Perspective Isoconjugate Triangle Pairs, Hofstadter Pairs, and Crosssums on the Nine-Point Circle

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Abstract. The \(r\)-Hofstadter triangle and the \((1 - r)\)-Hofstadter triangle are proved perspective, and homogeneous trilinear coordinates are found for the perspector. More generally, given a triangle \(DEF\) inscribed in a reference triangle \(ABC\), triangles \(A'B'C'\) and \(A''B''C''\) derived in a certain manner from \(DEF\) are perspective to each other and to \(ABC\). Trilinears for the three perspectors, denoted by \(P^*, P_1, P_2\) are found (Theorem 1) and used to prove that these three points are collinear. Special cases include (Theorems 4 and 5) this: if \(X\) and \(X'\) are an antipodal pair on the circumcircle, then the perspector \(P^* = X \oplus X'\), where \(\oplus\) denotes crosssum, is on the nine-point circle. Taking \(X\) to be successively the vertices of a triangle \(DEF\) inscribed in the circumcircle thus yields a triangle \(D'E'F'\) inscribed in the nine-point circle. For example, if \(DEF\) is the circumtangential triangle, then \(D'E'F'\) is an equilateral triangle.

1. Introduction and main theorem

We begin with a very general theorem about three triangles, one being the reference triangle \(ABC\) with sidelengths \(a, b, c\), and the other two, denoted by \(A'B'C'\) and \(A''B''C''\), which we shall now proceed to define. Suppose \(DEF\) is a triangle inscribed in \(ABC\); that is, the vertices are given by homogeneous trilinear coordinates (henceforth simply trilinears) as follows:

\[
D = 0 : y_1 : z_1, \quad E = x_2 : 0 : z_2, \quad F = x_3 : y_3 : 0, \tag{1}
\]

where \(y_1z_1x_2z_2x_3y_3 \neq 0\) (this being a quick way to say that none of the points is \(A, B, C\)). For any point \(P = p : q : r\) for which \(pqr \neq 0\), define

\[
D' = 0 : \frac{1}{qy_1} : \frac{1}{rz_1}, \quad E' = \frac{1}{px_2} : 0 : \frac{1}{rz_2}, \quad F' = \frac{1}{px_3} : \frac{1}{qy_3} : 0.
\]

Define \(A'B'C'\) and introduce symbols for trilinears of the vertices \(A', B', C'\):

\[
A' = CF \cap BE' = u_1 : v_1 : w_1, \\
B' = AD \cap CF' = u_2 : v_2 : w_2, \\
C' = BE \cap AD' = u_3 : v_3 : w_3,
\]

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and similarly,

\[ A'' = CF' \cap BE = \frac{1}{pu_1} : \frac{1}{qv_1} : \frac{1}{rw_1}, \]

\[ B'' = AD' \cap CF = \frac{1}{pu_2} : \frac{1}{qv_2} : \frac{1}{rw_2}, \]

\[ C'' = BE' \cap AD = \frac{1}{pu_3} : \frac{1}{qv_3} : \frac{1}{rw_3}. \]

Thus, triangles \( A'B'C' \) and \( A''B''C'' \) are a \( P \)-isoconjugate pair, in the sense that every point on each is the \( P \)-isoconjugate of a point on the other (except for points on a sideline of \( ABC \)). (The \( P \)-isoconjugate of a point \( X = x : y : z \) is the point \( 1/(px) : 1/(qy) : 1/(rz) \); this is the isogonal conjugate of \( X \) if \( P \) is the incenter, and the isotomic conjugate of \( X \) if \( P = X_{31} = a^2 : b^2 : c^2 \). Here and in the sequel, the indexing of triangles centers in the form \( X_t \) follows that of [4].)

**Theorem 1.** The triangles \( ABC, A'B'C', A''B''C'' \) are pairwise perspective, and the three perspectors are collinear.

