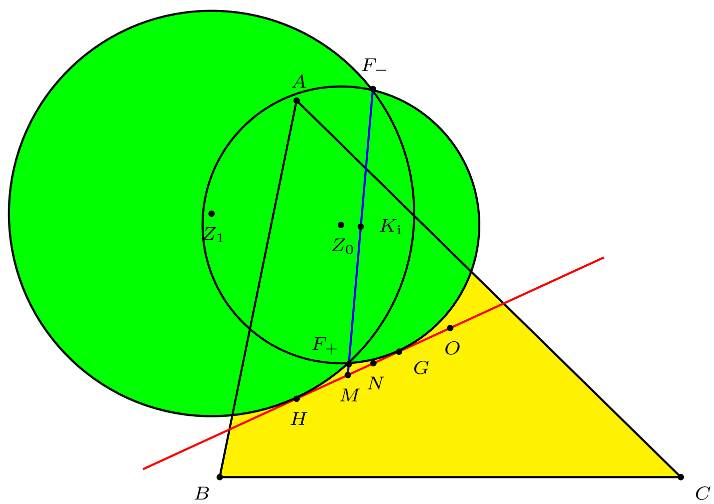

Figure 5. The circles F_+F_-G and F_+F_-H

[1]It is easy to see that this circle contains H and G. The center of the circle (see [15, §10.7.2]) is the point $M=(S_A(S_B+S_C)+4S_{BC}:S_B(S_C+S_A)+4S_{CA}:S_C(S_A+S_B)+4S_{AB})$ on the Euler line, which is necessarily the midpoint of HG.

Remarks. (1) The tangency of the circle F_+F_-G and the Euler line was noted in [9, pp.229–230].

(2) The symmedian point K and the Kiepert center K_i (which is the midpoint of F_+F_-) are inverse in the orthocentroidal circle.

4. Gibert's generalization of the first Lester circle

Bernard Gibert [7] has found an interesting generalization of the first Lester circle, which we explain as a natural outgrowth of an attempt to compute the equations of the circles F_+F_-G and F_+F_-H.

Theorem 6 (Gibert)**.** *Every circle whose diameter is a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points.*

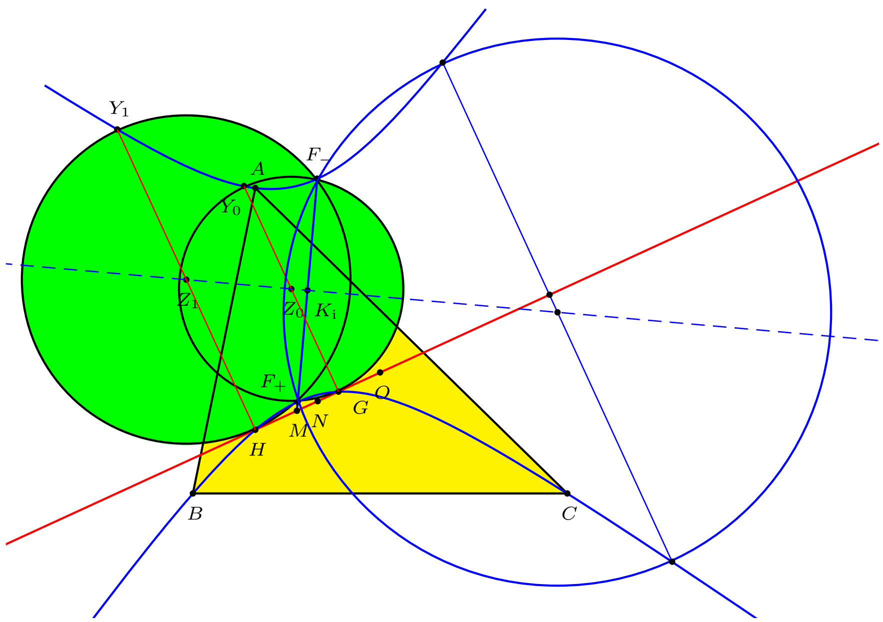

Figure 6. Gibert's generalization of the first Lester circle

Proof. Since $F_\pm$ and G are on the Kiepert hyperbola, and the center of the circle F_+F_-G is on the perpendicular to the Euler line at G, this line intersects the Kiepert hyperbola at a fourth point Y_0 (see Figure 6), and the circle is a member of the pencil of conics through F_+, F_-, G and Y_0. Let $L(x, y, z) = 0$ and $L_0(x, y, z) = 0$ represent the lines F_+F_- and GY_0 respectively. We may assume the circle given by

$$k_0((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_0(x, y, z) = 0$$

for an appropriately chosen constant k_0.

Replacing G by H and Y_0 by another point Y_1, the intersection of the Kiepert hyperbola with the perpendicular to the Euler line at H, we write the equation of the circle F_+F_-H in the form

$$k_1((b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy)-L(x,y,z)\cdot L_1(x,y,z)=0,$$

where $L_1(x,y,z)=0$ is the equation of the line HY_1.

The midpoints of the two chords GY_0 and HY_1 are the centers of the two circles F_+F_-G and F_+F_-H. The line joining them is therefore the perpendicular bisector of F_+F_-.

Every line perpendicular to the Euler line is represented by an equation

$$L_t(x,y,z):=tL_0(x,y,z)+(1-t)L_1(x,y,z)=0$$

for some real number t. Let $k_t := tk_0+(1-t)k_1$ correspondingly. Then the equation

$$k_t((b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy)-L(x,y,z)\cdot L_t(x,y,z)=0$$

represents a circle $\mathcal{C}_t$ through the Fermat points and the intersections of the line $L_t(x,y,z)=0$ and the Kiepert hyperbola. The perpendicular bisector of F_+F_- is the diameter of the family of parallel lines $L_t(x,y,z)=0$. Therefore the center of the circle is the midpoint of the chord cut out by $L_t(x,y,z)=0$. □

Remark. If the perpendicular to the Euler line intersects it outside the segment HG, then the circle intersects the Euler line at two points dividing the segment HG harmonically, say in the ratio $\tau : 1-\tau$ for $\tau<0$ or $\tau>1$. In this case, the line divides HG in the ratio $-\tau^2 : (1-\tau)^2$.

5. Center of the first Lester circle

Since the circumcenter O and the nine-point center N divides the segment HG in the ratio $3 : \mp 1$, the diameter of the first Lester circle perpendicular to the Euler line intersects the latter at the point L dividing HG in the ratio $9 : -1$. This is the midpoint of ON (see Figure 7), and has coordinates

$$(f_{4,6}(a,b,c) : f_{4,6}(b,c,a) : f_{4,6}(c,a,b)).$$

As such it is the nine-point center of the medial triangle, and appears as X_{140} in [10].

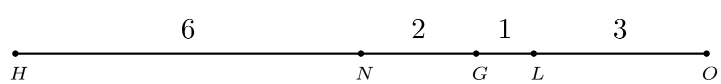

Figure 7. The Euler line

Proposition 7. (a) *Lines perpendicular to the Euler line have infinite point*

$$X_{523}=(b^2-c^2,\ c^2-a^2,\ a^2-b^2).$$

(b) *The diameter of the first Lester circle perpendicular to the Euler line is along the line*

$$\sum_{\text{cyclic}} f_{6,1}(a,b,c)x = 0. \tag{4}$$

Proposition 8. (a) *The equation of the line F_+F_- is*

$$\sum_{\text{cyclic}} (b^2-c^2)f_{4,4}(a,b,c)x = 0. \tag{5}$$

(b) *The perpendicular bisector of F_+F_- is the line*

$$\frac{x}{b^2-c^2} + \frac{y}{c^2-a^2} + \frac{z}{a^2-b^2} = 0. \tag{6}$$

Proof. (a) The line F_+F_- contains the symmedian point K and the Kiepert center

$$K_{\rm i} = ((b^2-c^2)^2,\ (c^2-a^2)^2,\ (a^2-b^2)^2).$$

(b) The perpendicular bisector of F_+F_- is the perpendicular at $K_{\rm i}$ to the line $KK_{\rm i}$, which has infinite point

$$X_{690} = ((b^2-c^2)(b^2+c^2-2a^2),\ (c^2-a^2)(c^2+a^2-2b^2),\ (a^2-b^2)(a^2+b^2-2c^2)).$$

□

Proposition 9. *The center of the first Lester circle has homogeneous barycentric coordinates*

$$((b^2-c^2)f_{8,3}(a,b,c) : (c^2-a^2)f_{8,3}(b,c,a) : (a^2-b^2)f_{8,3}(c,a,b)).$$

Proof. This is the intersection of the lines (4) and (6). □

Remarks. (1) The center of the first Lester circle appears as X_{1116} in [10].

(2) The perpendicular bisector of F_+F_- also contains the Jerabek center

$$J_{\rm e} = ((b^2-c^2)^2(b^2+c^2-a^2), (c^2-a^2)^2(c^2+a^2-b^2), (a^2-b^2)^2(a^2+b^2-c^2)),$$

which is the center of the Jerabek hyperbola, the isogonal conjugate of the Euler line. It follows that $J_{\rm e}$ is equidistant from the Fermat points. The points $K_{\rm i}$ and $J_{\rm e}$ are the common points of the nine-point circle and the pedal circle of the centroid.

