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The Incenter and an Excenter as Solutions to an Extremal Problem

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Abstract. Given a triangle ABC with a point X on the bisector of angle A, we show that the extremal values of $\frac{BX}{CX}$ occur at the incenter and the excenter on the opposite side of A.

Many centers of the triangle are solutions to a variety of extremal problems. For example, the Fermat point minimizes the sum of the distances of a point to the vertices of a triangle (provided the angles are all less than $120°$), and the centroid minimizes the sum of the squares of the distances to the vertices (see [1]). For recent results along these lines, see [3], and [2]. The following problem was brought to Dora's attention about 35 years ago.

Problem. Let ABC be a triangle and AT the angle bisector of angle A. Determine the points X on the line AT for which the ratio $\frac{BX}{CX}$ is extremal.

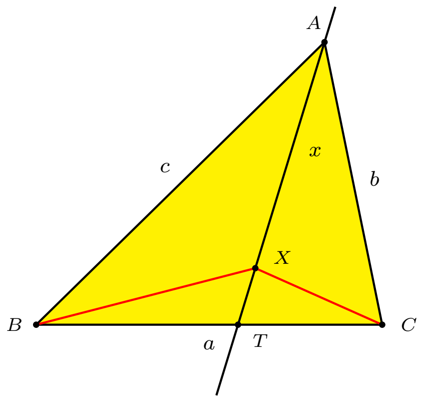

Figure 1.

Denote the lengths of the sides BC, CA, AB by a, b, c respectively. Let X be a point on AT and denote by x the directed length of AX. It is sufficient to consider

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This paper was conceived by Dora Bialostocki who passed away on July 25, 2009. After a failed attempt to find the results in the literature, her notes were supplemented and edited by her surviving husband Arie Bialostocki.

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the function

$$g(x) := \frac{BX^2}{CX^2} = \frac{x^2 - 2cx\cos\frac{A}{2} + c^2}{x^2 - 2bx\cos\frac{A}{2} + b^2}.$$

The derivative of $g(x)$ is given by

$$g'(x) = \frac{-2(b-c)\left(x^2\cos\frac{A}{2} - (b+c)x + bc\cos\frac{A}{2}\right)}{\left(x^2 - 2bx\cos\frac{A}{2} + b^2\right)^2}. \tag{1}$$

This is zero if and only if

$$x = \frac{b + c \pm \sqrt{(b+c)^2 - 4bc\cos^2\frac{A}{2}}}{2\cos\frac{A}{2}}.$$

Now,

$$\begin{aligned}(b+c)^2 - 4bc\cos^2\frac{A}{2} &= b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right)\\ &= b^2 + c^2 - 2bc\cos A\\ &= a^2. \end{aligned} \tag{2}$$

This means that g is extremal when

$$x = \frac{b+c-a}{2\cos\frac{A}{2}} = \frac{s-a}{\cos\frac{A}{2}} \quad \text{or} \quad x = \frac{b+c+a}{2\cos\frac{A}{2}} = \frac{s}{\cos\frac{A}{2}}, \tag{3}$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle ABC.

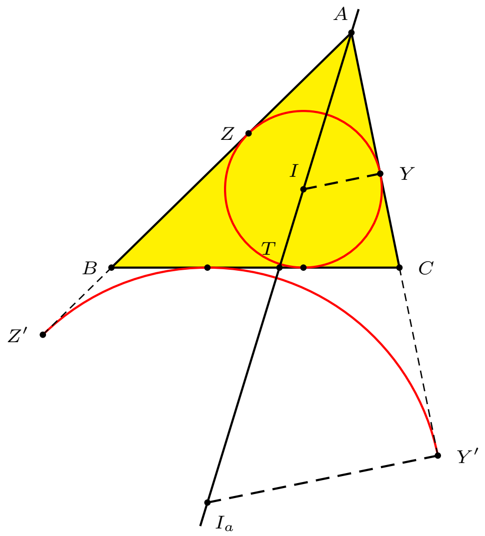

Figure 2. The incenter I and A-excenter I_a

Consider the incircle of the triangle tangent to the sides AC and AB at Y and Z respectively (see Figure 2). It is well known that

$$AY = AZ = \frac{1}{2}(b + c - a) = s - a.$$

Likewise, if the excircle on BC touches AC and AB at Y' and Z' respectively, then

$$AY' = AZ' = \frac{1}{2}(a+b+c) = s.$$

Therefore, according to (3), the extremal values of $g(x)$ occur at the incenter and A-excenter, whose orthogonal projections on the line AC are Y and Y', respectively. Note that

$$\frac{BI}{CI} = \frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} \quad \text{and} \quad \frac{BI_a}{CI_a} = \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}.$$

These are in fact the global maximum and minimum.

