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Translation of Fuhrmann's "Sur un nouveau cercle associé à un triangle"

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Abstract. We provide a translation (from the French) of Wilhelm Fuhrmann's influential 1890 paper in which he introduced what is today called the Fuhrmann triangle. We have added footnotes that indicate current terminology and references, ten diagrams to assist the reader, and an appendix.

Introductory remarks

Current widespread interest concerning the Fuhrmann triangle motivated us to learn what, exactly, Fuhrmann did in his influential 1890 article, "Sur un nouveau cercle associé à un triangle" [5]. We were quite surprised to see how much he accomplished in that paper. It was clear to us that anybody who has an interest in triangle geometry might appreciate studying the work for himself. We provide here a faithful translation of Fuhrmann's article (originally in French), supplementing it with footnotes that indicate current terminology and references. Two of the footnotes, however, (numbers 2 and 12) were provided in the original article by Joseph Neuberg (1840-1926), the cofounder (in 1881) of *Mathesis* and its first editor. Instead of reproducing the single diagram from the original paper, we have included ten figures showing that portion of the configuration relevant to the task at hand. At the end of our translation we have added an appendix containing further comments. Most of the background results that Fuhrmann assumed to be known can be found in classic geometry texts such as [7].

Biographical sketch.[1] Wilhelm Ferdinand Fuhrmann was born on February 28, 1833 in Burg bei Magdeburg. Although he was first attracted to a nautical career, he soon yielded to his passion for science, graduating from the Altstädtischen Gymnasium in Königsberg in 1853, then studying mathematics and physics at the University of Königsberg, from which he graduated in 1860. From then until his

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[1]Translated from Franz Kössler, *Personenlexikon von Lehrern des 19 Jahrhunderts*: *Berufsbiographien aus Schul-Jahresberichten und Schulprogrammen 1825-1918 mit Veröffentlichungsverzeichnissen*. We thank Professor Rudolf Fritsch of the Ludwig-Maximilians-University of Munich and his contacts on the genealogy chat line, Hans Christoph Surkau in particular, for providing us with this reference.

death 44 years later he taught at the Oberrealschule auf der Burg in Königsberg, receiving the title distinguished teacher (Oberlehrer) in 1875 and professor in 1887. In 1894 he obtained the Order of the Red Eagle, IV Class. He died in Königsberg on June 11, 1904.

On a New Circle Associated with a Triangle

Wilhelm Fuhrmann [2]

Preliminaries. Let ABC be a triangle; O its circumcenter; I, r its incenter and inradius; I_a, I_b, I_c its excenters; H_a, H_b, H_c, H the feet of its altitudes and the orthocenter; A', B', C' the midpoints of its sides; $A''B''C''$ the triangle whose sides pass through the vertices of ABC and are parallel to the opposite sides.

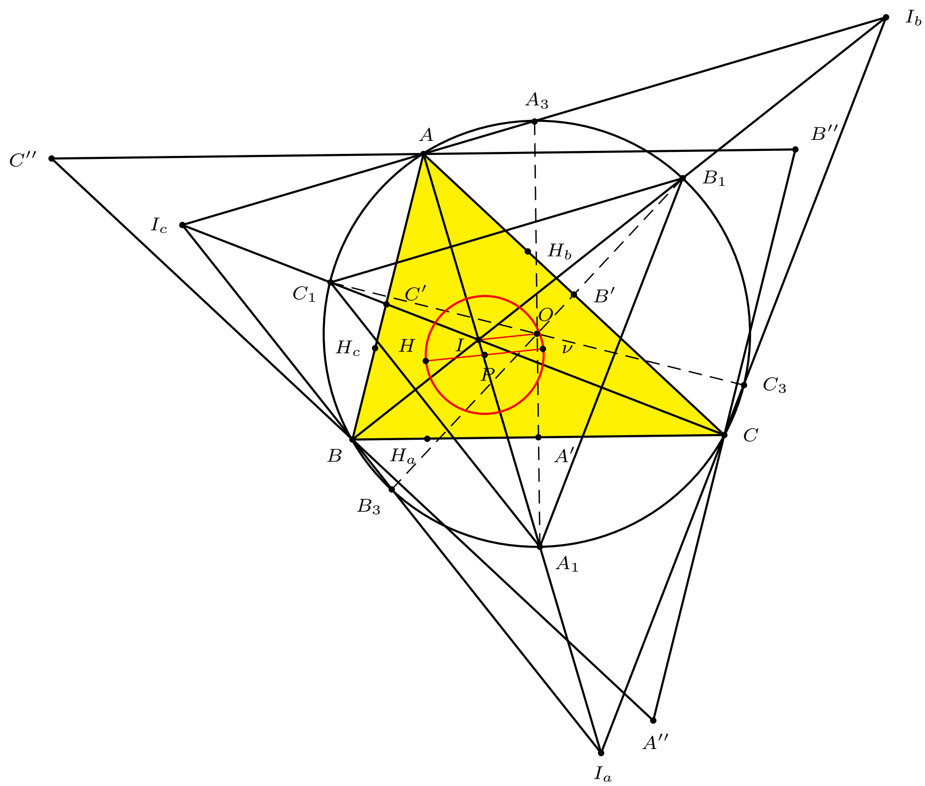

Figure 1. The initial configuration

The internal angle bisectors AI, BI, CI meet the circle O at the midpoints A_1, B_1, C_1 of the arcs that subtend the angles of triangle ABC. One knows that the triangles $A_1B_1C_1$ and $I_aI_bI_c$ are homothetic with respect to their common orthocenter I.

[2][Neuberg's footnote.] Mr. Fuhrmann has just brought out, under the title *Synthetische Beweise planimetrischer Sätze* (Berlin, Simion, 1890, XXIV-190p. in-8°, 14 plates), an excellent collection of results related, in large part, to the recent geometry of the triangle. We will publish subsequently the results of Mr. Mandart, which complement the article of Mr. Fuhrmann. (J.N.)