6. Equations of circles

6.1. *The circle F_+F_-G.* In the proof of Theorem 6, we take

$$L(x,y,z) = \sum_{\text{cyclic}} (b^2-c^2)f_{4,4}(a,b,c)x, \tag{7}$$

$$L_0(x,y,z) = \sum_{\text{cyclic}} (b^2+c^2-2a^2)x \tag{8}$$

for the equation of the line F_+F_- (Proposition 8(a)) and the perpendicular to the Euler line at G. Now, we seek a quantity k_0 such that the member

$$k_0((b^2-c^2)yz + (c^2-a^2)zx + (a^2-b^2)xy) - L(x,y,z)\cdot L_0(x,y,z) = 0$$

of the pencil of conic through the four points $F_{\pm}$, G, Y_0 is a circle. For this,

$$k_0 = -3(a^2(c^2-a^2)(a^2-b^2) + b^2(a^2-b^2)(b^2-c^2) + c^2(b^2-c^2)(c^2-a^2)),$$

and the equation can be reorganized as

$$9(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}}(b^2-c^2)(b^2+c^2-2a^2)f_{4,4}(a,b,c)x\right) = 0. \tag{9}$$

The center of the circle F_+F_-G is the point

$$Z_0 := ((b^2-c^2)f_{4,7}(a,b,c) : (c^2-a^2)f_{4,7}(b,c,a) : (a^2-b^2)f_{4,7}(c,a,b)).$$

The point Y_0 has coordinates $\left(\frac{b^2-c^2}{b^2+c^2-2a^2} : \cdots : \cdots\right)$.

6.2. *The circle* F_+F_-H. With the line

$$L_1(x,y,z) = \sum_{\text{cyclic}}(b^2+c^2-a^2)(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)x = 0$$

perpendicular to the Euler line at H, we seek a number k_1 such that

$$k_1((b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy) - L(x,y,z)\cdot L_1(x,y,z) = 0$$

of the pencil of conic through the four points $F_{\pm}$, H, Y_1 is a circle. For this,

$$k_1 = 16\Delta^2(a^4(b^2+c^2-a^2)+b^4(c^2+a^2-b^2)+c^4(a^2+b^2-c^2)-3a^2b^2c^2),$$

and the equation can be reorganized as

$$48(b^2-c^2)(c^2-a^2)(a^2-b^2)\Delta^2(a^2yz+b^2zx+c^2xy) - (x+y+z)\left(\sum_{\text{cyclic}}(b^2-c^2)(b^2+c^2-a^2)f_{4,4}(a,b,c)f_{4,5}(a,b,c)x\right) = 0. \tag{10}$$

This is the equation of the circle F_+F_-H. The center is the point

$$Z_1 := ((b^2-c^2)f_{8,2}(a,b,c) : (c^2-a^2)f_{8,2}(b,c,a) : (a^2-b^2)f_{8,2}(c,a,b)).$$

The triangle center

$$Y_1 = \left(\frac{b^2-c^2}{f_{4,5}(a,b,c)} : \frac{c^2-a^2}{f_{4,5}(b,c,a)} : \frac{a^2-b^2}{f_{4,5}(c,a,b)}\right)$$

is X_{2394}.

6.3. *The first Lester circle.* Since the line joining the Fermat points has equation $L(x,y,z) = 0$ with L given by (7), every circle through the Fermat points is represented by

$$9(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}}(b^2-c^2)(b^2+c^2-2a^2+t)f_{4,4}(a,b,c)x\right) = 0 \quad (11)$$

for an appropriate choice of t. The value of t for which this circle passes through the circumcenter is

$$t = \frac{a^2(c^2-a^2)(a^2-b^2)+b^2(a^2-b^2)(b^2-c^2)+c^2(b^2-c^2)(c^2-a^2)}{32\Delta^2}.$$

The equation of the circle is

$$96\Delta^2(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}}(b^2-c^2)f_{4,4}(a,b,c)f_{6,1}(a,b,c)x\right) = 0.$$

7. The Brocard axis and the Brocard circle

7.1. *The Brocard axis* . The isogonal conjugate of the Kiepert perspector $K(\theta)$ is the point

$$K^*(\theta) = (a^2(S_A+S_\theta),\ b^2(S_B+S_\theta),\ c^2(S_C+S_\theta)),$$

which lies on the line joining the circumcenter O and the symmedian point K. The line OK is called the Brocard axis. It is represented by the equation

$$\sum_{\text{cyclic}} b^2c^2(b^2-c^2)x = 0. \quad (12)$$

7.2. *The Brocard circle.* The Brocard circle is the circle with OK as diameter. It is represented by the equation

$$(a^2+b^2+c^2)(a^2yz+b^2zx+c^2xy)-(x+y+z)(b^2c^2x+c^2a^2y+a^2b^2z) = 0. \quad (13)$$

It is clear from

$$K^*(\theta) = (a^2S_A,\ b^2S_B,\ c^2S_C) + S_\theta(a^2,\ b^2,\ c^2),$$
$$K^*(-\theta) = (a^2S_A,\ b^2S_B,\ c^2S_C) - S_\theta(a^2,\ b^2,\ c^2)$$

that $K^*(\theta)$ and $K^*(-\theta)$ divide O and K harmonically, and so are inverse in the Brocard circle. The points $K^*\left(\pm\frac{\pi}{3}\right)$ are called the isodynamic points, and are more simply denoted by $J_\pm$.

Proposition 10. *$K^*(\pm\theta)$ are inverse in the circumcircle if and only if they are the isodynamic points.*

7.3. *The isodynamic points.* The isodynamic points $J_\pm$ are also the common points of the three Apollonian circles, each orthogonal to the circumcircle at a vertex (see Figure 8). Thus, the A-Apollonian circle has diameter the endpoints of the bisectors of angle A on the sidelines BC. These are the points $(b,\ \pm c)$. The center of the circle is the midpoint of these, namely, $(b^2,\ -c^2)$. The circle has equation

$$(b^2-c^2)(a^2yz+b^2zx+c^2xy)+a^2(x+y+z)(c^2y-b^2z)=0.$$

Similarly, the B- and C-Apollonian circles have equations

$$(c^2-a^2)(a^2yz+b^2zx+c^2xy)+b^2(x+y+z)(a^2z-c^2x)=0,$$
$$(a^2-b^2)(a^2yz+b^2zx+c^2xy)+c^2(x+y+z)(b^2x-a^2y)=0.$$

These three circles are coaxial. Their centers lie on the Lemoine axis

$$\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0, \tag{14}$$

which is the perpendicular bisector of the segment J_+J_-.

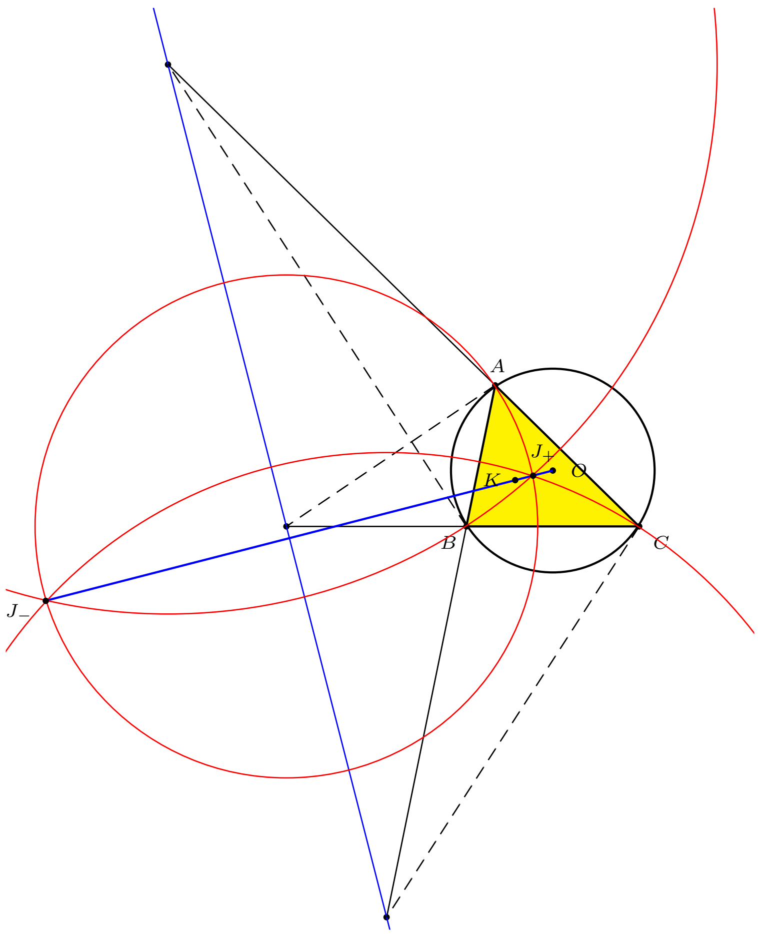

Figure 8. The Apollonian circles and the isodynamic points

Proposition 11. *Every circle through the isodynamic points can be represented by an equation*

$$3(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2+t)x\right) = 0 \qquad (15)$$

for some choice of t.

Proof. Combining the above equations for the three Apollonian circles, we obtain

$$3(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\sum_{\text{cyclic}} a^2(c^2-a^2)(a^2-b^2)(c^2y-b^2z) = 0.$$

A simple rearrangement of the terms brings the radical axis into the form

$$3(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)x\right) = 0. \qquad (16)$$

Now, the line containing the isodynamic points is the Brocard axis given by (12). It follows that every circle through $J_\pm$ is represented by (15) above for some choice of t (see §2.2). □

Remark. As is easily seen, equation (16) is satisfied by $x = y = z = 1$, and so represents the circle through $J_\pm$ and G. Since the factors $b^2 - c^2$ and $b^2 + c^2 - 2a^2$ yield infinite points, applying Proposition 1, we conclude that this circle intersects the circumcircle at the Euler reflection point $E = \left(\frac{a^2}{b^2-c^2} : \cdots : \cdots\right)$ and the Parry point $\left(\frac{a^2}{b^2+c^2-2a^2} : \cdots : \cdots\right)$.

This is the Parry circle we consider in §10 below.