**Proof.** The lines \( CF \) given by \(-y_3\alpha + x_3\beta = 0\) and \( BE' \) given by \( px_2\alpha - rz_2\gamma = 0\), and cyclic permutations, give

\[ A' = CF \cap BE' = u_1 : v_1 : w_1 = rx_3z_2 : ry_3z_2 : px_2x_3, \]

\[ B' = AD \cap CF' = u_2 : v_2 : w_2 = qy_3y_1 : px_3y_1 : px_3z_1, \]

\[ C' = BE \cap AD' = u_3 : v_3 : w_3 = qy_1x_2 : rz_2z_1 : qy_1z_2; \]

\[ A'' = CF' \cap BE = qx_2y_3 : px_2x_3 : qy_3z_2, \]

\[ B'' = AD' \cap CF = rz_1x_3 : ry_3z_1 : qy_3y_1, \]

\[ C'' = BE' \cap AD = rz_1z_2 : px_2y_1 : pz_1x_2. \]

Then the line \( B'B'' \) is given by \( d_1\alpha + d_2\beta + d_3\gamma = 0 \), where

\[ d_1 = px_3y_3(qy_2^2 - rz_1^2), \quad d_2 = px_3y_1^2 - qy_1^2y_3^2, \quad d_3 = ry_1z_1(qy_3^2 - px_3^2), \]

and the line \( C'C'' \) by \( d_4\alpha + d_5\beta + d_6\gamma = 0 \), where

\[ d_4 = px_2z_2(rz_1^2 - qy_2^2), \quad d_5 = qy_1z_1(rz_2^2 - px_3^2), \quad d_6 = pqx_2y_2^2 - r^2z_1^2z_2^2. \]

The perspector of \( A'B'C' \) and \( A''B''C'' \) is \( B'B'' \cap C'C'' \), with trilinears

\[ d_2d_6 - d_3d_5 : d_3d_4 - d_1d_5 - d_2d_4. \]

These coordinates share three common factors, which cancel, leaving the perspector

\[ P^* = qx_2y_1y_3 + rx_3z_1z_2 : ry_3z_1z_2 + px_2x_3y_1 : px_2x_3z_1 + qy_1y_3z_2. \]  \( \text{(2)} \)

Next, we show that the lines \( AA', BB', CC' \) concur. These lines are given, respectively, by

\[-px_2z_3\beta + ry_3z_2\gamma = 0, \quad px_2y_3\alpha - qy_3y_1\gamma = 0, \quad -rz_1z_2\alpha + qx_2y_1\beta = 0, \]

from which it follows that the perspector of \( ABC \) and \( A'B'C' \) is the point

\[ P_1 = AA' \cap BB' \cap CC' = qx_2y_1y_3 : ry_3z_1z_2 : px_2x_3z_1. \]  \( \text{(3)} \)
The same method shows that the lines $AA'', BB'', CC''$ concur in the $P$-isoconjugate of $P_1$:

$$P_2 = AA'' \cap BB'' \cap CC'' = rx_3z_1z_2 : py_1x_2x_3 : qy_1y_3z_2.$$ (4)

Obviously,

$$\begin{vmatrix}
q x_2 y_1 y_3 + r x_3 z_1 z_2 & r y_3 z_1 z_2 + p x_2 x_3 y_1 & p x_2 x_3 z_1 + q y_1 y_3 z_2 \\
q x_2 y_1 y_3 & r y_3 z_1 z_2 & p x_2 x_3 z_1 \\
r x_3 z_1 z_2 & p y_1 x_2 x_3 & q y_1 y_3 z_2 \\
\end{vmatrix} = 0,$$

so that the three perspectors are collinear.

**Example 1.** Let $P = 1 : 1 : 1$ (the incenter), and let $DEF$ be the cevian triangle of the centroid, so that $D = 0 : ca : ab$, etc. Then

$$P_1 = \frac{c}{b} : \frac{a}{c} : \frac{b}{a} \quad \text{and} \quad P_2 = \frac{b}{c} : \frac{c}{a} : \frac{a}{b},$$

these being the 1st and 2nd Brocard points, and

$$P^* = a(b^2 + c^2) : b(c^2 + a^2) : c(a^2 + b^2),$$

the midpoint $X_{39}$ of segment $P_1P_2$.

2. Hofstadter triangles

Suppose $r$ is a nonzero real number. Following ([3], p 176, 241), regard vertex $B$ as a pivot, and rotate segment $BC$ toward vertex $A$ through angle $rB$. Let $L_{BC}$ denote the line containing the rotated segment. Similarly, obtain line $L_{CB}$ by rotating segment $BC$ about $C$ through angle $rC$. Let $A' = L_{BC} \cap L_{CB}$, and obtain similarly points $B'$ and $C'$. The $r$-Hofstadter triangle is $A'B'C'$, and the $(1-r)$-Hofstadter triangle $A''B''C''$ is formed in the same way using angles $(1-r)A$, $(1-r)B$, $(1-r)C$.