The circle O passes through the midpoints A_3, B_3, C_3 of the sides of triangle $I_aI_bI_c$; the lines A_1A_3, B_1B_3, C_1C_3 are the perpendicular bisectors of the sides BC, CA, AB of triangle ABC.

We denote by ν the incenter of the anticomplementary triangle $A''B''C''$; this point is called the *Nagel point* of triangle ABC (see *Mathesis*, v. VII, p. 57).[3]

We denote the circle whose diameter is $H\nu$ by the letter P, which represents its center;[4] $H\nu$ and OI, homologous segments of the triangles $A''B''C''$, ABC are parallel and in the ratio $2:1$.

1. *Let A_2, B_2, C_2 be the reflections of the points A_1, B_1, C_1 in the lines BC, CA, AB: triangle $A_2B_2C_2$ is inscribed in the circle P and is oppositely similar to triangles $A_1B_1C_1$, $I_aI_bI_c$.*[5]

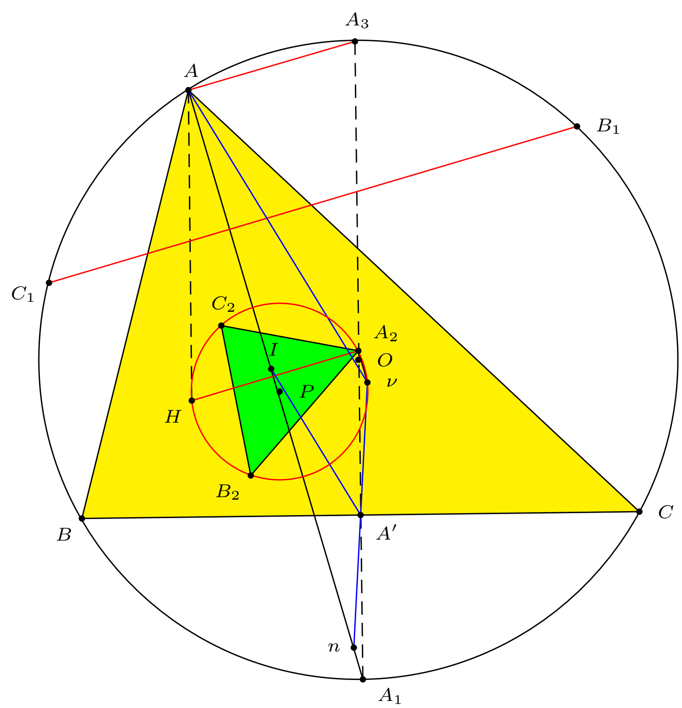

Figure 2. Proof of Theorem 1

One has

$$A_2A_3 = A'A_3 - A'A_1 = 2A'O = AH;$$

[3]Em. Vigarié, sur les points complémentaires. *Mathesis* VII (1887) 6-12, 57-62, 84-89, 105-110; Vigarié produced in French, Spanish, and German, more than a dozen such summaries of the latest results in triangle geometry during the final decades of the nineteenth century.

[4]Nowadays the circle P is called the *Fuhrmann circle* of triangle ABC. Its center P appears as X_{355} in Kimberling's *Encyclopedia of Triangle Centers* [11].

[5]Triangle $A_2B_2C_2$ is now called the *Fuhrmann triangle* of triangle ABC.

consequently, HA_2 is parallel to AA_3 and B_1C_1. $A\nu$ and $A'I$, homologous segments of the homothetic triangles $A''B''C''$, ABC, are parallel and $A\nu = 2A'I$; thus $\nu A'$ meets AI in a point n such that $\nu A' = A'n$, and as one also has $A_2A' = A'A_1$, the figure $A_1\nu A_2 n$ is a parallelogram.

Thus, the lines $A_2H, A_2\nu$ are parallel to the perpendicular lines AA_3, AA_1; consequently, $\angle HA_2\nu$ is right, and the circle P passes through A_2.

If, in general, through a point on a circle one takes three chords that are parallel to the sides of a given triangle, the noncommon ends of these chords are the vertices of a triangle that is oppositely similar to the given triangle. From this observation, because the lines HA_2, HB_2, HC_2 are parallel to the sides of triangle $A_1B_1C_1$, triangle $A_2B_2C_2$ is oppositely similar to $A_1B_1C_1$.

Remark. The points A_2, B_2, C_2 are the projections of H on the angle bisectors $A''\nu, B''\nu, C''\nu$ of triangle $A''B''C''$.

2. *From the altitudes AH, BH, CH of triangle ABC, cut segments AA_4, BB_4, CC_4 that are equal to the diameter $2r$ of the incircle: triangle $A_4B_4C_4$ is inscribed in circle P, and is oppositely similar to ABC.*