Proposition 12. *The circle through the isodynamic points and the orthocenter has equation*

$$16\Delta^2 \cdot (b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-a^2)f_{4,1}(a,b,c)x\right) = 0.$$

Its center is the point

$$Z_3 := (a^2(b^2-c^2)(b^2+c^2-a^2)f_{4,1}(a,b,c) : \cdots : \cdots).$$

8. The excentral triangle

The excentral triangle $I_aI_bI_c$ has as vertices the excenters of triangle ABC. It has circumradius $2R$ and circumcenter I', the reflection of I in O (see Figure 9). Since the angles of the excentral triangle are $\frac{1}{2}(B+C)$, $\frac{1}{2}(C+A)$, and $\frac{1}{2}(A+B)$, its sidelengths $a' = I_bI_c$, $b' = I_cI_a$, $c' = I_aI_b$ satisfy

$$\begin{aligned} a'^2 : b'^2 : c'^2 &= \cos^2\frac{A}{2} : \cos^2\frac{B}{2} : \cos^2\frac{C}{2} \\ &= a(b+c-a) : b(c+a-b) : c(a+b-c). \end{aligned}$$

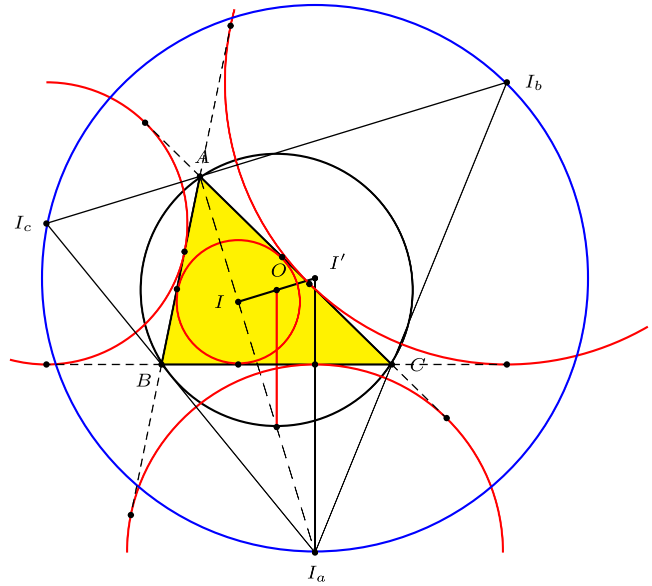

Figure 9. The excentral triangle and its circumcircle

8.1. *Change of coordinates.* A point with homogeneous barycentric coordinates (x, y, z) with reference to ABC has coordinates

$$(x', y', z') = (a(b+c-a)(cy+bz),\ b(c+a-b)(az+cx),\ c(a+b-c)(bx+ay))$$

with reference to the excentral triangle.

Consider, for example, the Lemoine axis of the excentral triangle, with equation

$$\frac{x'}{a'^2} + \frac{y'}{b'^2} + \frac{z'}{c'^2} = 0.$$

With reference to triangle ABC, the same line is represented by the equation

$$\frac{a(b+c-a)(cy+bz)}{a(b+c-a)} + \frac{b(c+a-b)(az+cx)}{b(c+a-b)} + \frac{c(a+b-c)(bx+ay)}{c(a+b-c)} = 0,$$

which simplifies into

$$(b+c)x + (c+a)y + (a+b)z = 0. \qquad (17)$$

On the other hand, the circumcircle of the excentral triangle, with equation

$$\frac{a'^2}{x'} + \frac{b'^2}{y'} + \frac{c'^2}{z'} = 0,$$

is represented by

$$\frac{1}{cy + bz} + \frac{1}{az + cx} + \frac{1}{bx + ay} = 0$$

with reference to triangle ABC. This can be rearranged as

$$a^2yz + b^2zx + c^2ay + (x + y + z)(bcx + cay + abz) = 0. \tag{18}$$

9. The first Evans circle

9.1. *The Evans perspector W.* Let A^*, B^*, C^* be respectively the reflections of A in BC, B in CA, C in AB. The triangle $A^*B^*C^*$ is called the triangle of reflections of ABC. Larry Evans has discovered the perspectivity of the excentral triangle and $A^*B^*C^*$.

Theorem 13. *The excentral triangle and the triangle of reflections are perspective at a point which is the inverse image of the incenter in the circumcircle of the excentral triangle.*

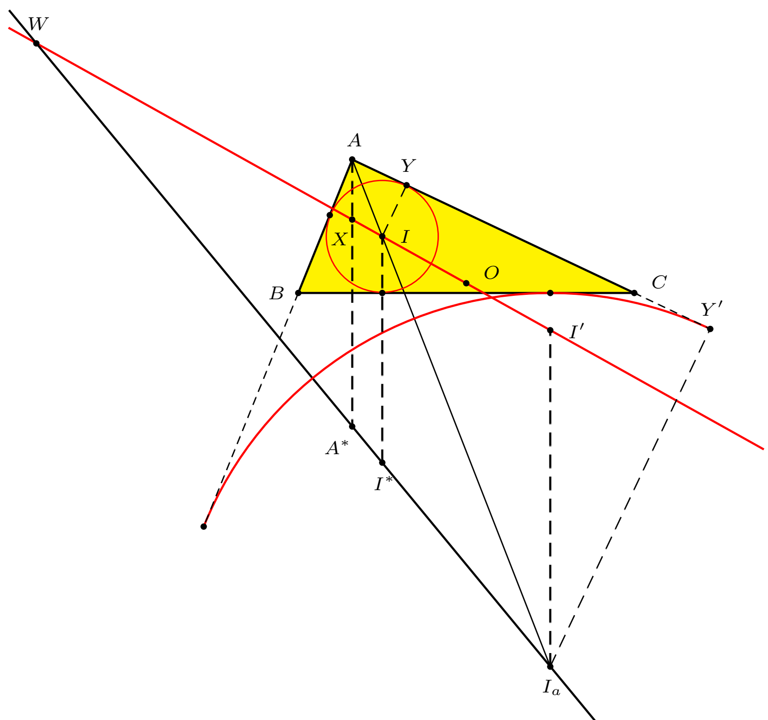

Figure 10. The Evans perspector W

Proof. We show that the lines I_aA^*, I_bB^*, and I_cC^* intersect the line OI at the same point. Let X be the intersection of the lines AA^* and OI (see Figure 10). If h_a is the A-altitude of triangle ABC, and the parallel from I to AA^* meets I_aA^* at I^*, then since

$$\frac{II^*}{2h_a} = \frac{II^*}{AA^*} = \frac{I_aI}{I_aA} = \frac{YY'}{AY'} = \frac{a}{s},$$

we have $II^* = \frac{2ah_a}{s} = 4r$ and $\frac{WI'}{WI} = \frac{I'I_a}{II^*} = \frac{2R}{4r}$. Therefore, W divides II' in the ratio

$$I'W : WI = R : -2r.$$

Since this ratio is a symmetric function of the sidelengths, we conclude that the same point W lies on the lines I_bB^* and I_cC^*. Moreover, since $I'W = \frac{R}{R-2r} \cdot I'I$, by the famous Euler formula $OI^2 = R(R-2r)$, we have

$$I'W \cdot I'I = \frac{R}{R-2r} \cdot I'I^2 = \frac{4R}{R-2r} \cdot OI^2 = \frac{4R}{R-2r} \cdot R(R-2r) = (2R)^2.$$

This shows that W and I are inverse in the circumcircle of the excentral triangle. □

The point W is called the Evans perspector; it has homogeneous barycentric coordinates,

$$\begin{aligned} W =& (a(a^3 + a^2(b+c) - a(b^2+bc+c^2) - (b+c)(b-c)^2) : \cdots : \cdots) \\ =& (a((a+b+c)(c+a-b)(a+b-c) - 3abc) : \cdots : \cdots). \end{aligned}$$

It appears as X_{484} in [10].

9.2. *Perspectivity of the excentral triangle and Kiepert triangles.*

Lemma 14. *Let XBC and $X'I_bI_c$ be oppositely oriented similar isosceles triangles with bases BC and I_bI_c respectively. The lines I_aX and I_aX' are isogonal with respect to angle I_a the excentral triangle* (see Figure 11).

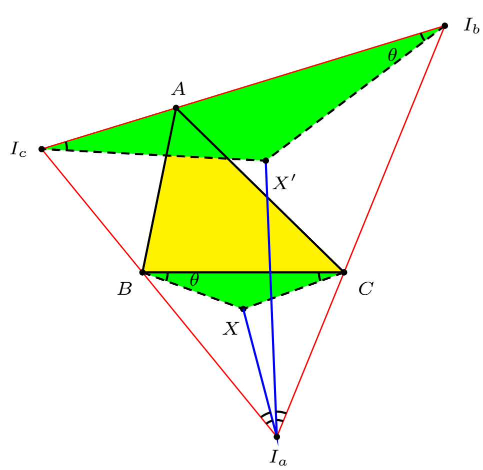

Figure 11. Isogonal lines joining I_a to apices of similar isosceles on BC and I_bI_c

Proof. The triangles I_aBC and $I_aI_bI_c$ are oppositely similar since BC and I_bI_c are antiparallel. In this similarity X and X' are homologous points. Hence, the lines I_aX and I_aX' are isogonal in the excentral triangle. $\square$

We shall denote Kiepert perspectors with reference to the excentral triangle by $K_\text{e}(-)$.

Theorem 15. *The excentral triangle and the Kiepert triangle $\mathcal{K}(\theta)$ are perspective at the isogonal conjugate of $K_\text{e}(-\theta)$ in the excentral triangle.*

Proof. Let XYZ be a Kiepert triangle $\mathcal{K}(\theta)$. Construct X', Y', Z' as in Lemma 14 (see Figure 11).

(i) I_aX', I_bY', I_cZ' concur at the Kiepert perspector $K_\text{e}(-\theta)$ of the excentral triangle.

