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On the Foci of Circumparabolas

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Abstract. We establish some results about the foci of the infinitely many parabolas passing through three given points A, B, C. A very simple construction directly leads to their barycentric coordinates which provide, besides their locus, a nice and unexpected link with the foci of the parabolas tangent to the sidelines of triangle ABC.

1. Introduction

The parabolas tangent to the sidelines of a triangle are well-known: the pair focus-directrix of such an *in*parabola is formed by a point of the circumcircle other than the vertices and its Steiner line (see [2] for example). In view of such a simple result, one is encouraged to consider the parabolas passing through the vertices or *circum*parabolas. The purpose of this note is to show how an elementary construction of their foci leads to some interesting results.

2. A construction

Given a triangle ABC, a ruler and compass construction of the focus of a circumparabola follows from two well-known results that we recall as lemmas.

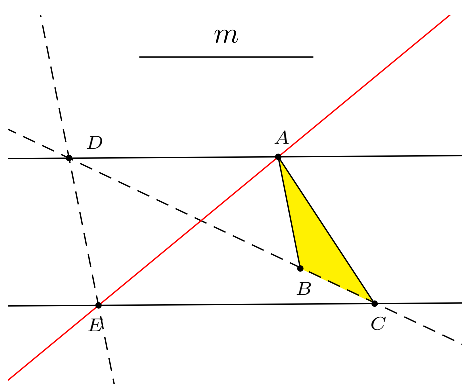

Figure 1.

Lemma 1. *Let P be a point on a parabola. The symmetric of the diameter through P in the tangent at P passes through the focus.*

Lemma 2. *Given three points A, B, C of a parabola and the direction m of its axis, the tangents to the parabola at A, B, C are constructible with ruler and compass.*

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Lemma 1 is an elementary property of the tangents to a parabola. Lemma 2 follows by applying Pascal's theorem to the six points m, B, A, A, m, C on the parabola. The intersections $D = BC \cap Am$, $E = mC \cap AA$ and $F = mm \cap BA$ are collinear. Note that F is the infinite point of AB. Therefore, by constructing
(i) the parallel to m through A to meet BC at D,
(ii) the parallels to AB through D to meet the parallel to m through C at E,
we obtain the line AE as the tangent to the parabola at A (see Figure 1).

Now, let ABC be a triangle and m be a direction other than the directions of its sides. It is an elementary fact that a unique parabola $\mathcal{P}_m$ with axis of direction m passes through A, B, C. The tangents to $\mathcal{P}_m$ at A, B, C can be drawn in accordance with Figure 1. Then, Lemma 1 indicates how to complete the construction of the focus F of this circumparabola $\mathcal{P}_m$ (see Figure 2). Note that the directrix is the line through the symmetrics $F_a,\ F_b,\ F_c$ of F in the tangents at A, B, C.