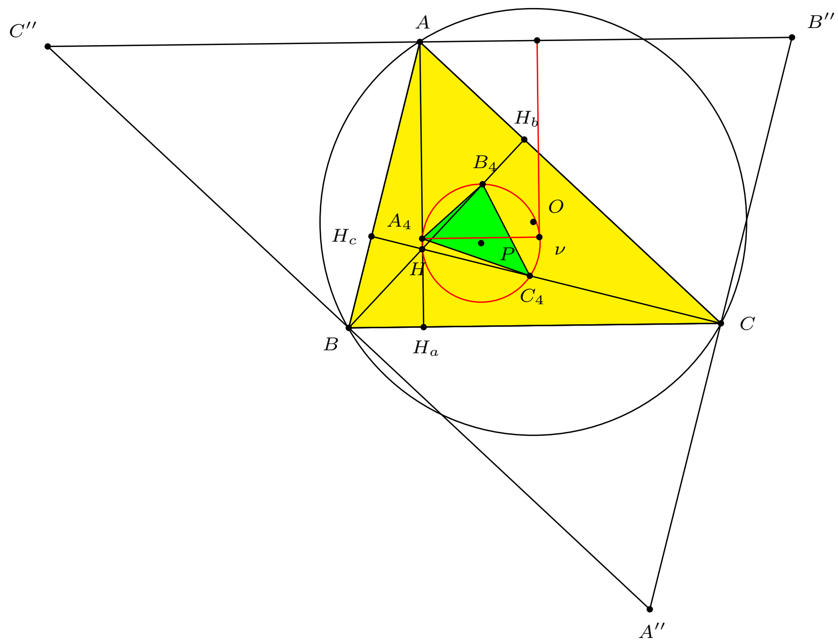

Figure 3. Proof of Theorem 2

Because the distances from ν to the sides of triangle $A''B''C''$ equal $2r$, the lines $\nu A_4, \nu B_4, \nu C_4$ are parallel to the sides of triangle ABC, and the angles νA_4H, etc. are right. Circle P passes, therefore, through the points A_4, B_4, C_4 and triangle $A_4B_4C_4$ is oppositely similar to ABC.

Remark. The points A_4, B_4, C_4 are the projections of the incenter of triangle $A''B''C''$ on the perpendicular bisectors of this triangle.

3. *The triangles $A_2B_2C_2$ and $A_4B_4C_4$ are perspective from I.*

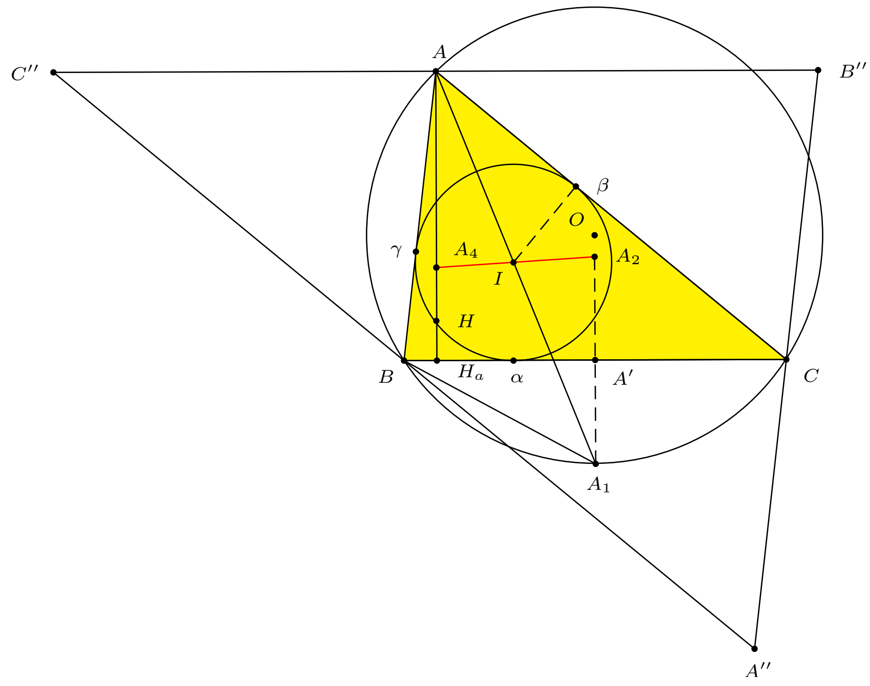

Figure 4. The proof of Theorems 3 and 4

Let α, β, γ be the points where the circle I touches BC, CA, AB. Recalling that triangles $AI\beta$, BA_1A' are similar, one finds that

$$\frac{AA_4}{A_2A_1} = \frac{2r}{2A_1A'} = \frac{I\beta}{A_1A'} = \frac{AI}{BA_1} = \frac{AI}{IA_1}.$$

The triangles AA_4I, A_1A_2I are therefore similar, and the points A_4, I, A_2 are collinear.

Remark. The axis of the perspective triangles ABC, $A_1B_1C_1$ is evidently the polar of the point I with respect to the circle O; that of triangles $A_2B_2C_2$, $A_4B_4C_4$ is the polar of I with respect to the circle P.

4. *I is the double point*[6] *of the oppositely similar triangles ABC, $A_4B_4C_4$, and also of the triangles $A_1B_1C_1$, $A_2B_2C_2$.*

Triangles ABC, $A_4B_4C_4$ are oppositely similar, and the lines that join their vertices to the point I meet their circumcircles in the vertices of the two oppositely similar triangles $A_1B_1C_1$, $A_2B_2C_2$. It follows that I is its own homologue in ABC and $A_4B_4C_4$, $A_1B_1C_1$ and $A_2B_2C_2$.

[6]That is, I is the center of the opposite similarity that takes the first triangle to the second. See the appendix for another proof of Theorem 4.

Remarks. (1) The two systems of points $ABCA_1B_1C_1$, $A_4B_4C_4A_2B_2C_2$ are oppositely similar; their double point is I, their double lines are the bisectors (interior and exterior) of angle AIA_4.