(ii) Since I_aX and I_aX' are isogonal with respect to I_a, and similarly for the pairs I_bY, I_bY' and I_cZ and I_cZ', the lines I_aX, I_bY, I_cZ concur at the isogonal conjugate of $K_\text{e}(-\theta)$ in the excentral triangle. $\square$

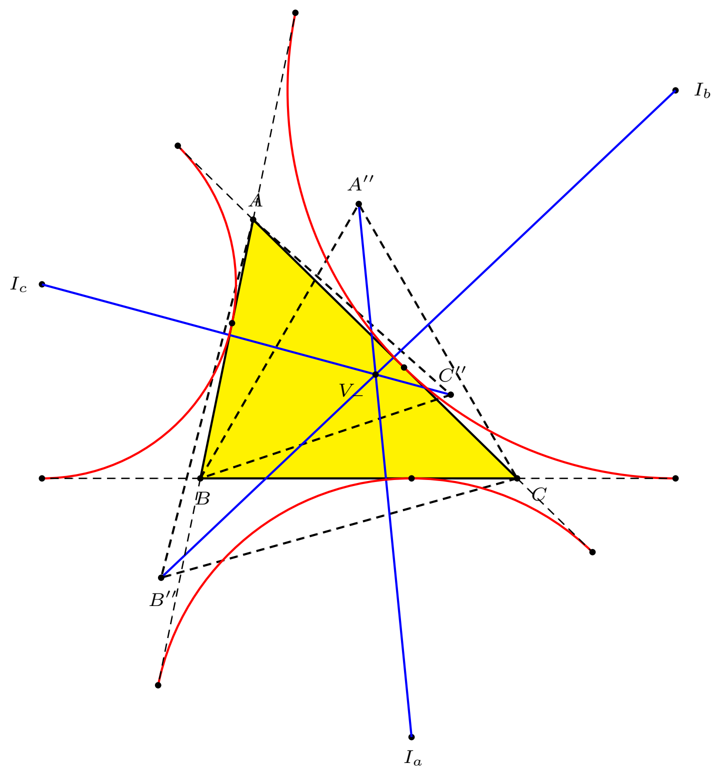

Figure 12. Evans' perspector V_- of $\mathcal{K}\left(-\frac{\pi}{3}\right)$ and excentral triangle

We denote the perspector in Theorem 15 by $V(\theta)$, and call this a generalized Evans perspector. In particular, $V\left(\frac{\pi}{3}\right)$ and $V\left(-\frac{\pi}{3}\right)$ are the isodynamic points of the excentral triangle, and are simply denoted by V_+ and V_- respectively (see

Figure 12 for V_-). These are called the second and third Evans perspectors respectively. They are X_{1276} and X_{1277} of [10].

Proposition 16. *The line V_+V_- has equation*

$$\sum_{\text{cyclic}} (b-c)(b^2+c^2-a^2)x = 0. \tag{19}$$

Proof. The line V_+V_- is the Brocard axis of the excentral triangle, with equation

$$\sum_{\text{cyclic}} \frac{b'^2-c'^2}{a'^2}\cdot x' = 0$$

with reference to the excentral triangle (see §7.1). Replacing these by parameter with reference to triangle ABC, we have $\sum_{\text{cyclic}}(b-c)(b+c-a)(cy+bz)=0$. Rearranging terms, we have the form (19) above. □

Proposition 17. *The Kiepert triangle $\mathcal{K}(\theta)$ is perspective with the triangle of reflections $A^*B^*C^*$ if and only if $\theta = \pm\frac{\pi}{3}$. The perspector is $K^*(-\theta)$, the isogonal conjugate of $K(-\theta)$.*

This means that for $\varepsilon = \pm 1$, the Fermat triangle $\mathcal{K}\left(\varepsilon\cdot\frac{\pi}{3}\right)$ and the triangle of reflections are perspective at the isodynamic point $J_{-\varepsilon}$ (see Figure 13 for the case $\varepsilon = -1$).

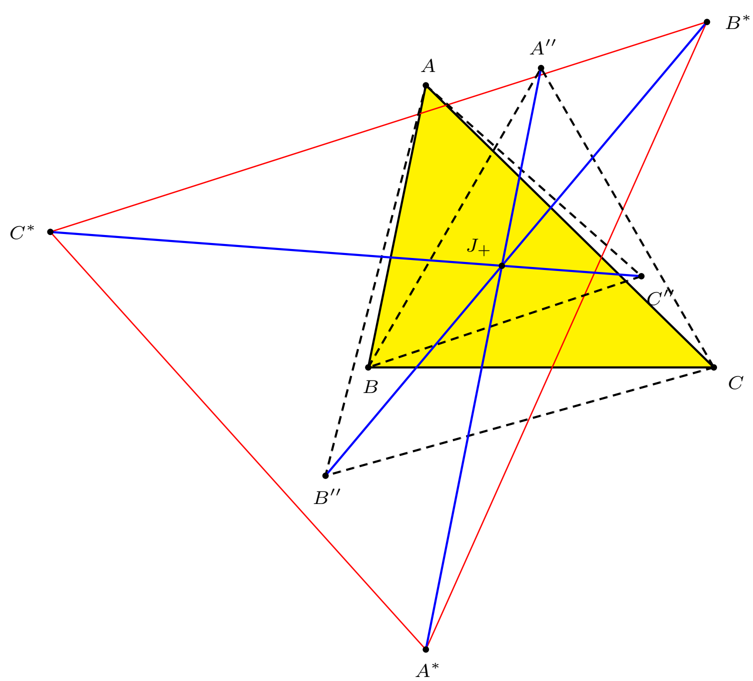

Figure 13. $\mathcal{K}\left(-\frac{\pi}{3}\right)$ and $A^*B^*C^*$ perspective at J_+

9.3. *The first Evans circle.* Since $V_{\pm}$ are the isodynamic points of the excentral triangle, they are inverse in the circumcircle of the excentral triangle. Since W and I are also inverse in the same circle, we conclude that V_+, V_-, I, and W are concyclic (see [1, Theorem 519]). We call this the first Evans circle. A stronger result holds in view of Proposition 10.

Theorem 18. *The four points $V(\theta)$, $V(-\theta)$, I, W are concyclic if and only if $\theta = \pm\frac{\pi}{3}$.*

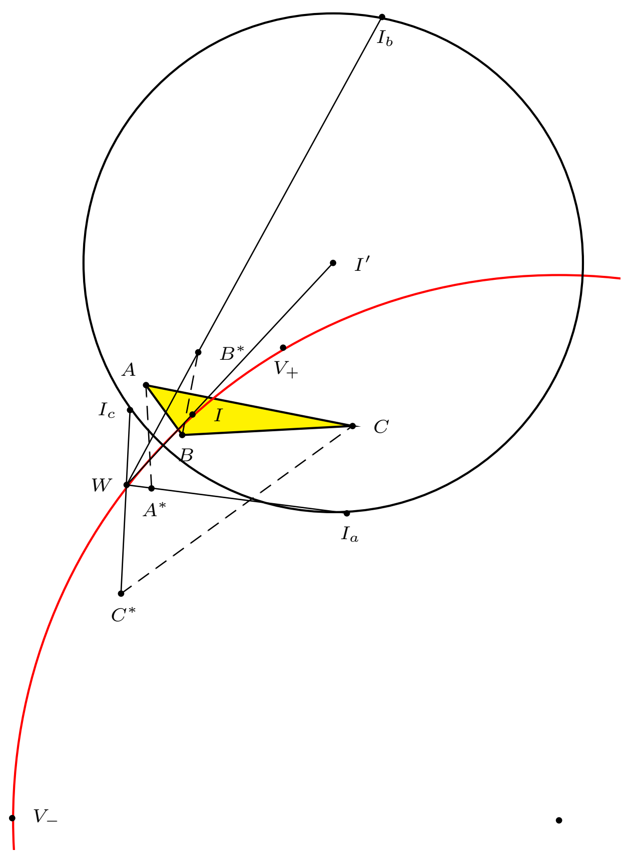

Figure 14. The first Evans circle

We determine the center of the first Evans circle as the intersection of the perpendicular bisectors of the segments IW and V_+V_-.

Lemma 19. *The perpendicular bisector of the segment IW is the line*

$$bc(b+c)x + ca(c+a)y + ab(a+b)z = 0. \tag{20}$$

Proof. If M' is the midpoint of IW, then since O is the midpoint of II', from the degenerate triangle $II'W$ we have $OM' = \frac{I'W}{2} = \frac{R^2}{OI}$. This shows that the midpoint of IW is the inverse of I in the circumcircle. Therefore, the perpendicular bisector of IW is the polar of I in the circumcircle. This is the line

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is the same as (20). □

Remark. $M' = (a^2(a^2-b^2+bc-c^2) : b^2(b^2-c^2+ca-a^2) : c^2(c^2-a^2+ab-b^2))$ is the triangle center X_{36} in [10].

Lemma 20. *The perpendicular bisector of the segment V_+V_- is the line*

$$(b+c)x + (c+a)y + (a+b)z = 0. \tag{21}$$

Proof. Since V_+ and V_- are the isodynamic points of the excentral triangle, the perpendicular bisector of V_+V_- is the polar of the symmedian point of the excentral triangle with respect to its own circumcircle. With reference to the excentral triangle, its Lemoine axis has equation

$$\frac{x'}{a'^2} + \frac{y'}{b'^2} + \frac{z'}{c'^2} = 0.$$

Changing coordinates, we have, with reference to ABC, the same line represented by the equation

$$\frac{a(b+c-a)(cy+bz)}{a(b+c-a)} + \frac{b(c+a-b)(az+cx)}{b(c+a-b)} + \frac{c(a+b-c)(bx+ay)}{c(a+b-c)} = 0,$$

which simplifies into (21). □

Proposition 21. *The center of the first Evans circle is the point*

$$\left(\frac{a(b-c)}{b+c} : \frac{b(c-a)}{c+a} : \frac{c(a-b)}{a+b}\right).$$

Proof. This is the intersection of the lines (20) and (21). □

Remark. The center of the first Evans circle is the point X_{1019} in [10]. It is also the perspector of excentral triangle and the cevian triangle of the Steiner point.

