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The Droz-Farny Circles of a Convex Quadrilateral

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Abstract. The Droz-Farny circles of a triangle are a pair of circles of equal radii obtained by particular geometric constructions. In this paper we deal with the problem to see whether and how analogous properties of concyclicity hold for convex quadrilaterals.

1. Introduction

The Droz-Farny circles of a triangle are a pair of circles of equal radii obtained by particular geometric constructions [4]. Let **T** be a triangle of vertices A_1, A_2, A_3, with circumcenter O and orthocenter H. Let H_i be the foot of the altitude of **T** at A_i, and M_i the middle point of the side A_iA_{i+1} (with indices taken modulo 3); see Figure 1.

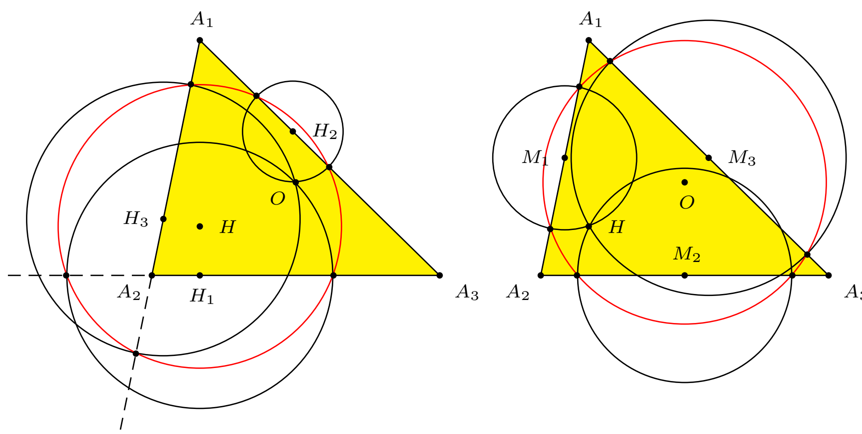

Figure 1.

(a) If we consider the intersections of the circle $H_i(O)$ (center H_i and radius H_iO) with the line $A_{i+1}A_{i+2}$, then we obtain six points which all lie on a circle with center H (first Droz-Farny circle).

(b) If we consider the intersections of the circle $M_i(H)$ (center M_i and radius M_iH) with the line A_iA_{i+1}, then we obtain six points which all lie on a circle with center O (second Droz-Farny circle).

The property of the first Droz-Farny circle is a particular case of a more general property (first given by Steiner and then proved by Droz-Farny in 1901 [2]). Fix a segment of length r, if for $i = 1, 2, 3$, the circle with center A_i and radius r

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intersects the line M_iM_{i+2} in two points, then we obtain six points all lying on a circle Γ with center H. When r is equal to the circumradius of $\mathbf{T}$ we obtain the first Droz-Farny circle (see Figure 2).

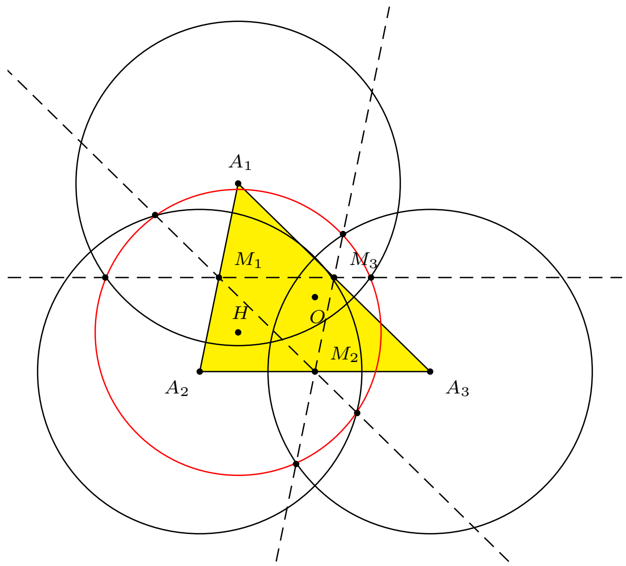

Figure 2.

In this paper we deal with the problem to see whether and how analogous properties of concyclicity hold for convex quadrilaterals.