(2) I is the incenter of triangle $A_4B_4C_4$, and the orthocenter of triangle $A_2B_2C_2$.[7]

5. *Let I'_a, I'_b, I'_c be the reflections of the points I_a, I_b, I_c in the lines BC, CA, AB, respectively. The lines AI'_a, BI'_b, CI'_c, and the circumcircles of the triangles BCI'_a, CAI'_b, ABI'_c all pass through a single point R. The sides of triangle $A_2B_2C_2$ are the perpendicular bisectors of AR, BR, CR.*

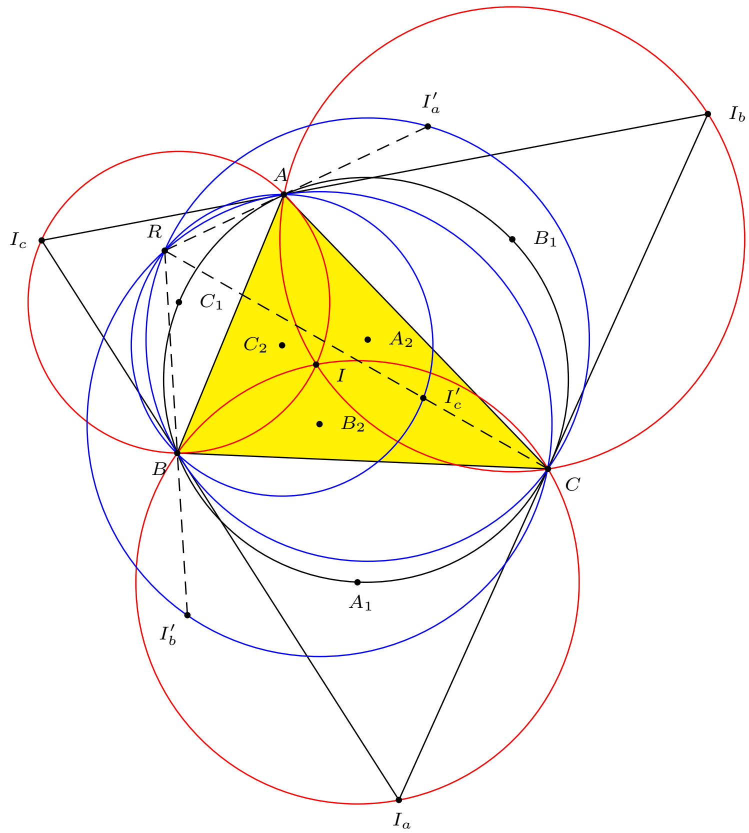

Figure 5. The proof of Theorem 5

Triangles I_aBC, AI_bC, ABI_c are directly similar; their angles equal

$$90^\circ - \frac{A}{2},\ 90^\circ - \frac{B}{2},\ 90^\circ - \frac{C}{2}.$$

The circumcircles of these triangles have centers A_1, B_1, C_1, and they pass through the same point I.

Triangles I'_aBC, AI'_bC, ABI'_c are oppositely similar to I_aBC, AI_bC, ABI_c; their circumcircles have, evidently, for centers the points A_2, B_2, C_2, and they pass

[7]Milorad Stevanovic [13] recently rediscovered that I is the orthocenter of the Fuhrmann triangle. Yet another recent proof can be found in [1].

through a single point R, say, which is the center of a pencil $R(ABC)$ that is congruent and oppositely oriented to the pencil $I(ABC)$. One shows easily, by considering the cyclic quadrilaterals $ABRI'_c$, $BCRI'_a$, that AR and RI'_a lie along the same line.

The line of centers C_2B_2 of the circles ABI'_c, ACI'_b is the perpendicular bisector of the common chord AR.

Remark. Because the pencils $R(ABC), I(ABC)$ are symmetric, their centers R, I are, in the terminology proposed by Mr. Artzt, *twin points* with respect to triangle ABC.[8]

6. *The axis of the perspective triangles $A_1B_1C_1$, $A_2B_2C_2$ is perpendicular to IR at its midpoint, and it touches the incircle of triangle ABC at this point.*[9]

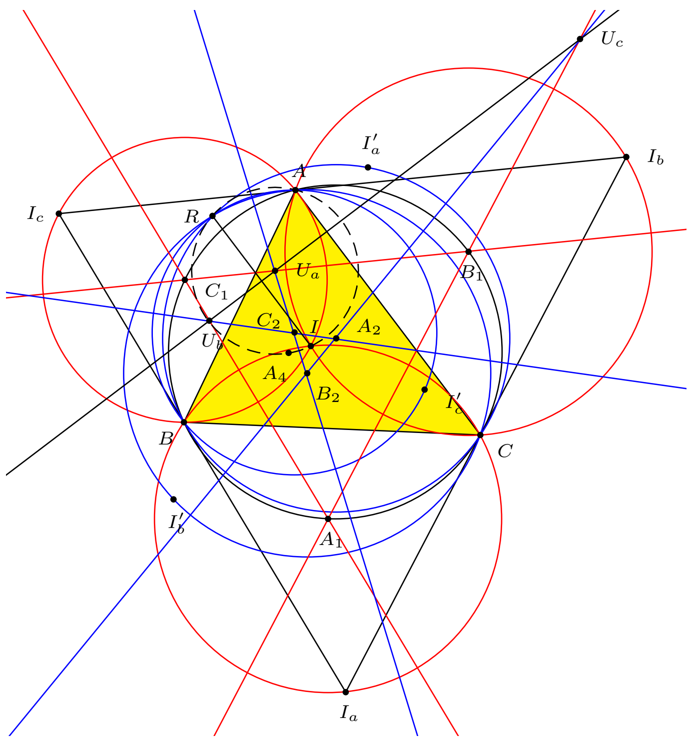

Figure 6. The proof of Theorems 6 and 7

Let U_a, U_b, U_c be the intersection points of the homologous sides of the two triangles; Since I is the orthocenter of both $A_1B_1C_1$ and $A_2B_2C_2$, B_1C_1 is the

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[8]August Artzt (1835-1899) is known today for the Artzt parabolas of a triangle. (See, for example, [4, p.201].) Instead of *twin points*, Darij Grinberg uses the terminology *antigonal conjugates* [9]. Hatzipolakis and Yiu call these reflection conjugates; see [6, §3].