Proposition 22. *The equation of the first Evans circle is*

$$\begin{aligned} &(a-b)(b-c)(c-a)(a+b+c)(a^2yz+b^2zx+c^2xy) \\ -&(x+y+z)\left(\sum_{\text{cyclic}} bc(b-c)(c+a)(a+b)(b+c-a)x\right) = 0. \end{aligned} \tag{22}$$

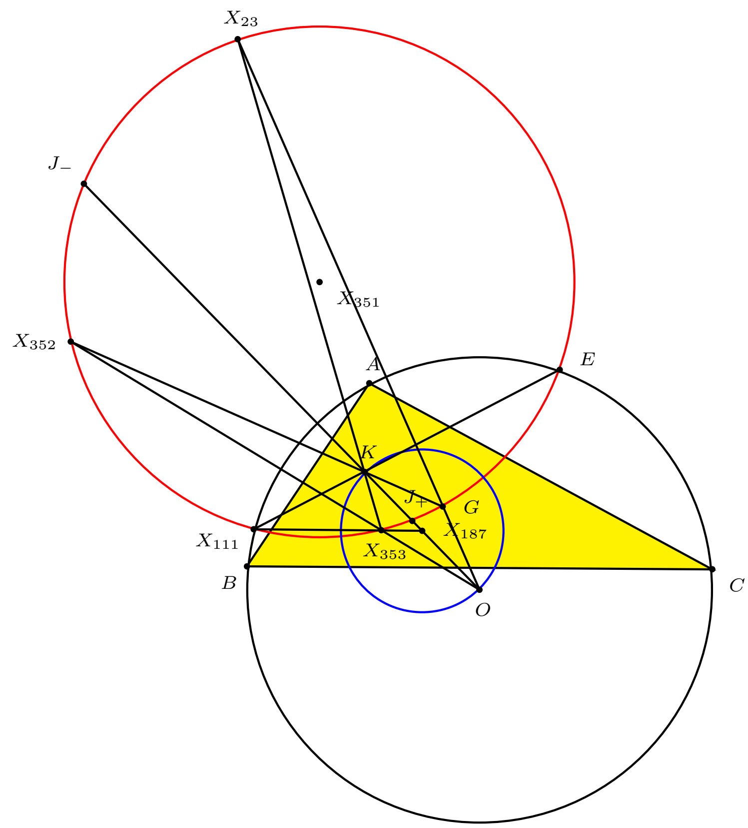

Figure 15. The Parry circle

10. The Parry circle and the Parry point

The Parry circle $\mathcal{C}_{\mathrm{P}}$, according to [9, p.227], is the circle through a number of interesting triangle centers, including the isodynamic points and the centroid. We shall define the Parry circle as the circle through these three points, and seek to explain the incidence of the other points.

First of all, since the isodynamic points are inverse in each of the circumcircle and the Brocard circle, the Parry circle is orthogonal to each of these circles. In particular, it contains the inverse of G in the circumcircle. This is the triangle center

$$X_{23} = (a^2(a^4-b^4+b^2c^2-c^4),\ b^2(b^4-c^4+c^2a^2-a^4),\ c^2(c^4-a^4+a^2b^2-b^4)). \tag{23}$$

The equation of the Parry circle has been computed in §7.3, and is given by (16). Applying Proposition 1, we see that this circle contains the Euler reflection point

$$E = \left(\frac{a^2}{b^2-c^2}, \frac{b^2}{c^2-a^2}, \frac{c^2}{a^2-b^2}\right) \tag{24}$$

and the point

$$P = \left(\frac{a^2}{b^2 + c^2 - 2a^2}, \frac{b^2}{c^2 + a^2 - 2b^2}, \frac{c^2}{a^2 + b^2 - 2c^2} \right) \tag{25}$$

which we call the Parry point.

Remark. The line EP also contains the symmedian point K.

Lemma 23. *The line EG is parallel to the Fermat line F_+F_-.*

Proof. The line F_+F_- is the same as the line $KK_{\rm i}$, with equation given by (5). The line EG has equation

$$\sum_{\text{cyclic}} (b^2 - c^2) f_{4,2}(a, b, c)x = 0. \tag{26}$$

Both of these lines have the same infinite point

$$X_{542} = (f_{6,2}(a, b, c) : f_{6,2}(b, c, a) : f_{6,2}(c, a, b)).$$

□

Proposition 24. *The Euler reflection point E and the centroid are inverse in the Brocard circle.*

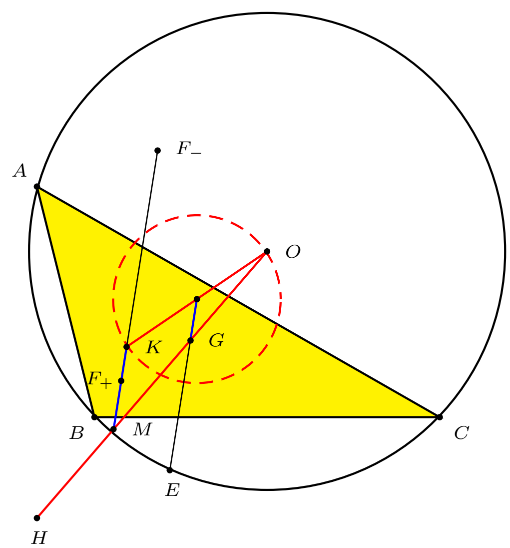

Figure 16. E and G are inverse in the Brocard circle

Proof. Note that the line F_+F_- intersects the Euler line at the midpoint M of HG, and G is the midpoint of OM. Since EG is parallel to MK, it intersects OK at its midpoint, the center of the Brocard circle. Since the circle through E, G, $J_\pm$ is orthogonal to the Brocard circle, E and G are inverse to each other with respect to this circle. □

The following two triangle centers on the Parry circle are also listed in [9]:
(i) the second intersection with the line joining G to K, namely,

$$X_{352} = (a^2(a^4 - 4a^2(b^2+c^2) + (b^4 + 5b^2c^2 + c^4)), \cdots, \cdots),$$

(ii) the second intersection with the line joining X_{23} to K, namely,

$$X_{353} = (a^2(4a^4 - 4a^2(b^2+c^2) - (2b^4 + b^2c^2 + 2c^4)), \cdots, \cdots),$$

which is the inverse of the Parry point P in the Brocard circle, and also the inverse of X_{352} in the circumcircle.

10.1. *The center of the Parry circle.*

Proposition 25. *The perpendicular bisector of the segment GE is the line*

$$\frac{x}{b^2+c^2-2a^2} + \frac{y}{c^2+a^2-2b^2} + \frac{z}{a^2+b^2-2c^2} = 0. \qquad (27)$$

Proof. The midpoint of EG is the point

$$Z_4 := ((b^2+c^2-2a^2)f_{4,5}(a,b,c) : (c^2+a^2-2b^2)f_{4,5}(b,c,a) : (a^2+b^2-2c^2)f_{4,5}(c,a,b)).$$

By Lemma 23 and Proposition 8(b), the perpendicular bisector of EG has infinite point X_{690}. The line through Z_4 with this infinite point is the perpendicular bisector of EG. □

Proposition 26. *The center of the Parry circle $\mathcal{C}_P$ is the point*

$$(a^2(b^2-c^2)(b^2+c^2-2a^2), \cdots, \cdots).$$

Proof. This is the intersection of the line (27) above and the the Lemoine axis (14). □

Remark. The center of the Parry circle appears in [10] as X_{351}.

11. The generalized Parry circles

We consider the generalized Parry circle $\mathcal{C}_P(\theta)$ passing through the centroid and the points $K^*(\pm\theta)$ on the Brocard axis. Since $K^*(\theta)$ and $K^*(-\theta)$ are inverse in the Brocard circle (see §7.2), the generalized Parry circle $\mathcal{C}_P(\theta)$ is orthogonal to the Brocard circle, and must also contain the Euler reflection point E. Its equation is

$$3(b^2-c^2)(c^2-a^2)(a^2-b^2)(16\Delta^2\sin^2\theta - (a^2+b^2+c^2)^2\cos^2\theta)(a^2yz+b^2zx+c^2xy)$$
$$+ (x+y+z)\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(f_{6,2}(a,b,c)\sin^2\theta + f_{6,3}(a,b,c)\cos^2\theta)x = 0.$$

The second intersection with the circumcircle is the point

$$Q(\theta) = \left(\frac{a^2}{f_{6,2}(a,b,c)\sin^2\theta + f_{6,3}(a,b,c)\cos^2\theta}, \cdots, \cdots\right).$$

The Parry point P is $Q(\theta)$ for $\theta = \frac{\pi}{3}$. Here are two more examples.

(i) With $\theta = \frac{\pi}{2}$, we have the circle GEO tangent to the Brocard axis and with center

$$Z_5 := (a^2(b^2-c^2)(b^2+c^2-2a^2)f_{4,4}(a,b,c) : \cdots : \cdots).$$

It intersects the circumcircle again at the point

$$\left(\frac{a^2}{f_{6,2}(a,b,c)}, \frac{b^2}{f_{6,2}(b,c,a)}, \frac{c^2}{f_{6,2}(c,a,b)}\right). \tag{28}$$

This is the triangle center X_{842}.