2. An eight-point circle

Let $A_1A_2A_3A_4$ be a convex quadrilateral, which we denote by $\mathbf{Q}$, and let G be its centroid. Let $\mathbf{V}$ be the Varignon parallelogram of $\mathbf{Q}$, *i.e.*, the parallelogram $M_1M_2M_3M_4$, where M_i is the middle point of the side A_iA_{i+1}. Let H_i be the foot of the perpendicular drawn from M_i to the line $A_{i+2}A_{i+3}$. The quadrilateral $H_1H_2H_3H_4$, which we denote by $\mathbf{H}$, is called the principal orthic quadrilateral of $\mathbf{Q}$ [5], and the lines M_iH_i are the maltitudes of $\mathbf{Q}$. We recall that the maltitudes of $\mathbf{Q}$ are concurrent if and only if $\mathbf{Q}$ is cyclic [6]. If $\mathbf{Q}$ is cyclic, the point of concurrency of the maltitudes is called anticenter of $\mathbf{Q}$ [7]. Moreover, if $\mathbf{Q}$ is cyclic and orthodiagonal, the anticenter is the common point of the diagonals of $\mathbf{Q}$ (Brahmagupta theorem) [4]. In general, if $\mathbf{Q}$ is cyclic, O is its circumcenter and G its centroid, the anticenter H is the symmetric of O with respect to G, and the line containing the three points H, O and G is called the Euler line of $\mathbf{Q}$.

Theorem 1. *The vertices of the Varignon parallelogram and those of the principal orthic quadrilateral of* $\mathbf{Q}$*, that lie on the lines containing two opposite sides of* $\mathbf{Q}$*, belong to a circle with center* G*.*

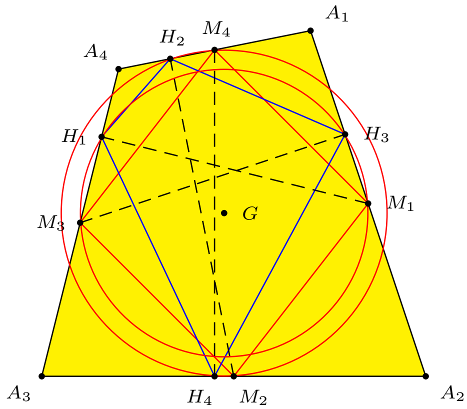

Figure 3.

Proof. The circle with diameter M_1M_3 passes through H_1 and H_3, because $\angle M_1H_1M_3$ and $\angle M_1H_3M_3$ are right angles (see Figure 3). Analogously, the circle with diameter M_2M_4 passes through H_2 and H_4. □

Theorem 1 states that the vertices of **V** and those of **H** lie on two circles with center G.

Corollary 2. *The vertices of the Varignon parallelogram and those of the principal orthic quadrilateral of* **Q** *all lie on a circle* (*with center* G) *if and only if* **Q** *is orthodiagonal.*

Proof. The two circles containing the vertices of V and H coincide if and only if $M_1M_3 = M_2M_4$, *i.e.*, if and only if V is a rectangle. This is the case if and only if **Q** is orthodiagonal. □

If **Q** is orthodiagonal (see Figure 4), the circle containing all the vertices of **V** and **H** is the eight-point circle of **Q** (see [1, 3]).

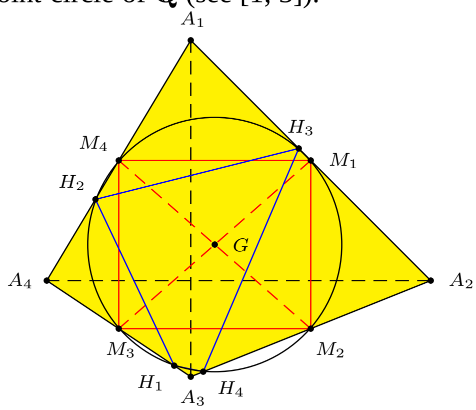

Figure 4.

3. The first Droz-Farny circle

Let **Q** be cyclic and let O and H be the circumcenter and the anticenter of **Q**, respectively. Consider the principal orthic quadrilateral **H** with vertices H_1, H_2, H_3, H_4. Let X_i and X_i' be the intersections of the circle $H_i(O)$ with the line $A_{i+2}A_{i+3}$ (indices taken modulo 4). Altogether there are eight points.