[9]The following Theorem 7 states that the midpoint of IR is what is now called the Feuerbach point of triangle ABC. The point R appears in [11] as X_{80}, which is called the *anti-Gray point* of triangle ABC in [8]. See the appendix for further information.

perpendicular bisector of IA and B_2C_2 the perpendicular bisector of IA_4; B_2C_2 is also the perpendicular bisector of RA. Thus, U_a is the center of a circle through the four points A, I, A_4, R; consequently, it lies on the perpendicular bisector of IR. Similarly, the points U_b, U_c lie on this line.

The lines AA_4, IR are symmetric with respect to B_2C_2; whence, $IR = 2r$. The midpoint Q of IR therefore lies on the incircle of ABC, and the tangent at this point is the line $U_aU_bU_c$.

7. *The incircle of ABC and the nine-point circle are tangent at the midpoint of IR.*

In the oppositely similar triangles $A_4B_4C_4$ and ABC, I is its own homologue and P corresponds to O: thus $IAO,\ IA_4P$ are homologous angles and, therefore, equal and oppositely oriented. As AI bisects angle A_4AO, the angles $IA_4P,\ IAA_4$ are directly equal, whence the radius PA_4 of circle P touches at A_4 circle AIA_4R, and the two circles are orthogonal. Circles $BIB_4R,\ CIC_4R$ are likewise orthogonal to circle P; moreover, the common points I, R harmonically divide a diameter of circle P. Consequently,

$$\overline{PH}^2 = PI \cdot PR = PR(PR - 2r);$$

from a known theorem,

$$\overline{PH}^2 = \overline{OI}^2 = OA(OA - 2r).$$

Comparison of these equations gives the relation $PR = OA$. Let O_9 be the center of the nine-point circle of triangle ABC; this point is the common midpoint of HO, IP (HP and IO are equal and parallel). Because Q is the midpoint of IR, the distance $O_9Q = \frac{1}{2}PR = \frac{1}{2}OA$. Thus, Q belongs to the circle O_9, and as it is on the line of centers of the circles I and O_9, these circles are tangent at Q.

Remark. The point R lies on the axis of the perspective triangles $A_2B_2C_2,\ A_4B_4C_4$ (no. **3**, *Remark*).

8. *Lemma*. If X is the harmonic conjugate of X_1 with respect to Y and Y_1, while M is the midpoint of XX_1 and N is the harmonic conjugate of Y with respect to X and M, then one can easily show that[10]

$$NY_1 \cdot NM = \overline{NX}^2.$$

9. *The lines A_2A_4, B_2B_4, C_2C_4 meet respectively BC, CA, AB in three points V_a, V_b, V_c situated on the common tangent to the circles I and O_9.*

Let D be the intersection of AI with BC, and α, α' the points where the circles I, I_a are tangent to BC. The projection of the harmonic set $(ADII_a)$[11] on BC is the harmonic set $(H_aD\alpha\alpha')$. Since A' is the midpoint of A_1A_2, one has a harmonic pencil $I(A_1A_2A'\alpha)$, whose section by BC is the harmonic set $(DV_aA'\alpha)$. The two

[10] One way to show this is with coordinates: With $-1, 1, y$ attached to the points X, X_1, Y we would have $Y_1 = \frac{1}{y}$, $M = 0$, and $N = \frac{-y}{2y+1}$.

[11] meaning A is the harmonic conjugate of D with respect to I and I_a.

sets $(H_a D\alpha\alpha')$, $(DV_a A'\alpha)$ satisfy the hypothesis of Lemma 8 because A' is the midpoint of $\alpha\alpha'$. Consequently,

$$\overline{V_a\alpha}^2 = V_aH_a \cdot V_aA';$$

whence, V_a is of equal power with respect to the circles I, O_9. This point belongs, therefore, to the common tangent through Q.

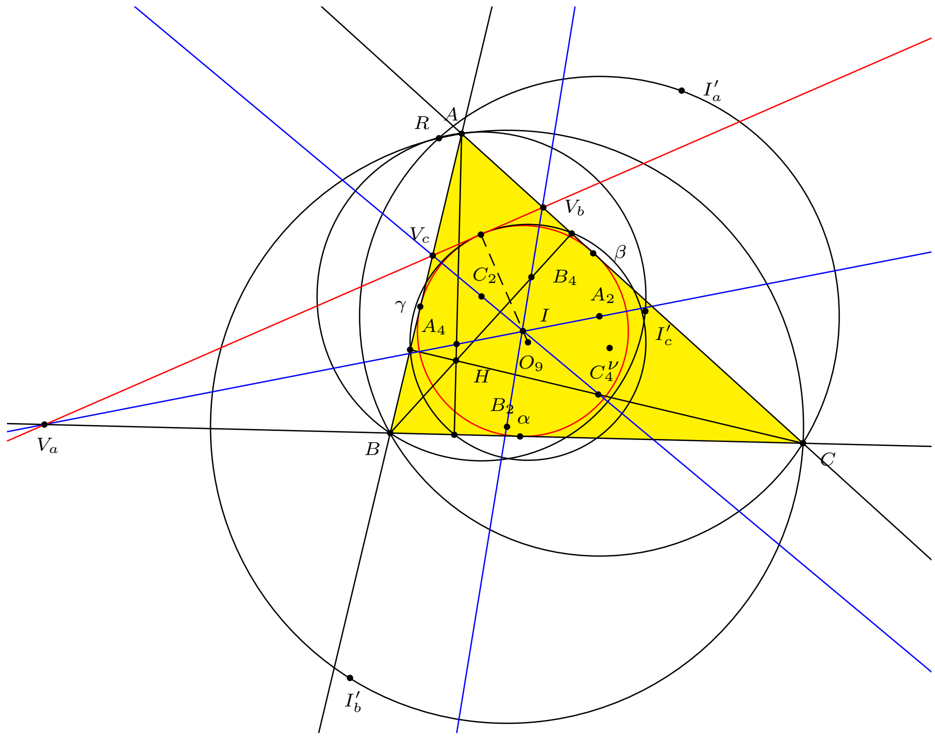

Figure 7. Theorem 9

Remark. From the preceding equality one deduces that

$$\overline{V_aI}^2 = V_aA_4 \cdot V_aA_2;$$

this says that the circle with center V_a, radius V_aI, is orthogonal to the circle P, a property that follows also from V_a being equidistant from the points I, R.