(ii) With $\theta = 0$, we have the circle GEK tangent to the Brocard axis and with center

$$Z_6 := (a^2(b^2-c^2)(b^2+c^2-2a^2)((a^2+b^2+c^2)^2-9b^2c^2) : \cdots : \cdots).$$

It intersects the circumcircle again at the point

$$Z_7 := \left(\frac{a^2}{f_{6,3}(a,b,c)} : \frac{b^2}{f_{6,3}(b,c,a)} : \frac{c^2}{f_{6,3}(c,a,b)}\right). \tag{29}$$

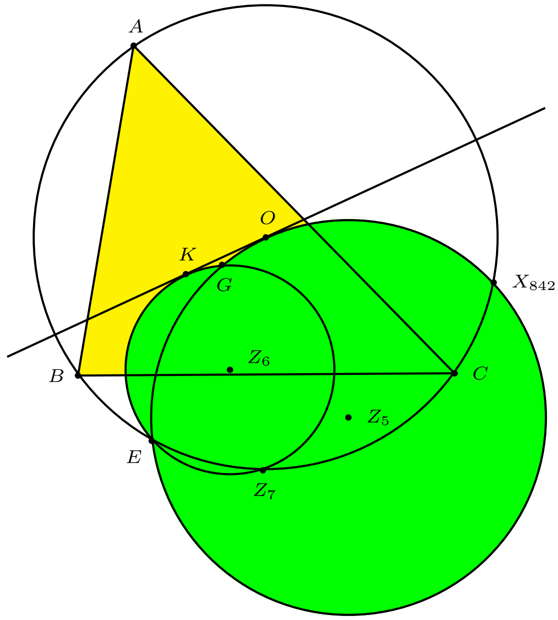

Figure 17. The circles GEO and GEK

12. Circles containing the Parry point

12.1. *The circle* F_+F_-G. The equation of the circle F_+F_-G has been computed in §6.1, and is given by (9). Applying Proposition 1, we see that the circle F_+F_-G contains the Parry point and the point

$$Q = \left(\frac{1}{(b^2-c^2)f_{4,4}(a,b,c)} : \frac{1}{(c^2-a^2)f_{4,4}(b,c,a)} : \frac{1}{(a^2-b^2)f_{4,4}(c,a,b)}\right).$$

This is the triangle center X_{476} in [10]. It is the reflection of the Euler reflection point in the Euler line. [2]

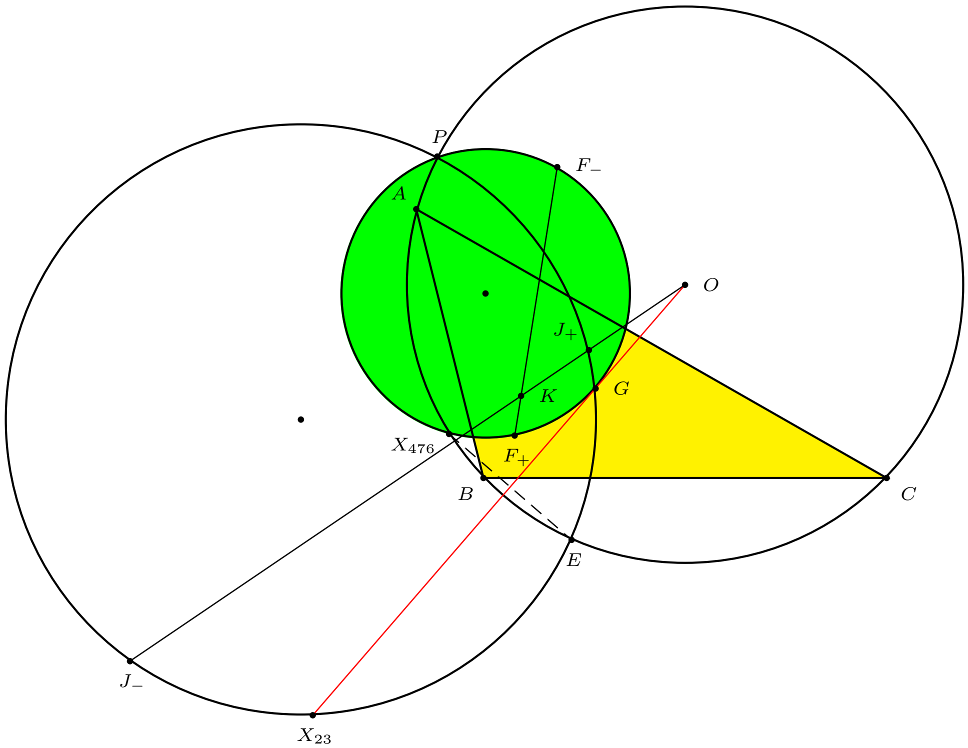

Figure 18. Intersections of F_+F_-G and the circumcircle

12.2. *The circle GOK.* Making use of the equations (13) of the Brocard circle and (12) of the Brocard axis, we find equations of circles through O and K in the form

$$(a^2+b^2+c^2)(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2((b^2-c^2)t+1)x\right) = 0 \tag{30}$$

for suitably chosen t.

With $t = -\frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{3(b^2-c^2)(c^2-a^2)(a^2-b^2)}$, and clearing denominators, we obtain the equation of the circle GOK.

$$3(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2+b^2+c^2)(a^2yz+b^2zx+c^2xy)$$
$$+(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)^2x\right) = 0.$$

[2] To justify this, one may compute the infinite point of the line EQ and see that it is $X_{523} = (b^2-c^2 : c^2-a^2 : a^2-b^2)$. This shows that EQ is perpendicular to the Euler line.

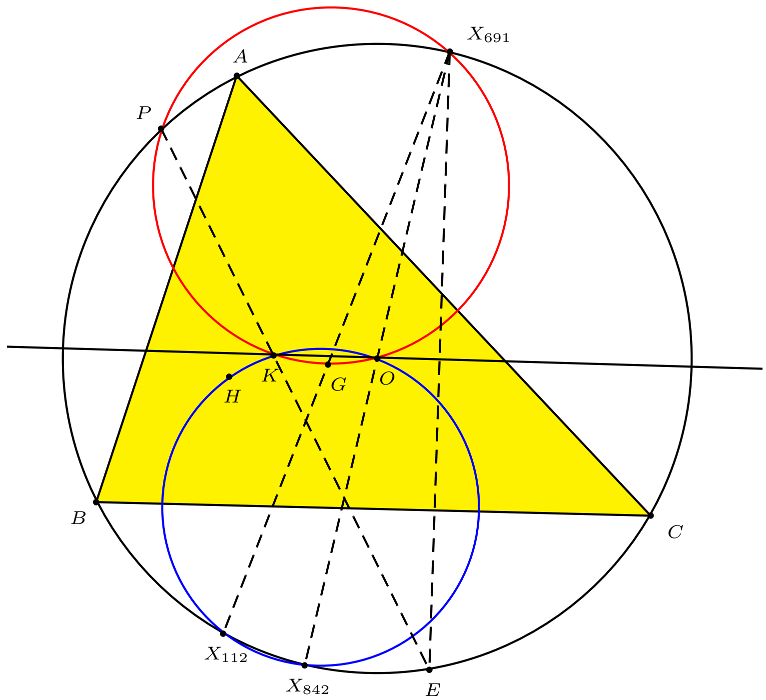

Figure 19. The circles GOK and HOK

This circle GOK contains the Parry point P and the point

$$Q' = \left(\frac{a^2}{(b^2-c^2)(b^2+c^2-2a^2)} : \cdots : \cdots\right),$$

which is the triangle center X_{691}. It is the reflection of E in the Brocard axis.[3]

Remark. The line joining P to X_{691} intersects
(i) the Brocard axis at X_{187}, the inversive image of K in the circumcircle,
(ii) the Euler line at X_{23}, the inversive image of the centroid in the circumcircle.

13. Some special circles

13.1. *The circle HOK.* By the same method, with

$$t = -\frac{a^4(c^2-a^2)(a^2-b^2)+b^4(a^2-b^2)(b^2-c^2)+c^4(b^2-c^2)(c^2-a^2)}{16\Delta^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}$$

[3]This may be checked by computing the infinite point of the line EQ' as $X_{512} = (a^2(b^2-c^2),\ b^2(c^2-a^2), c^2(a^2-b^2)$, the one of lines perpendicular to OK.

in (30), we find the equation of the circle HOK as

$$16\Delta^2(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2yz+b^2zx+c^2xy)$$
$$+(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-a^2)f_{6,2}(a,b,c)x\right)=0.$$

Therefore, the circle HOK intersects the circumcircle at

$$\left(\frac{a^2}{(b^2-c^2)(b^2+c^2-a^2)}, \frac{b^2}{(c^2-a^2)(c^2+a^2-b^2)}, \frac{c^2}{(a^2-b^2)(a^2+b^2-c^2)}\right),$$

which is the triangle center X_{112}, and X_{842} given by (28).

Remarks. (1) The circle HOK has center

$$Z_8 := (a^2(b^2-c^2)f_{4,2}(a,b,c)f_{4,3}(a,b,c) : \cdots : \cdots).$$

(2) X_{112} is the second intersection of the circumcircle with the line joining X_{74} with the symmedian point.

(3) X_{842} is the second intersection of the circumcircle with the parallel to OK through E. It is also the antipode of X_{691}, which is the reflection of E in the Brocard axis.

(4) The radical axis with the circumcircle intersects the Euler line at X_{186} and the Brocard axis at X_{187}. These are the inverse images of H and K in the circumcircle.

13.2. *The circle NOK.* The circle NOK contains the Kiepert center because both O, N and K, K_i are inverse in the orthocentroidal circle.

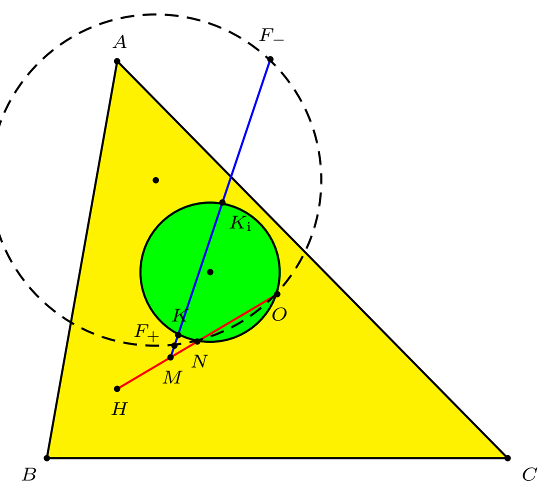

Figure 20. The circle NOK

This circle has equation

$$32(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2+b^2+c^2)\Delta^2(a^2yz+b^2zx+c^2xy)$$
$$+\sum_{\text{cyclic}} b^2c^2(b^2-c^2)f_{8,4}(a,b,c)x=0.$$

Its center is the point

$$Z_9 := (a^2(b^2-c^2)f_{8,1}(a,b,c):\cdots:\cdots).$$

14. The second Evans circle

Evans also conjectured that the perspectors $V_\pm = V(\pm\frac{\pi}{3})$ and X_{74}, X_{399} are concyclic. Recall that

$$X_{74}=\left(\frac{a^2}{f_{4,5}(a,b,c)}:\frac{b^2}{f_{4,5}(b,c,a)}:\frac{c^2}{f_{4,5}(c,a,b)}\right)$$

is the antipode on the circumcircle of the Euler reflection point E and X_{339}, the Parry reflection point, is the reflection of O in E.