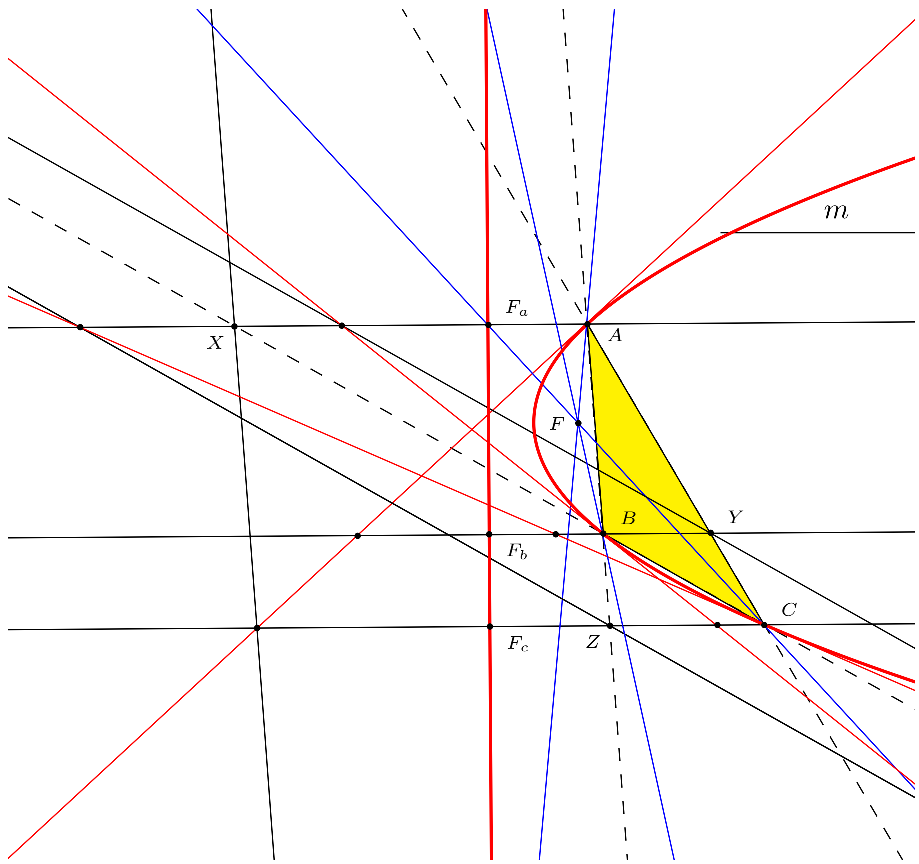

Figure 2.

3. The barycentric coordinates of F

In what follows, all barycentric coordinates are relatively to (A, B, C), and as usual, $a = BC, b = CA, c = AB$. First, we give a short proof of a result that will be needed later:

Lemma 3. *If $(f : g : h)$ is the infinite point of the line ℓ, then the infinite point $(f' : g' : h')$ of the perpendiculars to ℓ is given by*

$$f' = gS_B - hS_C, \quad g' = hS_C - fS_A, \quad h' = fS_A - gS_B$$

where $S_A = \frac{b^2+c^2-a^2}{2}$, $S_B = \frac{c^2+a^2-b^2}{2}$, $S_C = \frac{a^2+b^2-c^2}{2}$.

Proof. Note that $f + g + h = 0 = f' + g' + h'$ and that S_A, S_B, S_C are just the dot products $\overrightarrow{AB} \cdot \overrightarrow{AC}, \overrightarrow{BC} \cdot \overrightarrow{BA}, \overrightarrow{CA} \cdot \overrightarrow{CB}$, respectively. Expressing that the vectors $g\overrightarrow{AB} + h\overrightarrow{AC}$ and $g'\overrightarrow{AB} + h'\overrightarrow{AC}$ are orthogonal yields

$$0 = (g\overrightarrow{AB} + h\overrightarrow{AC}) \cdot (g'\overrightarrow{AB} + h'\overrightarrow{AC}) = g'(gc^2 + hS_A) + h'(gS_A + hb^2)$$

so that

$$\frac{g'}{gS_A + hb^2} = \frac{-h'}{hS_A + gc^2} = \frac{f'}{-gS_A - hb^2 + hS_A + gc^2}$$

or, as it is easily checked,

$$\frac{f'}{gS_B - hS_C} = \frac{g'}{hS_C - fS_A} = \frac{h'}{fS_A - gS_B}.$$

□

For an alternative proof of this lemma, see [3].

From now on, we identify direction and infinite point and denote by $\mathcal{P}_m$ or $\mathcal{P}_{u,v,w}$ the circumparabola whose axis has direction $m = (u : v : w)$ (distinct from the directions of the sides of triangle ABC). Translating the construction of the previous paragraph analytically, we will obtain the coordinates of F.