10. *The perpendiculars from A_1, B_1, C_1 to the opposite sides of triangle $A_2B_2C_2$ concur in a point S on the circle O. This point lies on the line $O\nu$ and is, with respect to triangle ABC, the homologue of the point ν with respect to triangle $A_4B_4C_4$.* [12]

[12][Neuberg's footnote.] The normal coordinates of ν with respect to triangle $A_4B_4C_4$ are inversely proportional to the distances $\nu A_4, \nu B_4, \nu C_4$ or to the projections of OI on BC, CA, AB; namely, $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$. These are also the coordinates of S with respect to triangle ABC. (J.N.)

The first part of the theorem follows from the triangles $A_1B_1C_1$, $A_2B_2C_2$ being oppositely similar. One knows (§1) that the perpendiculars from the vertices of $A_2B_2C_2$ to the sides of $A_1B_1C_1$ concur at ν; thus ν, S are homologous points of $A_2B_2C_2$, $A_1B_1C_1$. P and O are likewise homologous points, and I corresponds to itself; consequently, $IP\nu$ and IOS are homologous angles, and as $PIO\nu$ is a parallelogram, the points O, ν, S are collinear.

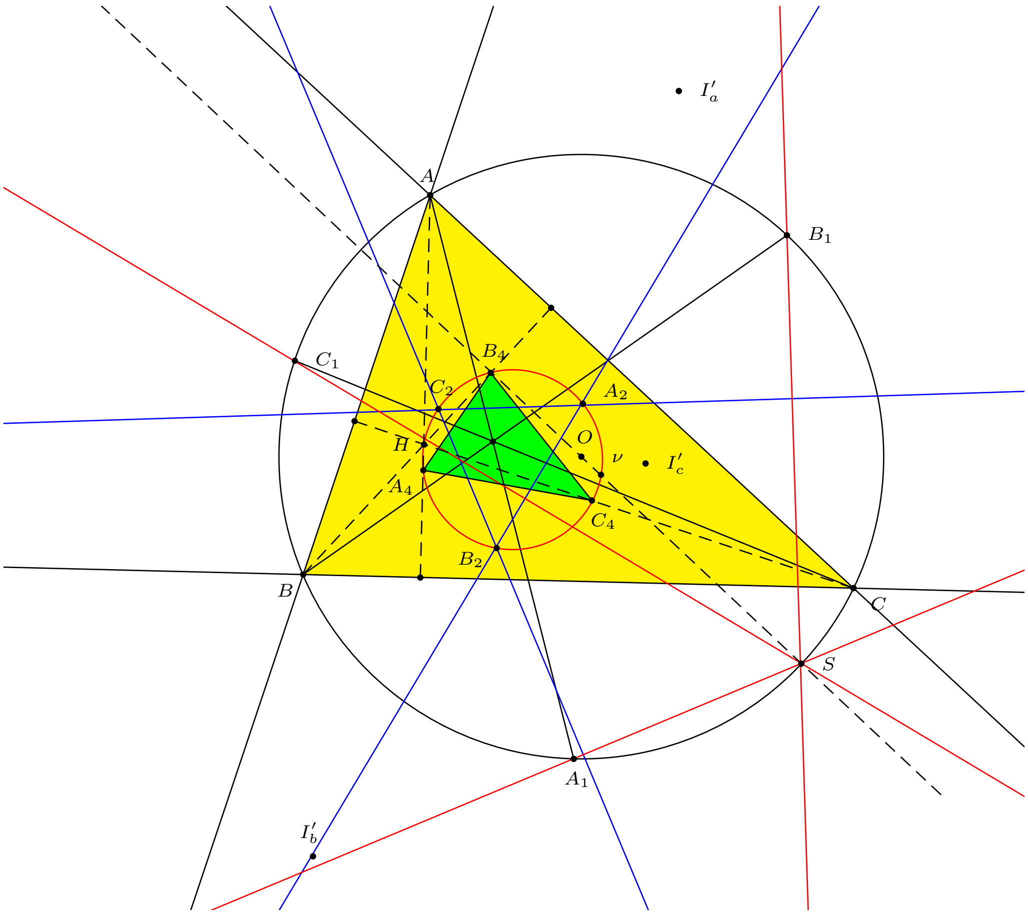

Figure 8. Theorem 10

11. *The perpendiculars from A, B, C to the opposite sides of triangle $A_4B_4C_4$ concur in a point T on the circle O; T and H are homologous points of the triangles ABC, $A_4B_4C_4$.*

The proof of this theorem is similar to the preceding proof.

Remarks. (1) The inverse [13] of the point S with respect to triangle ABC is the point at infinity in the direction perpendicular to the line $H\nu$. Indeed, let SS' be

[13]In today's terminology, the *inverse* of a point with respect to a triangle is called the *isogonal conjugate* of the point.