We confirm Evans' conjecture indirectly, by first finding the circle through $V_\pm$ and the point

$$X_{101}=(a^2(c-a)(a-b):\ b^2(a-b)(b-c):\ c^2(b-c)(c-a))$$

on the circumcircle. Making use of the equation (22) of the first Evans circle, and the equation (19) of the line V_+V_-, we seek a quantity t such that

$$(a-b)(b-c)(c-a)(a+b+c)(a^2yz+b^2zx+c^2xy)$$
$$-(x+y+z)\left(\sum_{\text{cyclic}}(bc(b-c)(c+a)(a+b)(b+c-a)+t(b-c)(b^2+c^2-a^2))x\right)=0,$$

represents a circle through the point X_{101}. For this, we require

$$t=-\frac{abc(a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)+abc)}{(b+c-a)(c+a-b)(a+b-c)},$$

and the equation of the circle through $V_\pm$ and X_{101} is

$$16\Delta^2(a-b)(b-c)(c-a)(a^2yz+b^2zx+c^2xy)$$
$$-(x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b-c)f_{4,5}(a,b,c)x\right)=0.$$

It is clear that this circle does also contain the point X_{74}.

The center of the circle is the point

$$Z_{10}=(a^2(b-c)((b^2+c^2-a^2)^2-b^2c^2),\ \cdots,\ \cdots).$$

Now, the perpendicular bisector of the segment OE is the line

$$\sum\frac{x}{a^2f_{4,4}(a,b,c)}=0,$$

which clearly contains the center of the circle. Therefore, the circle also contains the point which is the reflection of X_{74} in the midpoint of OE. This is the same as the reflection of O in E, the Parry reflection point X_{399}.

Theorem 27 (Evans)**.** *The four points $V_{\pm}$, the antipode of the Euler reflection point E on the circumcircle, and the reflection of O in E are concyclic* (see Figure 21).

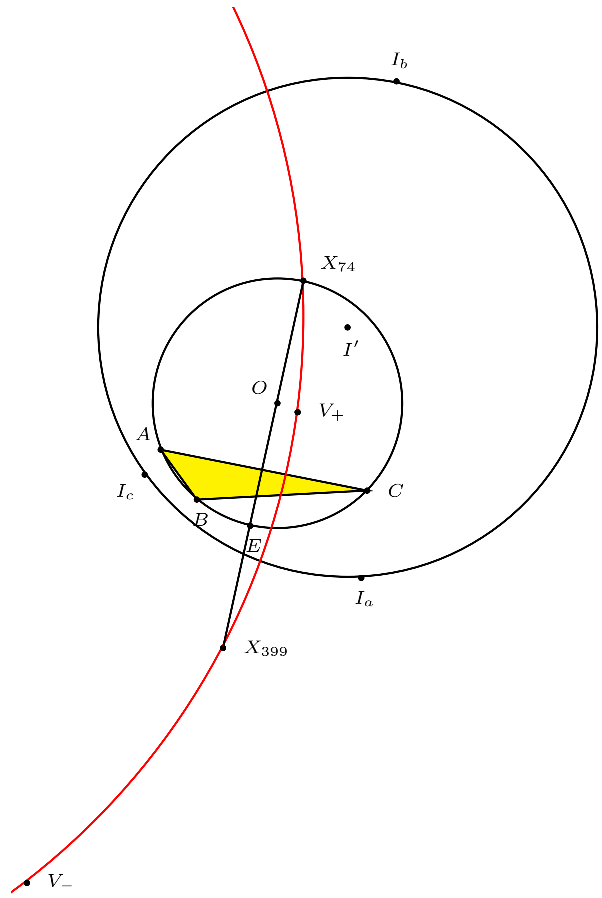

Figure 21. The second Evans circle

15. The second Lester circle

In Kimberling's first list of triangle centers [8], the point X_{19}, the homothetic center of the orthic triangle and the triangular hull of the three excircles, was called the crucial point. Kimberling explained that this name "derives from the name of the publication [13] in which the point first appeared". In the expanded list in [9], this point was renamed after J.W. Clawson. Kimberling gave the reference [2], and commented that this is "possibly the earliest record of this point".[4]

$$C_{\mathrm{w}} = \left(\frac{a}{b^2+c^2-a^2}, \frac{b}{c^2+a^2-b^2}, \frac{c}{a^2+b^2-c^2} \right). \tag{31}$$

Proposition 28. *The Clawson point C_{w} is the perspector of the triangle bounded by the radical axes of the circumcircle with the three excircles* (see Figure 24).

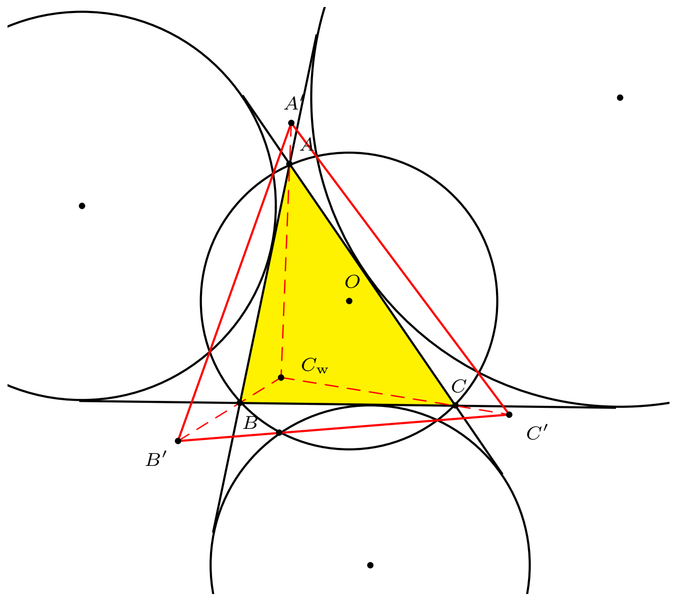

Figure 22. The Clawson point

Proof. The equations of the excircles are given in [15, §6.1.1]. The radical axes with circumcircle are the lines

$$\begin{aligned} L_a &:= \; s^2x+(s-c)^2y+(s-b)^2z=0,\\ L_b &:= \; (s-c)^2x+s^2y+(s-a)^2z=0,\\ L_c &:= \; (s-b)^2x+(s-a)^2y+s^2z=0. \end{aligned}$$

These lines intersect at

$$\begin{aligned} A' &= \; (0 : b(a^2+b^2-c^2) : c(c^2+a^2-b^2)),\\ B' &= \; (a(a^2+b^2-c^2) : 0 : c(b^2+c^2-a^2)),\\ C' &= \; (a(c^2+a^2-b^2) : b(b^2+c^2-a^2) : 0). \end{aligned}$$

It is clear that the triangles ABC and $A'B'C'$ are perspective at a point whose coordinates are given by (31). □

[4]According to the current edition of [10], this point was studied earlier by E. Lemoine [11].

Apart from the circle through the circumcenter, the nine-point center and the Fermat points, Lester has discovered another circle through the symmedian point, the Clawson point, the Feuerbach point and the homothetic center of the orthic and the intangents triangle. The intangents are the common separating tangents of the incircle and the excircles apart from the sidelines. These are the lines

$$\begin{aligned} L'_a &:= \; bcx + (b-c)cy - (b-c)bz = 0, \\ L'_b &:= \; -(c-a)cx + cay + (c-a)az = 0, \\ L'_c &:= \; (a-b)bx - (a-b)ay + abz = 0. \end{aligned}$$

These lines are parallel to the sides of the orthic triangles, namely,

$$\begin{aligned} -(b^2+c^2-a^2)x + (c^2+a^2-b^2)y + (a^2+b^2-c^2)z = &\; 0, \\ (b^2+c^2-a^2)x - (c^2+a^2-b^2)y + (a^2+b^2-c^2)z = &\; 0, \\ (b^2+c^2-a^2)x + (c^2+a^2-b^2)y - (a^2+b^2-c^2)z = &\; 0. \end{aligned}$$

The two triangles are therefore homothetic. The homothetic center is

$$T_{\rm o} = \left(\frac{a(b+c-a)}{b^2+c^2-a^2}, \frac{b(c+a-b)}{c^2+a^2-b^2}, \frac{c(a+b-c)}{a^2+b^2-c^2}\right). \qquad (32)$$

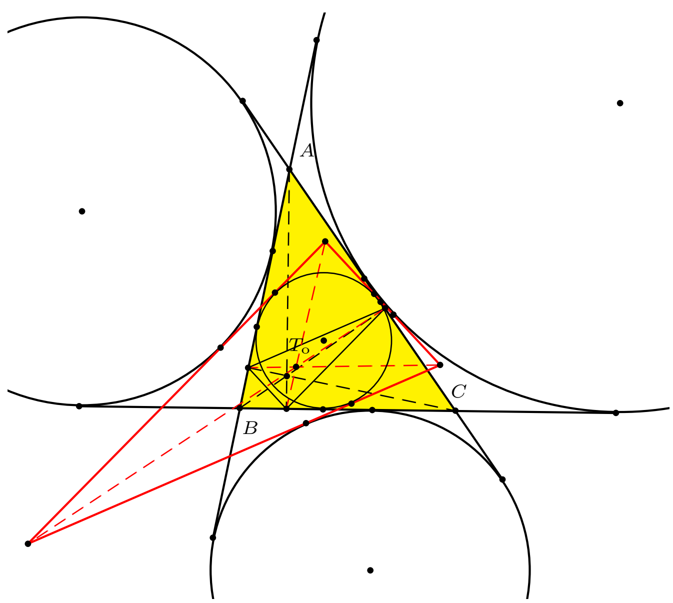

Figure 23. The intangents triangle

Theorem 29 (Lester)**.** *The symmedian point, the Feuerbach point, the Clawson point, and the homothetic center of the orthic and the intangent triangles are concyclic.*