Theorem 4. *Let u, v, w be real numbers with $u, v, w \neq 0$ and $u + v + w = 0$. Barycentric coordinates of the focus F of the circumparabola $\mathcal{P}_{u,v,w}$ are*

$$\left(\frac{u^2}{vw} + \frac{a^2vw}{\rho} \ : \ \frac{v^2}{wu} + \frac{b^2wu}{\rho} \ : \ \frac{w^2}{uv} + \frac{c^2uv}{\rho}\right)$$

where $\rho = a^2vw + b^2wu + c^2uv$.

Proof. First, consider the conic with equation

$$u^2yz + v^2zx + w^2xy = 0. \qquad (1)$$

Clearly, this conic passes through A, B, C and also through the infinite point $m = (u : v : w)$. Moreover, the tangent at this point is $x(wv^2 + vw^2) + y(uw^2 + wu^2) + z(vu^2 + uv^2) = 0$ that is, the line at infinity (since $wv^2 + vw^2 = -uvw = uw^2 + wu^2 = vu^2 + uv^2$). It follows that (1) is the equation of the circumparabola $\mathcal{P}_m$.

From (1), standard calculations give the tangent t_A to $\mathcal{P}_m$ at A and the diameter m_A through A:

$$t_A: \quad w^2y + v^2z = 0, \qquad m_A: \quad wy - vz = 0.$$

Since the infinite point of t_A is $(w^2 - v^2 : v^2 : -w^2)$, the lemma readily provides the normal n_A to $\mathcal{P}_m$ at A:

$$y(w^2S_A - v^2c^2) - z(v^2S_A - w^2b^2) = 0.$$

Now, the symmetric m'_A of m_A in t_A is the polar of the point $(0 : v : w)$ of m_A with respect to the pair of lines (t_A, n_A). The equation of this pair being $(w^2y + v^2z)(y(w^2S_A - v^2c^2) - z(v^2S_A - b^2w^2)) = 0$ that is,

$$y^2w^2(w^2S_A - v^2c^2) + yz(w^4b^2 - v^4c^2) + z^2v^2(b^2w^2 - v^2S_A) = 0,$$

the equation of m'_A is easily found to be

$$y[w^4(ub^2 + va^2) + c^2uvw(uv + w^2)] = z[v^4(uc^2 + wa^2) + b^2uvw(uw + v^2)]$$

From similar equations for the corresponding lines m'_B and m'_C, we immediately see that the common point F of m'_A, m'_B, m'_C has barycentric coordinates

$$x_1 = u^4(vc^2+wb^2)+a^2uvw(vw+u^2), \qquad y_1 = v^4(uc^2+wa^2)+b^2uvw(uw+v^2),$$

$$z_1 = w^4(ub^2 + va^2) + c^2uvw(uv + w^2)$$

or, observing that $u^4(vc^2 + wb^2) = \rho u^3 - a^2u^3vw$,

$$x_1 = \rho u^3 + a^2uv^2w^2, \quad y_1 = \rho v^3 + b^2u^2vw^2, \quad z_1 = \rho w^3 + c^2u^2v^2w. \tag{2}$$

Using $u^3 + v^3 + w^3 = 3uvw$ (since $u + v + w = 0$), an easy computation yields $x_1 + y_1 + z_1 = 4uvw\rho$ and dividing out by $uvw\rho$ in (2) leads to

$$4F = \left(\frac{u^2}{vw} + \frac{a^2vw}{\rho}\right)A + \left(\frac{v^2}{wu} + \frac{b^2wu}{\rho}\right)B + \left(\frac{w^2}{uv} + \frac{c^2uv}{\rho}\right)C. \tag{3}$$

□

4. The locus of F

Theorem 4 gives a parametric representation of the locus of F, a not well-known curve, to say the least! (see Figure 3 below). Clearly, this curve must have asymptotic directions parallel to the sides of triangle ABC. Actually, it is a quintic that can be found in [1] under the reference Q077 with more information, in particular about the asymptotes. A barycentric equation of the curve is also given, but this equation is almost two-page long!