Theorem 3. *If* **Q** *is cyclic, the points X_i, X_i' that belong to the lines containing two opposite sides of* **Q** *lie on a circle with center H.*

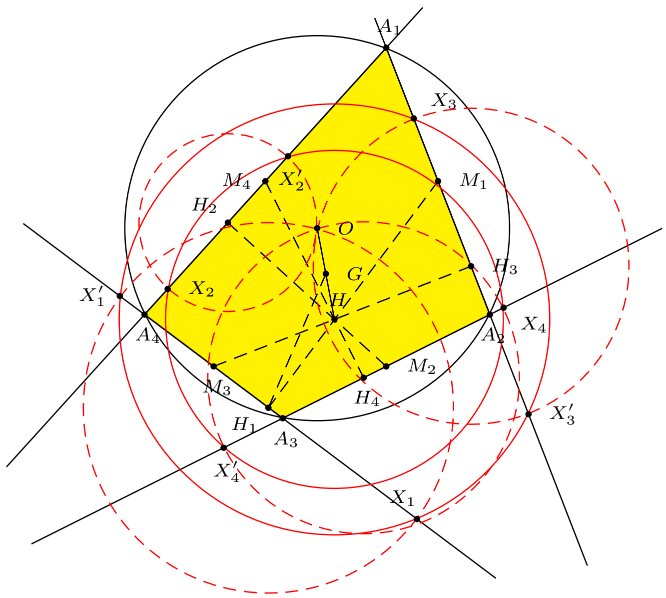

Figure 5.

Proof. Let us prove that the points X_1, X_1', X_3, X_3' are on a circle with center H (see Figure 5). Since H is on the perpendicular bisector of the segment X_1X_1', we have $HX_1 = HX_1'$. Moreover, since X_1 lies on the circle with center H_1 and radius OH_1, $H_1X_1 = OH_1$. By applying Pythagoras' theorem to triangle HH_1X_1, and Apollonius' theorem to the median H_1G of triangle OHH_1, we have

$$HX_1^2 = HH_1^2 + H_1X_1^2 = HH_1^2 + OH_1^2 = 2H_1G^2 + \frac{1}{2}OH^2.$$

Analogously,

$$HX_3^2 = 2H_3G^2 + \frac{1}{2}OH^2.$$

But from Theorem 1, H_1 and H_3 are on a circle with center G, then $H_1G = H_3G$. Consequently, $HX_1 = HX_3$, and it follows that the points X_1, X_1', X_3, X_3' are on a circle with center H.

The same reasoning shows that the points X_2, X_2', X_4, X_4' also lie on a circle with center H. □

Theorem 3 states that the points X_i, X_i', $i = 1, 2, 3, 4$, lie on two circles with center H.

Corollary 4. *For a cyclic quadrilateral* **Q**, *the eight points* X_i, X_i', $i = 1, 2, 3, 4$, *all lie on a circle* (*with center* H) *if and only if* **Q** *is orthodiagonal.*

Proof. The two circles that contains the points X_i, X_i' and coincide if and only if $H_1G = H_2G = H_3G = H_4G$, *i.e.*, if and only if the principal orthic quadrilateral is inscribed in a circle with center G. From Corollary 2, this is the case if and only if **Q** is orthodiagonal. □

As in the triangle case, if **Q** is cyclic and orthodiagonal, we call the circle containing the eight points X_i, X_i', $i = 1, 2, 3, 4$, the first Droz-Farny circle of **Q**.

Theorem 5. *If* **Q** *is cyclic and orthodiagonal, the radius of the first Droz-Farny circle of* **Q** *is the circumradius of* **Q**.

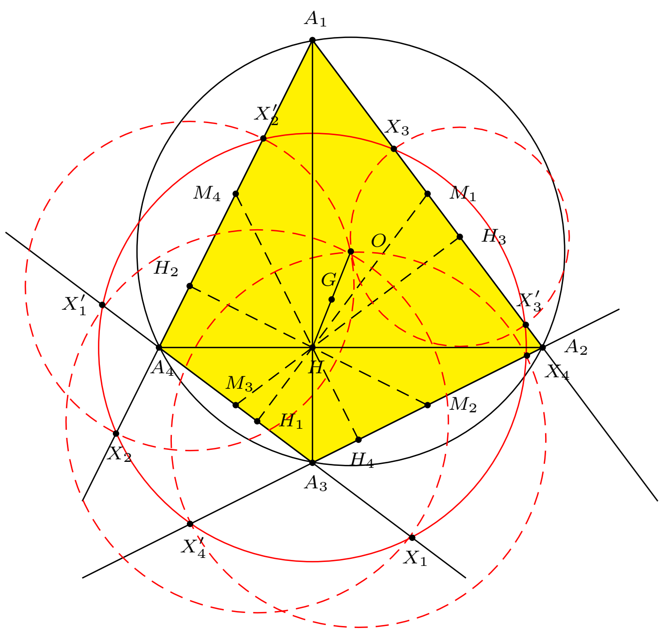

Figure 6.