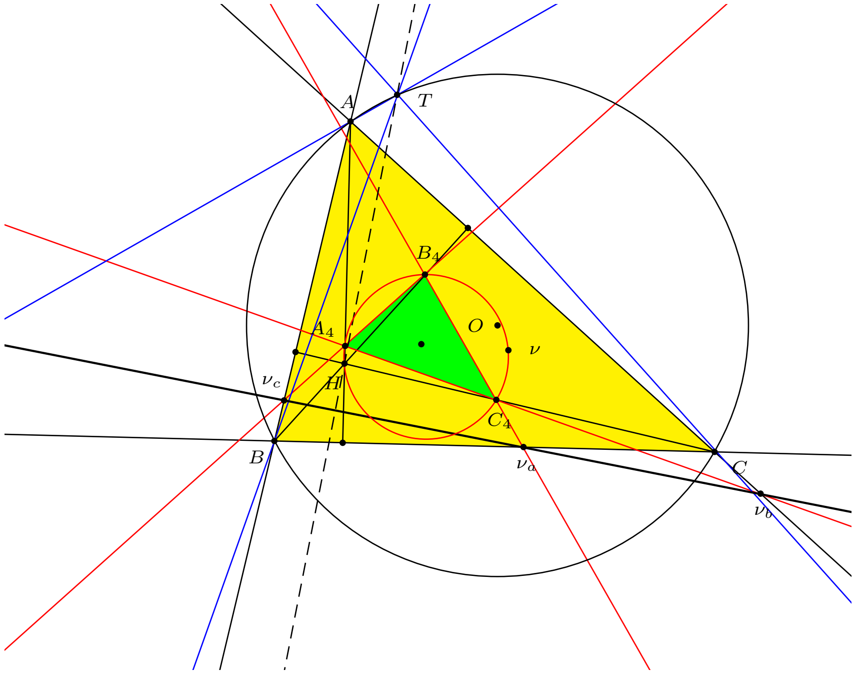

Figure 9. Theorems 11 and 12

the chord of circle O through S parallel to AB. In the oppositely similar figures $ABC, A_4B_4C_4$ the directions ST and $H\nu$, CT and C_4H correspond; thus angle $CH\nu = STC = SS'C$, and as SS' is perpendicular to CH, it follows that CS' is perpendicular to $H\nu$. But, CS' is the isogonal of CS, and the claim follows.

(2) Because $H\nu$ is a diameter of circle $A_4B_4C_4$, S and T are the ends of the diameter of circle ABC determined by the line $O\nu$. It follows that the lines AS, BS, CS are parallel to B_4C_4, C_4A_4, A_4B_4.

12. *The axis of the perspective triangles $ABC, A_4B_4C_4$ is perpendicular to the line HT.*

Let ν_a, ν_b, ν_c be the intersections of the corresponding sides of these triangles. The triangles $THB, CC_4\nu_a$ have the property that the perpendiculars from the vertices of the former to the sides of the latter concur at the point A; thus, the perpendiculars from C, C_4, ν_a, respectively on HB, TB, TH also concur; the first two of these lines meet at ν_b; thus, $\nu_a\nu_b$ is perpendicular to TH. [14]

More simply, the two complete quadrangles $ABHT, \nu_b\nu_aC_4C$ have five pairs of perpendicular sides; thus the remaining pair $HT, \nu_a\nu_b$ is perpendicular as well.

13. *The double lines of the oppositely similar triangles $ABC, A_4B_4C_4$ meet the altitudes of ABC in points (X_a, X_b, X_c), (X'_a, X'_b, X'_c) such that*

$$AX_a = BX_b = CX_c = R - d, \quad AX'_a = BX'_b = CX'_c = R + d,$$

[14] THB and $CC_4\nu_a$ are called *orthologic triangles*; see, for example, Dan Pedoe, *Geometry: A Comprehensive Course*, Dover (1970), Section 8.3.

where R is the circumradius and $d = OI$.

Indeed, the similar triangles ABC and $A_4B_4C_4$ are in the ratio of their circumradii, namely $R : d$; since IX_a bisects angle AIA_4, one has successively,

$$\frac{AX_a}{X_aA_4} = \frac{R}{d}, \qquad \frac{AX_a}{AX_a + X_aA_4} = \frac{R}{R+d}, \qquad AX_a = \frac{2Rr}{R+d} = R - d.$$

Analogously, $AX'_a = R + d$.

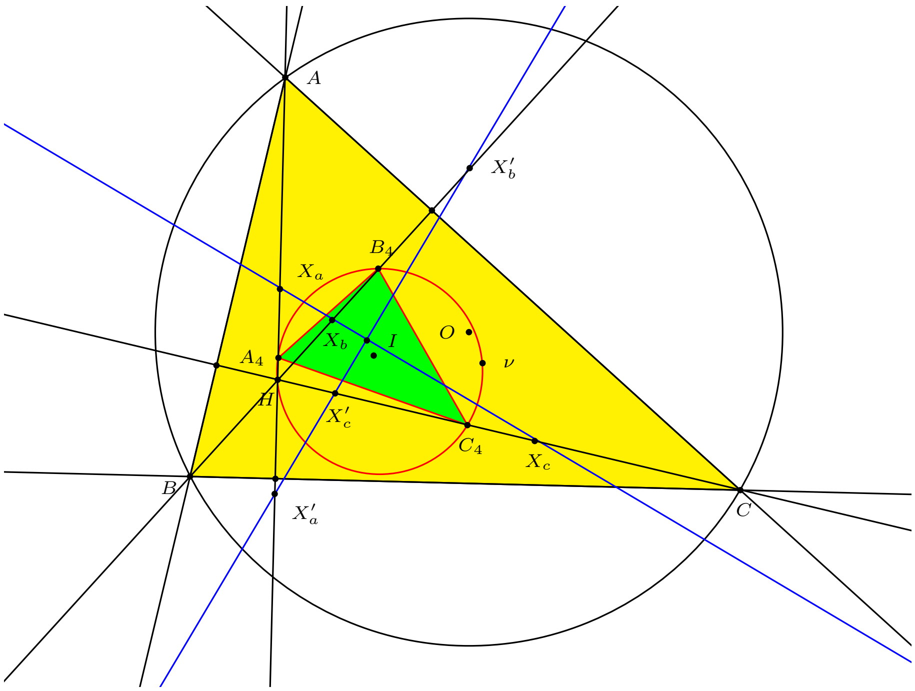

Figure 10. Theorem 13

Remarks. (1) Proposition 13 resolves the following problem: Take on the altitudes AH, BH, CH three equal segments $AX_a = BX_b = CX_c$ such that the points X_a, X_b, X_c are collinear. One sees that there exist two lines $X_aX_bX_c$ that satisfy these conditions; they are perpendicular and pass through the incenter of ABC.