Figure 3 shows the graph of $g(x)$ for $b = 5$, $c = 6$ and $A = \frac{\pi}{3}$.

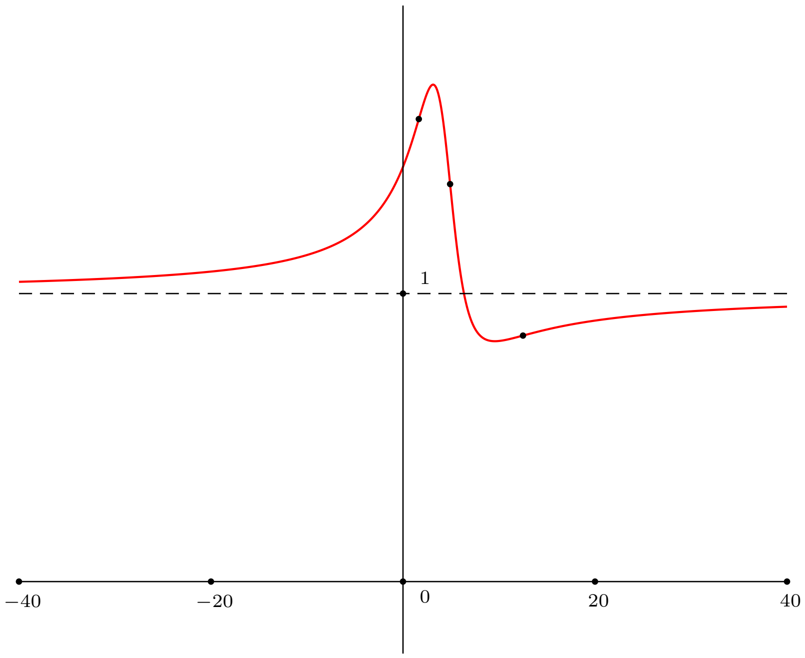

Figure 3. Graph of $g(x) = \frac{BX^2}{CX^2}$ with inflection points

We determine the inflection points of the graph of $g(x)$ (in the general case), and show that they are related to the problem of trisection of angles. Differentiating (1), we have

$$g''(x) = \frac{2(b-c)\left(2x^3\cos\frac{A}{2} - 3(b+c)x^2 + 6bcx\cos\frac{A}{2} + b^2\left(b+c-4c\cos^2\frac{A}{2}\right)\right)}{\left(x^2+b^2-2bx\cos\frac{A}{2}\right)^3}.$$

The inflection points are the roots of

$$\left(2\cos\frac{A}{2}\right)x^3 - 3(b+c)x^2 + \left(6bc\cos\frac{A}{2}\right)x + b^2\left(b+c-4c\cos^2\frac{A}{2}\right) = 0. \quad (4)$$

With the substitution

$$x = u + \frac{b+c}{2\cos\frac{A}{2}}, \tag{5}$$

equation (4) becomes

$$4u^3\cos^3\frac{A}{2} - \left((b+c)^2 - 4bc\cos^2\frac{A}{2}\right)\left(3u\cos\frac{A}{2} + b + c - 2b\cos^2\frac{A}{2}\right) = 0. \tag{6}$$

Making use of (2) and

$$b + c - 2b\cos^2\frac{A}{2} = c - b\cos A = a\cos B,$$

we rewrite (6) as

$$4u^3\cos^3\frac{A}{2} - a^2\left(3u\cos\frac{A}{2} + a\cos B\right) = 0. \tag{7}$$

If we further substitute

$$u\cos\frac{A}{2} = a\cos\theta, \tag{8}$$

the identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

transforms (7) into

$$\cos 3\theta = \cos B. \tag{9}$$

Combining (5) and (8), we conclude that the inflection points of $g(x)$ are at

$$\begin{aligned} x &= \frac{b + c + 2a\cos\theta}{2\cos\frac{A}{2}} \\ &= \frac{(s-a) + a\left(\frac{1}{2} + \cos\theta\right)}{\cos\frac{A}{2}}, \end{aligned}$$

where θ satisfies (9).

Let F_1, F_2, F_3 be the inflection points on AT and denote their projections on AC by P_1, P_2, P_3. The last expression implies that the distances of P_1, P_2, P_3 from Y are given by $\left(\frac{1}{2} + \cos\theta\right)a$, where the angle θ satisfies (9).

Remark. It would be interesting to find in physics an interpretation of the two extrema.

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[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.

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[3] C. Kimberling, Trilinear distance inequalities for the symmedian point, the centroid, and other triangle centers, *Forum Geom.*, 10 (2010) 135–139.

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