![Diagram of Hofstadter triangles](image-url)
Trilinears for \( A' \) and \( A'' \) are easily found, and appear here in rows 2 and 3 of an equation for line \( A'A'' \):

\[
\begin{vmatrix}
\alpha & \beta & \gamma \\
\sin B \sin C & \sin(B - rB) \sin(C - rC) & \sin(rA \sin(B - rB)) \\
\sin B \sin(C - rC) & \sin rC \sin(B - rB) & \sin rB \sin(C - rC) \\
\sin rB \sin(C - rC) & \sin rC \sin(B - rB) & \sin rB \sin(C - rC)
\end{vmatrix} = 0.
\]

Lines \( B'B'' \) and \( C'C'' \) are similarly obtained, or obtained from \( A'A'' \) by cyclic permutations of symbols. It is then found by computer that the perspector of the \( r \)- and \( (1 - r) \)-Hofstadter triangles is the point

\[ P(r) = \sin(A - rA) \sin rB \sin rC + \sin rA \sin(B - rB) \sin(C - rC) \]

\[ : \sin(B - rB) \sin rC \sin rA + \sin rB \sin(C - rC) \sin(A - rA) \]

\[ : \sin(C - rC) \sin rA \sin rB + \sin rC \sin(A - rA) \sin(B - rB). \]

The domain of \( P \) excludes 0 and 1. When \( r \) is any other integer, it can be checked that \( P(r) \), written as \( u : v : w \), satisfies

\[ u \sin A + v \sin B + w \sin C = 0, \]

which is to say that \( P(r) \) lies on the line \( L^\infty \) at infinity. For example, \( P(2) \), alias \( P(-1) \), is the point \( X_{30} \) in which the Euler line meets \( L^\infty \). Also, \( P\left(\frac{1}{2}\right) = X_1 \), the incenter, and \( P\left(-\frac{1}{2}\right) = P\left(\frac{3}{2}\right) = X_{1770} \). Regarding \( r = 1 \) and \( r = 0 \), we obtain, as limits, the Hofstadter one-point and Hofstadter zero-point:

\[ P(1) = X_{359} = \frac{a}{A} : \frac{b}{B} : \frac{c}{C}, \]

\[ P(0) = X_{360} = \frac{A}{a} : \frac{B}{b} : \frac{C}{c}, \]

remarkable because of the “exposed” vertex angles \( A, B, C \). Another example is \( P\left(\frac{1}{2}\right) = X_{356} \), the centroid of the Morley triangle.

This scattering of results can be supplemented by a more systematic view of selected points \( P(r) \). In Figure 2, the specific triangle \( \sigma \) is used to show the points \( P\left(r + \frac{1}{2}\right) \) for \( r = 0, 1, 2, 3, \ldots, 5000 \).

If the swing angles \( rA, rB, rC, (1 - r)B, (1 - r)B, (1 - r)C \) are generalized to \( rA + \theta, rB + \theta, rC + \theta, (1 - r)B + \theta, (1 - r)B + \theta, (1 - r)C + \theta \), then the perspector is given by

\[ P(r, \theta) = \sin(A - rA + \theta) \sin(rB - \theta) \sin(rC - \theta) \]

\[ + \sin(rA - \theta) \sin(B - rB + \theta) \sin(C - rC + \theta) \]

\[ : \sin(B - rB + \theta) \sin(rC - \theta) \sin(rA - \theta) \]

\[ + \sin(rB - \theta) \sin(C - rC + \theta) \sin(A - rA + \theta) \]

\[ : \sin(C - rC + \theta) \sin(rA - \theta) \sin(rB - \theta) \]

\[ = + \sin(rC + \theta) \sin(A - rA + \theta) \sin(B - rB + \theta). \]
Trilinears for the other two perspectors are given by

\[ P_1(r, \theta) = \sin(rA - \theta) \csc(A - rA + \theta) \]
\[ : \sin(rB - \theta) \csc(B - rB + \theta) : \sin(rC - \theta) \csc(A - rC + \theta); \]
\[ P_2(r, \theta) = \sin(A - rA + \theta) \csc(rA - \theta) \]
\[ : \sin(B - rB + \theta) \csc(rB - \theta) : \sin(A - rC + \theta) \csc(rC - \theta). \]