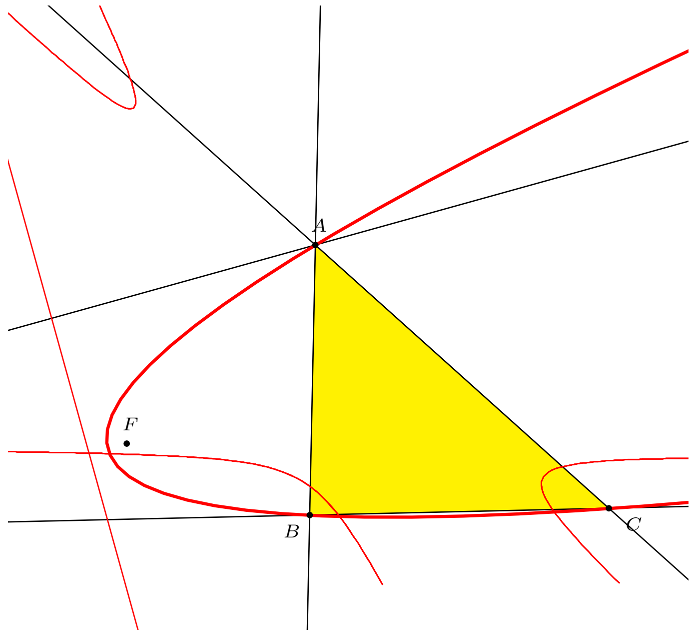

Figure 3.

5. A connection with the inparabolas

Let us write (3) as $4F = 3P + F'$ with

$$3P = \frac{u^2}{vw}A + \frac{v^2}{wu}B + \frac{w^2}{uv}C \quad \text{and} \quad F' = \frac{uvw}{\rho}\left(\frac{a^2}{u}A + \frac{b^2}{v}B + \frac{c^2}{w}C\right).$$

We observe that P is the centroid of the cevian triangle of the infinite point m (triangle XYZ in Figure 2), [1] and F' is the isogonal conjugate of this infinite point. Thus, F' is the focus of the inparabola $\mathcal{P}'_m$ whose axis has direction $m = (u : v : w)$. This point F' is easily constructed as the point of the circumcircle of triangle ABC whose Simson line is perpendicular to m. Since $4\overrightarrow{PF} = \overrightarrow{PF'}$, we see that F' is the image of F under the homothety $\mathsf{h} = \mathsf{h}(P, 4)$ with center P and ratio 4. More can be said.

Theorem 5. *With the notations above,*
(a) $\mathcal{P}'_m$ *is the image of* $\mathcal{P}_m$ *under* $\mathsf{h}(P, 4)$,
(b) *the locus of* P *is the cubic* $\mathcal{C}$ *with barycentric equation* $(x + y + z)^3 = 27xyz$, *the centroid* G *of triangle* ABC *being excluded.*

Proof. (a) As in §3, it is readily checked that a barycentric equation of $\mathcal{P}'_m$ is

$$u^2x^2 + v^2y^2 + w^2z^2 - 2vwyz - 2wuzx - 2uvxy = 0.$$

[1] I thank P. Yiu for this nice observation.

Let $A' = \mathrm{h}(A)$ so that $uvwA' = (4uvw - u^3)A - v^3B - w^3C$. Using $u = -v - w$ repeatedly, a straightforward computation yields

$$u^2(4uvw - u^3)^2 + v^8 + w^8 - 2v^4w^4 + 2uw^4(4uvw - u^3) + 2uv^4(4uvw - u^3) = 0$$

hence A' is on $\mathcal{P}'_m$. By symmetry, the homothetic B' and C' of B and C are on $\mathcal{P}'_m$ as well and therefore the parabolas $\mathcal{P}'_m$ and $\mathrm{h}(\mathcal{P}_m)$ both pass through A', B', C' and $(u : v : w)$. As a result, $\mathrm{h}(\mathcal{P}_m) = \mathcal{P}'_m$.