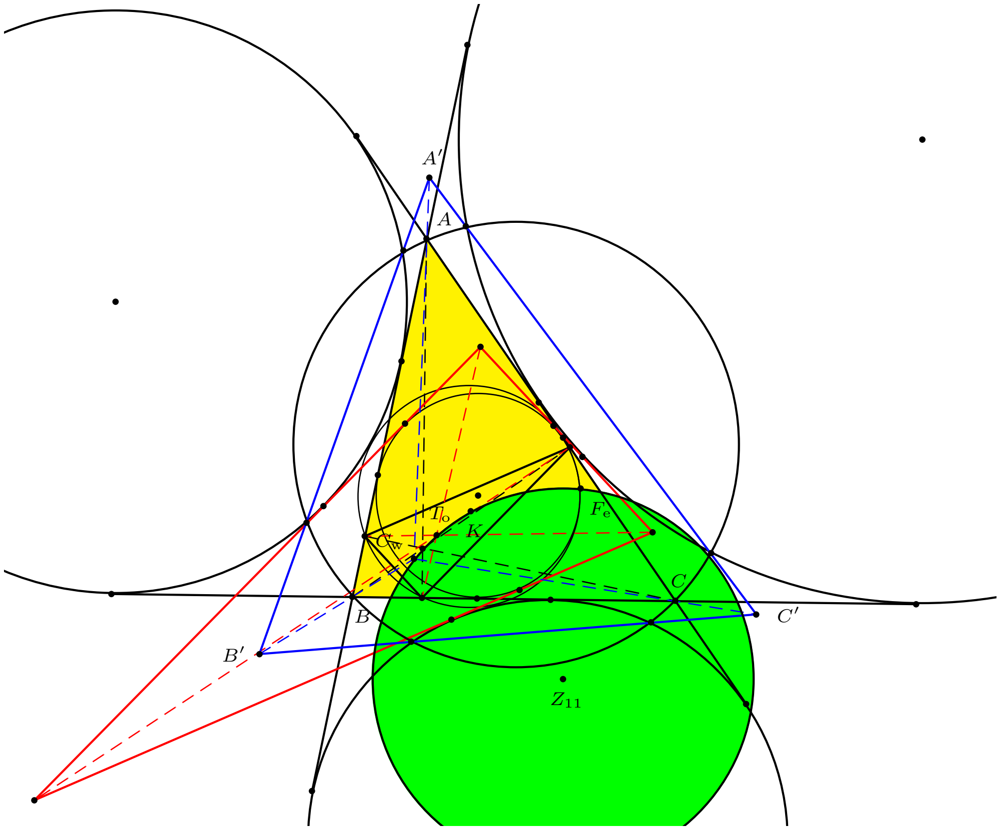

Figure 24. The second Lester circle

There are a number of ways of proving this theorem, all very tedious. For example, it is possible to work out explicitly the equation of the circle containing these four points. Alternatively, one may compute distances and invoke the intersecting chords theorem. These proofs all involve polynomials of large degrees. We present here a proof given by Nikolaos Dergiades which invokes only polynomials of relatively small degrees.

Lemma 30. *The equation of the circle passing through three given points* $P_1 = (u_1 : v_1 : w_1)$, $P_2(u_2 : v_2 : w_2)$ *and* $P_3 = (u_3 : v_3 : w_3)$ *is*

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0$$

where

$$p = \frac{D(u_1, u_2, u_3)}{s_1s_2s_3D(1,2,3)}, \quad q = \frac{D(v_1, v_2, v_3)}{s_1s_2s_3D(1,2,3)}, \quad r = \frac{D(w_1, w_2, w_3)}{s_1s_2s_3D(1,2,3)},$$

with

$$s_1 = u_1+v_1+w_1, \quad s_2 = u_2+v_2+w_2, \quad s_3 = u_3+v_3+w_3, \quad D(1,2,3) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix},$$

and

$$D(u_1,u_2,u_3)=\begin{vmatrix} a^2v_1w_1+b^2w_1u_1+c^2u_1v_1 & s_1v_1 & s_1w_1\\ a^2v_2w_2+b^2w_2u_2+c^2u_2v_2 & s_2v_2 & s_2w_2\\ a^2v_3w_3+b^2w_3u_3+c^2u_3v_3 & s_3v_3 & s_3w_3\end{vmatrix},$$

$$D(v_1,v_2,v_3)=\begin{vmatrix} s_1u_1 & a^2v_1w_1+b^2w_1u_1+c^2u_1v_1 & s_1w_1\\ s_2u_2 & a^2v_2w_2+b^2w_2u_2+c^2u_2v_2 & s_2w_2\\ s_3u_3 & a^2v_3w_3+b^2w_3u_3+c^2u_3v_3 & s_3w_3\end{vmatrix},$$

$$D(w_1,w_2,w_3)=\begin{vmatrix} s_1u_1 & s_1v_1 & a^2v_1w_1+b^2w_1u_1+c^2u_1v_1\\ s_2u_2 & s_2v_2 & a^2v_2w_2+b^2w_2u_2+c^2u_2v_2\\ s_3u_3 & s_3v_3 & a^2v_3w_3+b^2w_3u_3+c^2u_3v_3\end{vmatrix}.$$

Proof. This follows from applying Cramer's rule to the system of linear equations

$$\begin{aligned} a^2v_1w_1+b^2w_1u_1+c^2u_1v_1-s_1(pu_1+qv_1+rw_1)=&\ 0,\\ a^2v_2w_2+b^2w_2u_2+c^2u_2v_2-s_2(pu_2+qv_2+rw_2)=&\ 0,\\ a^2v_3w_3+b^2w_3u_3+c^2u_3v_3-s_3(pu_3+qv_3+rw_3)=&\ 0.\end{aligned}$$

□

Lemma 31. *Four points $P_i=(u_i:v_i:w_i)$, $i=1,2,3,4$, are concyclic if and only if*

$$\frac{D(u_1,u_2,u_4)}{D(u_1,u_2,u_3)}=\frac{D(v_1,v_2,v_4)}{D(v_1,v_2,v_3)}=\frac{D(w_1,w_2,w_4)}{D(w_1,w_2,w_3)}=\frac{s_4D(1,2,4)}{s_3D(1,2,3)}.$$

Proof. The circumcircles of triangles $P_1P_2P_3$ and $P_1P_2P_4$ have equations

$$\begin{aligned} a^2yz+b^2zx+c^2xy-(x+y+z)(px+qy+rz)=&\ 0,\\ a^2yz+b^2zx+c^2xy-(x+y+z)(p'x+q'y+r'z)=&\ 0\end{aligned}$$

where p, q, r are given in Lemma 30 above and p', q', r' are calculated with u_3, v_3, w_3 replaced by u_4, v_4, w_4 respectively. These two circles are the same if and only if $p=p'$, $q=q'$, $r=r'$. The condition $p=p'$ is equivalent to $\frac{D(u_1,u_2,u_4)}{D(u_1,u_2,u_3)}=\frac{s_4D(1,2,4)}{s_3D(1,2,3)}$; similarly for the remaining two conditions. □

Finally we complete the proof of the second Lester circle theorem. For

$$\begin{aligned} P_1=&\ K=(a^2:b^2:c^2),\\ P_2=&\ F_{\rm e}=((b-c)^2(b+c-a):(c-a)^2(c+a-b):(a-b)^2(a+b-c),\\ P_3=&\ C_{\rm w}=(aS_{BC}:bS_{CA}:cS_{AB}),\\ P_4=&\ T_{\rm o}=(a(b+c-a)S_{BC}:b(c+a-b)S_{CA}:c(a+b-c)S_{AB}),\end{aligned}$$

we have

$$\frac{D(u_1,u_2,u_4)}{D(u_1,u_2,u_3)}=\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}=\frac{s_4D(1,2,4)}{s_3D(1,2,3)}.$$

The cyclic symmetry also shows that

$$\frac{D(v_1,v_2,v_4)}{D(v_1,v_2,v_3)}=\frac{D(w_1,w_2,w_4)}{D(w_1,w_2,w_3)}=\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}.$$

It follows from Lemma 31 that the four points K, $F_{\rm e}$, $C_{\rm w}$, and $T_{\rm o}$ are concyclic. This completes the proof of Theorem 29.

For completeness, we record the coordinates of the center of the second Lester circle, namely,

$$Z_{11} := (a(b-c)f_5(a,b,c)f_{12}(a,b,c) : \cdots : \cdots),$$

where

$$\begin{aligned}
f_5(a,b,c) =\ & a^5 - a^4(b+c) + 2a^3bc - a(b^4 + 2b^3c - 2b^2c^2 + 2bc^3 + c^4) + (b-c)^2(b+c)^3,\\
f_{12}(a,b,c) =\ & a^{12} - 2a^{11}(b+c) + 9a^{10}bc + a^9(b+c)(2b^2 - 13bc + 2c^2)\\
& - a^8(3b^4 - 2b^3c - 22b^2c^2 - 2bc^3 + 3c^4) + 4a^7(b+c)((b^2-c^2)^2 - b^2c^2)\\
& - 10a^6bc(b^2-c^2)^2 - 2a^5(b+c)(b-c)^2(b^2 - 4bc + c^2)(2b^2 + 3bc + 2c^2)\\
& + a^4(b-c)^2(3b^6 + 2b^5c - 19b^4c^2 - 32b^3c^3 - 19b^2c^4 + 2bc^5 + 3c^6)\\
& - 2a^3(b+c)(b-c)^2(b^6 + 2b^5c - 3b^4c^2 - 2b^3c^3 - 3b^2c^4 + 2bc^5 + c^6)\\
& + a^2bc(b^4-c^4)^2 + a(b+c)(b-c)^4(b^2+c^2)^2(2b^2 + 3bc + 2c^2)\\
& - (b-c)^4(b+c)^2(b^2+c^2)^3.
\end{aligned}$$

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Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA

E-mail address: yiu@fau.edu