If \(0 < \theta < 2\pi\), then \(P(0, \theta)\) is defined, and taking the limit as \(\theta \to 0\) enables a definition of \(P(0, 0)\). Then, remarkably, the locus of \(P(0, \theta)\) for \(0 \leq \theta < 2\pi\) is the Euler line. Six of its points are indicated here:

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0</th>
<th>(\pi/6)</th>
<th>(\pi/4)</th>
<th>(\pi/3)</th>
<th>(\pi/2)</th>
<th>((1/2) \arccos(5/2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(0, \theta))</td>
<td>(X_2)</td>
<td>(X_5)</td>
<td>(X_4)</td>
<td>(X_{30})</td>
<td>(X_{20})</td>
<td>(X_{549})</td>
</tr>
</tbody>
</table>

In general,

\[ P \left(0, \frac{1}{2} \arccos t\right) = t \cos A + \cos B \cos C : t \cos B + \cos C \cos A : t \cos C + \cos A \cos B.\]

Among intriguing examples are several for which the angle \(\theta\), as a function of \(a, b, c\) (or \(A, B, C\)), is not constant:
if \( \theta = \arctan \frac{4\sigma}{\sigma^2 + b^2 + c^2} \) (the Brocard angle) then \( P(0, \theta) = \omega \)

\[
\frac{1}{2} \arccos \left( 3 - \frac{|OI|^2}{2R^2} \right) \quad X_{384} \\
\frac{1}{2} \arccos \frac{|OI|}{2R} \quad X_{21} \\
\frac{1}{2} \arccos \left( -1 - 2 \cos^2 A \cos^2 B \cos^2 C \right) \quad X_{144} \\
\frac{1}{2} \arccos \left( -\frac{1}{3} - \frac{4}{3} \cos^2 A \cos^2 B \cos^2 C \right) \quad X_{23}
\]

where

\[
\sigma = \text{area of } ABC, \\
|OI| = \text{distance between the circumcenter and the incenter, } \\
R = \text{circumradius of } ABC.
\]

3. Cevian triangles

The cevian triangle of a point \( X = x : y : z \) is defined by (1) on putting \((x_i, y_i, z_i) = (x, y, z)\) for \(i = 1, 2, 3\). Suppose that \( X \) is a triangle center, so that

\[ X = g(a, b, c) : g(b, c, a) : g(c, a, b) \]

for a suitable function \( g(a, b, c) \). Abbreviating this as \( X = g_a : g_b : g_c \), the cevian triangle of \( X \) is then given by

\[ D = 0 : g_b : g_c, \quad E = g_a : 0 : g_c, \quad F = g_a : g_b : 0, \]

and the perspector of the derived triangles \( A'B'C' \) and \( A''B''C'' \) in Theorem 1 is given by

\[ P^* = g_a \left( qg_b^2 + rg_c^2 \right) : g_b \left( pg_a^2 + rg_c^2 \right) : g_c \left( pg_a^2 + qg_b^2 \right), \]

which is a triangle center, namely the crosspoint (defined in the Glossary of [4]) of \( U \) and the \( P \)-isoconjugate of \( U \). In this case, the other two perspectors are

\[ P_1 = qg_a g_b^2 : rg_b g_c^2 : pg_c g_a^2 \text{ and } P_2 = rg_a g_c^2 : pg_c g_a^2 : qg_a g_b^2. \]