(b) Since $P(u^3 : v^3 : w^3)$ and $(u^3 + v^3 + w^3)^3 = (u^3 + v^3 + (-u - v)^3)^3 = (-3u^2v - 3uv^2)^3 = 27u^3v^3w^3$, P is on the cubic $\mathcal{C}$. Clearly, $P \neq G$.

Conversely, supposing that $P(x, y, z)$ is on $\mathcal{C}$ and $P \neq G$, we can set $x = u^3, y = v^3, z = w^3$. Then w is given by $w^3 - 3uvw + u^3 + v^3 = 0$ or $(w + u + v)(w^2 - (u + v)w + (u^2 - uv + v^2)) = 0$. If $u \neq v$, the second factor does not vanish, hence $w = -u - v$ and $u + v + w = 0$. If $u = v$, then $w \neq u$ (since $P \neq G$) and we must have $w = -2u = -u - v$ again. □

Note that setting $\frac{u}{w} = t, \frac{v}{w} = -1 - t$, the locus of P can be also be constructed as the set of points P defined by

$$\overrightarrow{CP} = -\frac{t^2}{3(1+t)}\overrightarrow{CA} + \frac{(1+t)^2}{3t}\overrightarrow{CB}.$$

Figure 4 shows $\mathcal{C}$ and the parabolas $\mathcal{P}_m$ and $\mathcal{P}'_m$.

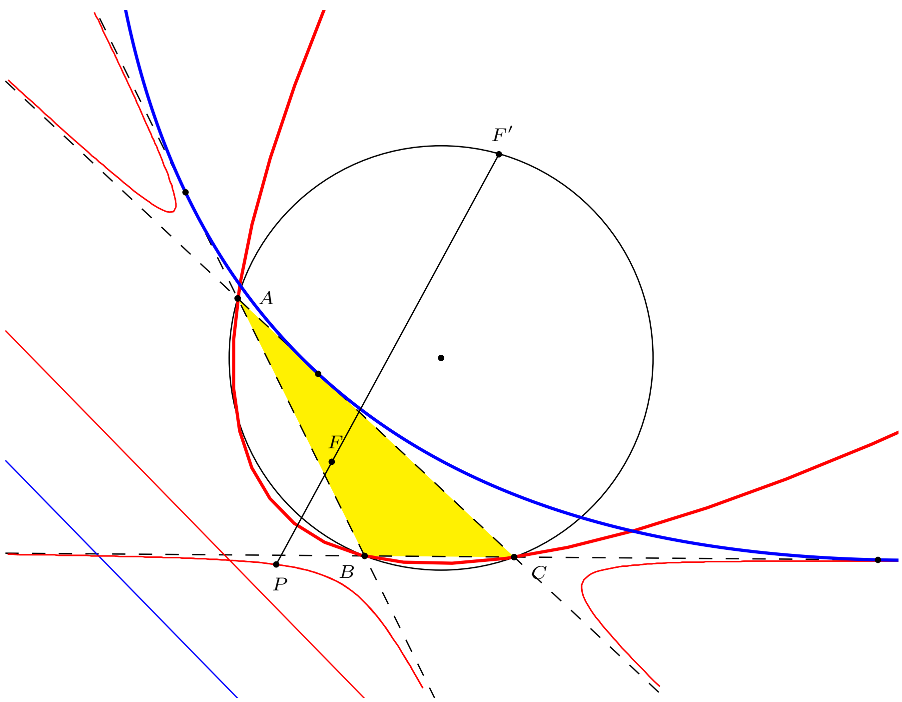

Figure 4.

In some way, the quintic of §4 can be considered as a combination of a quadratic (the circumcircle) and a cubic ($\mathcal{C}$).

References

[1] B. Gibert, http://bernard.gibert.pagesperso-orange.fr/curves/q077.html
[2] Y. and R. Sortais, *La Géométrie du Triangle*, Hermann, 1987, pp. 66-7.
[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic Univ., 2002, pp. 54–5. (available at http://math.fau./yiu/GeometryNotes020402.ps)

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