(2) Let us take the lengths $AY_a = BY_b = CY_c = OA$ on AH, BH, CH. The triangles $Y_aY_bY_c$, $A_2B_2C_2$ are symmetric with respect to O_9.

Appendix

Comments on Theorem 4. Here is an alternative proof that *I is the center of the opposite similarity that takes triangle $A_1B_1C_1$ to triangle $A_2B_2C_2$* (given that they are oppositely similar by Theorem 1). The isosceles triangles OA_1C and CA_2A_1 are similar because they share the angle at A_1; thus $\frac{OA_1}{A_1C} = \frac{A_1C}{A_1A_2}$. But $A_1C = A_1I$, so $\frac{OA_1}{A_1I} = \frac{A_1I}{A_1A_2}$. Because the angle at A_1 is shared, triangles OIA_1

and IA_2A_1 are similar (by SAS). Consequently,

$$\frac{IA_1}{IA_2} = \frac{OA_1}{OI} = \frac{R}{OI},$$

where R is the circumradius of triangle ABC. Similarly $\frac{R}{OI} = \frac{IB_1}{IB_2} = \frac{IC_1}{IC_2}$, whence I is the fixed point of the similarity, as claimed. Note that this argument proves, again, that the radius of the Fuhrmann circle $A_2B_2C_2$ equals OI.

The figure used in our argument suggests a way to construct the geometric mean of two given segments as a one-step-shorter alternative to the method handed down to us by Euclid. Copy the given lengths AC and BC along a line with B between A and C: Construct the perpendicular bisector of BC and call D either point where it meets the circle with center A and radius AC. Then CD is the geometric mean of AC and BC (because, $\frac{AC}{CD} = \frac{CD}{BC}$).

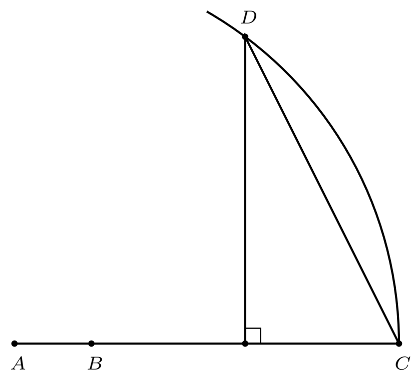

Figure 11. CD is the geometric mean of AC and BC

Further comments on the point R from Theorems 5 through 7.

From the definition of twin points in Section 5 we can immediately deduce that the twin of a point D with respect to triangle ABC is the point common to the three circles ABD_c, BCD_a, and CAD_b, where D_c, D_a, and D_b are the reflections of D in the sides of the triangle. From this construction we see that except for the orthocenter (whose reflection in a side of the triangle lies on the circumcircle), each point in the plane is paired with a unique twin. The point R, which by Theorem 5 can be defined to be the incenter's twin point (or the incenter's antigonal conjugate if you prefer Grinberg's terminology), is listed as X_{80} in [11], where it is defined to be the reflection of the incenter in the Feuerbach point (which agrees with Fuhrmann's Theorem 7).

Numerous properties of twin points are listed by Grinberg in his note [9]. There he quotes that the twin of a point D with respect to triangle ABC is the isogonal conjugate of the inverse (with respect to the circumcircle) of the isogonal conjugate of D. One can use that property to help show that O (the circumcenter of triangle ABC) and R are isogonal conjugates with respect to the Fuhrmann triangle $A_2B_2C_2$ as follows: Indeed, we see that P (the center of the Fuhrmann circle) and O are twin points with respect to the Fuhrmann triangle because $\angle B_2OC_2 =$

$\angle B_1OC_1 = -\angle B_2PC_2$, where the last equality follows from the opposite similarity of triangles $A_1B_1C_1$ and $A_2B_2C_2$; similarly $\angle A_2OC_2 = -\angle A_2PC_2$, and $\angle A_2OB_2 = -\angle A_2PB_2$. Because P is the circumcenter and I the orthocenter of triangle $A_2B_2C_2$, they are isogonal conjugates. According to the proof of Theorem 7, R is the inverse of I with respect to the Fuhrmann circle. It follows that R and O are isogonal conjugates with respect to the Fuhrmann triangle $A_2B_2C_2$.

The triangle $\alpha\beta\gamma$, whose vertices (which appeared in Sections 3 and 9) are the points where the incircle of triangle ABC touches the sides, is variously called the *Gergonne*, or *intouch*, or *contact triangle of triangle* ABC. A theorem attributed to the Japanese mathematician Kariya Yosojirou says that *for any real number* k*, if* XYZ *is the image of the contact triangle under the homothecy* $\mathsf{h}(I,k)$*, then triangles* ABC *and* XYZ *are perspective*. [15] Grinberg [8] calls their perspective center the k*-Kariya point of triangle* ABC. The 0-Kariya point is I and the 1-Kariya point is the Gergonne point X_7, while the -1-Kariya point is the Nagel point X_8, called ν by Fuhrmann. We will prove that R is the -2-Kariya point: In this case $|IX| = 2r = |AA_4|$ and $IX \parallel AA_4$, so it follows that $IA_4 \parallel XA$, which implies (by the final claim of Theorem 5) that $X \in AR$.

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[15] This result was independently discovered on at least three occasions: First by Auguste Boutin [2, 3], then by V. Retali [12], and then by J. Kariya [10]. Kariya's paper inspired numerous results appearing in *L'Enseignement mathématique* over the following two years. We thank Hidetoshi Fukagawa for his help in tracking down these references.