4. Pedal triangles

Suppose \( X = x : y : z \) is a point for which \( xyz \neq 0 \). The pedal triangle \( DEF \) of \( X \) is given by

\[ D = 0 : y + xc_1 : z + xb_1, \quad E = x + yc_1 : 0 : z + ya_1, \quad F = x + zb_1 : y + xa_1 : 0, \]

where

\[
(a_1, b_1, c_1) = (\cos A, \cos B, \cos C) = \left( \frac{b^2 + c^2 - a^2}{2bc}, \frac{c^2 + a^2 - b^2}{2ca}, \frac{a^2 + b^2 - c^2}{2ab} \right).
\]
The three perspectors as in Theorem 1 are given, as in (2)-(4) by
\[
\begin{align*}
P^* &= u + u' : v + v' : w + w', \\
P_1 &= u : v : w, \\
P_2 &= u' : v' : w',
\end{align*}
\]
where
\[
\begin{align*}
 u &= q(x + yc_1)(y + xc_1)(y + za_1), \\
v &= r(y + za_1)(z + ya_1)(z + xb_1), \\
w &= p(z + xb_1)(x + zb_1)(x + yc_1); \\
u' &= r(x + zb_1)(z + xb_1)(z + ya_1), \\
v' &= p(y + xc_1)(x + yc_1)(x + zb_1), \\
w' &= q(z + ya_1)(y + za_1)(y + xc_1).
\end{align*}
\]

The perspector $P^*$ is notable in two cases which we shall now consider: when $X$ is on the line at infinity, $L^\infty$, and when $X$ is on the circumcircle, $\Gamma$.

**Theorem 2.** Suppose $DEF$ is the pedal triangle of a point $X$ on $L^\infty$. Then the perspectors $P^*$, $P_1$, $P_2$ are invariant of $X$, and $P^*$ lies on $L^\infty$.

**Proof.** The three perspectors as in Theorem 1 are given as in (5)-(7) by
\[
\begin{align*}
P^* &= a(b^2r - c^2q) : b(c^2p - a^2r) : c(a^2q - b^2p), \\
P_1 &= qac^2 : rba^2 : pbc^2, \\
P_2 &= rab^2 : pbc^2 : qca^2.
\end{align*}
\]
Clearly, the trilinears in (8) satisfy the equation $a\alpha + b\beta + c\gamma = 0$ for $L^\infty$. □

**Example 2.** For $P = 1 : 1 : 1 = X_1$, we have $P^* = a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2)$, indexed in ETC as $X512$. This and other examples are included in the following table.

<table>
<thead>
<tr>
<th>$P$</th>
<th>661</th>
<th>1</th>
<th>6</th>
<th>32</th>
<th>663</th>
<th>649</th>
<th>667</th>
<th>19</th>
<th>25</th>
<th>184</th>
<th>48</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*$</td>
<td>511</td>
<td>512</td>
<td>513</td>
<td>514</td>
<td>517</td>
<td>518</td>
<td>519</td>
<td>520</td>
<td>521</td>
<td>522</td>
<td>523</td>
<td>788</td>
</tr>
</tbody>
</table>

We turn now to the case that $X$ is on $\Gamma$, so that pedal triangle of $X$ is degenerate, in the sense that the three vertices $D, E, F$ are collinear ([11], [3]). The line $DEF$ is known as the pedal line of $X$. We restrict the choice of $P$ to the point $X_{31}$:
\[
P = a^2 : b^2 : c^2,
\]
so that the $P$-isoconjugate of a point is the isotomic conjugate of the point.

**Theorem 3.** Suppose $X$ is a point on the circumcircle of $ABC$, and $P = a^2 : b^2 : c^2$. Then the perspector $P^*$, given by
\[
\begin{align*}
P^* &= bcx(y^2 - z^2) (ax(bz - cy) + yz(b^2 - c^2)) \\
&\quad : cay(z^2 - x^2) (by(cx - az) + zx(c^2 - a^2)) \\
&\quad : abz(x^2 - y^2) (cz(ay - bx) + xy(a^2 - b^2)),
\end{align*}
\]

\[
(9)
\]

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lies on the nine-point circle.

Proof. Since $X$ satisfies $a \frac{y}{x} + b \frac{z}{y} + c \frac{x}{z} = 0$, we can and do substitute $z = -\frac{cxy}{ay+bx}$ in (8), obtaining

$$P^* = \alpha : \beta : \gamma,$$

where

$$\alpha = ycb (ay + bx - cx) (ay + bx + cx) (2abx + a^2y + b^2y - c^2y),$$
$$\beta = xca (2aby + a^2x + b^2x - c^2x) (ay + bx - cy) (ay + bx + cy),$$
$$\gamma = ab (ay + bx) (b^3x - a^3y - a^2bx + ab^2y + ac^2y - bc^2x) (x + y) (y - x).$$

An equation for the nine-point circle [5] is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (1/2)(a\alpha + b\beta + c\gamma)(a_1\alpha + b_1\beta + c_1\gamma) = 0,$$

and using a computer, we find that $P^*$ indeed satisfies (11). Using $ay+by = -\frac{cxy}{z}$, one can verify that the trilinears in (10) yield those in (9). □

A description of the perspector $P^*$ in Theorem 3 is given in Theorem 4, which refers to the antipode $X'$ of $X$, defined as the reflection of $X$ in the circumcenter, $O$; i.e., $X'$ is the point on $\Gamma$ that is on the opposite side of the diameter that contains $X$. Theorem 4 also refers to the crosssum of two points, defined (Glossary of [4]) for points $U = u : v : w$ and $U' = u' : v' : w'$ by

$$U \oplus U' = vw' + wv' : vu' + uv' : uu' + uv'.$$

Theorem 4. Suppose $X$ is a point on the circumcircle of $ABC$, and let $X'$ denote the antipode of $X$. Then $P^* = X \oplus X'$.

Proof. 1 Since $X$ is an arbitrary point on $\Gamma$, there exists $\theta$ such that

$$X = \csc \theta : \csc(C - \theta) : -\csc(B + \theta),$$

where $\theta$, understood here a function of $a, b, c$, is defined ([16], [3, p. 39]) by

$$0 \leq 2\theta = \angle AOX < \pi,$$

so that the antipode of $X$ is

$$X' = \sec \theta : -\sec(C - \theta) : -\sec(B + \theta).$$

The crosssum of the two antipodes is the point $X \oplus X' = \alpha : \beta : \gamma$ given by

$$\alpha = -\csc(C - \theta) \sec(B + \theta) + \csc(B + \theta) \sec(C - \theta)$$
$$\beta = -\csc(B + \theta) \sec \theta - \csc \theta \sec(B + \theta)$$
$$\gamma = -\csc \theta \sec(C - \theta) + \csc(C - \theta) \sec \theta.$$

It is easy to check by computer that $\alpha : \beta : \gamma$ satisfies (11). □

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1This proof includes a second proof that $P^*$ lies on the nine-point circle.
Example 3. In Theorems 3 and 4, let \( X = X_{1113} \), this being a point of intersection of the Euler line and the circumcircle. The antipode of \( X \) is \( X_{1114} \), and we have
\[
X_{1113} \oplus X_{1114} = X_{125},
\]
the center of the Jerabek hyperbola, on the nine-point circle.

Example 4. The antipode of the Tarry point, \( X_{98} \), is the Steiner point, \( X_{99} \), and
\[
X_{98} \oplus X_{99} = X_{2679}.
\]

Example 5. The antipode of the point, \( X_{101} = \frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b} \) is \( X_{103} \), and
\[
X_{101} \oplus X_{103} = X_{1566}.
\]

Example 6. The Euler line meets the line at infinity in the point \( X_{30} \), of which the isogonal conjugate on the circumcircle is the point
\[
X_{74} = \frac{1}{\cos A - 2 \cos B \cos C} : \frac{1}{\cos B - 2 \cos C \cos A} : \frac{1}{\cos C - 2 \cos A \cos B}.
\]
The antipode of \( X_{74} \) is the center of the Kiepert hyperbola, given by
\[
X_{110} = \csc(B - C) : \csc(C - A) : \csc(A - B),
\]
and we have
\[
X_{74} \oplus X_{110} = X_{3258}.
\]

Our final theorem gives a second description of the perspector \( X \oplus X' \) in (9). The description depends on the point \( X(\text{medial}) \), this being functional notation, read “\( X \) of medial (triangle)”, in the same way that \( f(x) \) is read “\( f \) of \( x \)”; the variable triangle to which the function \( X \) is applied is the cevian triangle of the centroid, whose vertices are the midpoint of the sides of the reference triangle \( ABC \). Clearly, if \( X \) lies on the circumcircle of \( ABC \), then \( X(\text{medial}) \) lies on the nine-point circle of \( ABC \).

Theorem 5. Let \( X' \) be the antipode of \( X \). Then \( X \oplus X' \) is a point of intersection of the nine-point circle and the line of the following two points: the isogonal conjugate of \( X \) and \( X(\text{medial}) \).

Proof. Trilinears for \( X(\text{medial}) \) are given ([3, p. 86]) by
\[
(by + cz)/a : (cz + ax)/b : (ax + by)/c.
\]
Writing \( u : v : w \) for trilinears for \( X \oplus X' \) and \( yz : zx : xy \) for the isogonal conjugate of \( X \), and putting \( z = -cxy/(ay + bx) \) because \( X \in \Gamma \), we find
\[
\begin{vmatrix}
  u & v & w \\
  yz & zx & xy \\
  (by + cz)/a & (cz + ax)/b & (ax + by)/c
\end{vmatrix} = 0,
\]
so that the three points are collinear. \( \square \)
As a source of further examples for Theorems 4 and 5, suppose $D, E, F$ are points on the circumcircle. Let $D', E', F'$ be the respective antipodes of $D, E, F$, so that the triangle $D'E'F'$ is the reflection in the circumcenter of triangle $DEF$. Let

$$D'' = D \oplus D', \quad E'' = E \oplus E', \quad F'' = F \oplus F',$$

so that $D''E''F''$ is inscribed in the nine-point circle.

**Example 7.** If $DEF$ is the circumcevian triangle of the incenter, then $D''E''F''$ is the medial triangle.

**Example 8.** If $DEF$ is the circumcevian triangle of the circumcenter, then $D''E''F''$ is the orthic triangle.

**Example 9.** If $DEF$ is the circumtangential triangle, then $D''E''F''$ is homothetic to each of the three Morley equilateral triangles, as well as the circumtangential triangle (perspector $X_2$, homothetic ratio $-1/2$) the circumnormal triangle (perspector $X_4$, homothetic ratio $1/2$), and the Stammler triangle (perspector $X_{381}$). If $DEF$ is the circumnormal triangle, then $D''E''F''$ is the same as for the circumtangential. (For descriptions of the various triangles, see [5].) The triangle $D''E''F''$ is the second of two equilateral triangles described in the article on the Steiner deltoid at [5]; its vertices are given as follows:

$$D'' = \cos(B - C) - \cos\left(\frac{B - C}{3}\right),$$

$$E'' = \cos(B - C) - \cos\left(\frac{C - A}{3}\right),$$

$$F'' = \cos(B - C) - \cos\left(\frac{A - B}{3}\right).$$

5. Summary and concluding remarks

If the point $X$ in Section 4 is a triangle center, as defined at [5], then the perspector $P^*$ is a triangle center. If instead of the cevian triangle of $X$, we use in Section 4 a central triangle of type 1 (as defined in [3], pp. 53-54), then $P^*$ is clearly the same point as obtained from the cevian triangle of $X$.

Regarding pedal triangles, in Section 4, there, too, if $X$ is a triangle center, then so is $P^*$, in (8). The same perspector is obtained by various central triangles of type 2. In all of these cases, the other two perspectors, $P_1$ and $P_2$, as in (6) and (7) are a bicentric pair [5].
In Examples 3-6, the antipodal pairs are triangle centers. The 90° rotation of such a pair is a bicentric pair, as in the following example.

**Example 10.** The Euler line meets the circumcircle in the points $X_{1113}$ and $X_{1114}$. Let $X_{1113}^*$ and $X_{1114}^*$ be their 90° rotations about the circumcenter. Then $X_{1113}^* \oplus X_{1114}^*$ (the perspector of two triangles $A'B'C'$ and $A''B''C''$ as in Theorem 1) lies on the nine-point circle, in accord with Theorems 4 and 5. Indeed $X_{1113}^* \oplus X_{1114}^* = X_{113}$, which is the nine-point-circle-antipode of $X_{125} = X_{1113} \oplus X_{1114}$. Likewise, $X_{1379}^* \oplus X_{1380}^* = X_{114}$ and $X_{1381}^* \oplus X_{1382}^* = X_{119}$.

Example 10 illustrates the following theorem, which the interested reader may wish to prove: Suppose $X$ and $Y$ are circumcircle-antipodes, with 90° rotations $X^*$ and $Y^*$. Then $X^* \oplus Y^*$ is the nine-point-circle-antipode of the center of the rectangular circumhyperbola formed by the isogonal conjugates of the points on the line $XY$.

**References**


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