Proof. From the proof of Theorem 3 we have

$$HX_1^2 = 2H_1G^2 + \frac{1}{2}OH^2. \qquad (1)$$

Moreover, $M_3A_3 = M_3H$ since **Q** is orthodiagonal and $\angle A_3HA_4$ is a right angle (see Figure 6). By applying Pythagoras' theorem to the triangle OM_3A_3, and Apollonius' theorem to the median M_3G of triangle OM_3H, we have

$$OA_3^2 = OM_3^2 + M_3A_3^2 = OM_3^2 + M_3H^2 = 2M_3G^2 + \frac{1}{2}OH^2.$$

Since M_3G and H_1G are radii of the eight-points circle of **Q**,

$$OA_3^2 = 2H_1G^2 + \frac{1}{2}OH^2. \tag{2}$$

From (1) and (2) it follows that $HX_1 = OA_3$. □

4. The second Droz-Farny circle

Let **Q** be cyclic, with circumcenter O and anticenter H. For $i = 1, 2, 3, 4$, let Y_i and Y_i' be the intersection points of the line A_iA_{i+1} with the circle $M_i(H)$. Altogether there are eight points.

Theorem 6. *If* **Q** *is cyclic, the points* Y_i, Y_i' *that belong to the lines containing two opposite sides of* **Q** *lie on a circle with center* O.

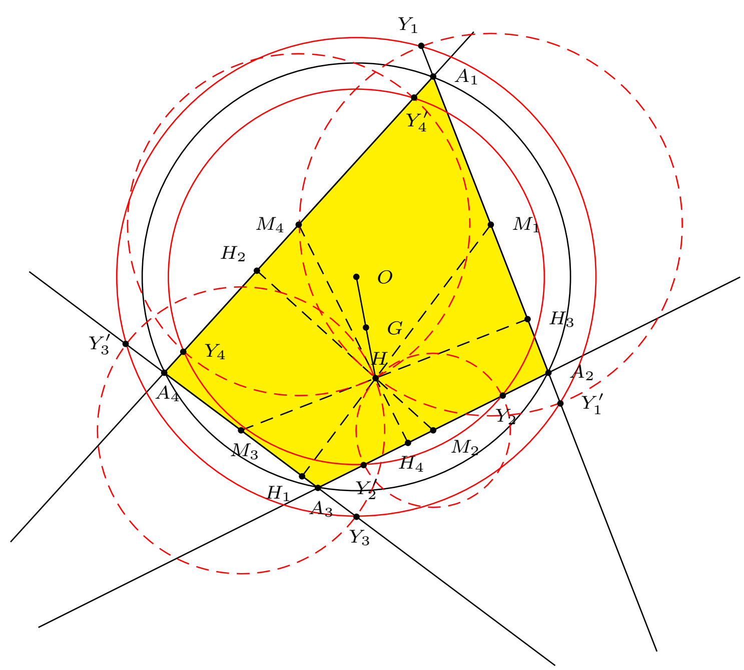

Figure 7.

Proof. Let us prove that the points Y_1, Y_1', Y_3, Y_3' are on a circle with center O (see Figure 7).

Since O is on the perpendicular bisector of the segment Y_1Y_1', $OY_1 = OY_1'$. Moreover, since Y_1 lies on the circle with center M_1 and radius HM_1, $M_1Y_1 = HM_1$. By applying Pythagoras' theorem to triangle OM_1Y_1, and Apollonius' theorem to the median M_1G of triangle OM_1H, we have

$$OY_1^2 = OM_1^2 + M_1Y_1^2 = OM_1^2 + HM_1^2 = 2M_1G^2 + \frac{1}{2}OH^2.$$

Analogously,

$$OY_3^2 = 2M_3G^2 + \frac{1}{2}OH^2.$$

Since G is the midpoint of the segment M_1M_3, $OY_1 = OY_3$. It follows that the points Y_1, Y_1', Y_3, Y_3' lie on a circle with center O.

The same reasoning shows that the points Y_2, Y_2', Y_4, Y_4' also lie on a circle with center O. □

Theorem 6 states that the points Y_i, Y_i', $i = 1, 2, 3, 4$, lie on two circles with center O.

Corollary 7. *For a cyclic quadrilateral* $\mathbf{Q}$, *the eight points* Y_i, Y_i', $i = 1, 2, 3, 4$, *all lie on a circle* (*with center* O) *if and only if* $\mathbf{Q}$ *is orthodiagonal.*

Proof. The two circles that contain the points Y_i', Y_i', $i = 1, 2, 3, 4$, coincide if and only if $M_1G = M_2G$, *i.e.*, if and only if $M_1M_3 = M_2M_4$. This is the case if and only if $\mathbf{Q}$ is orthodiagonal. □

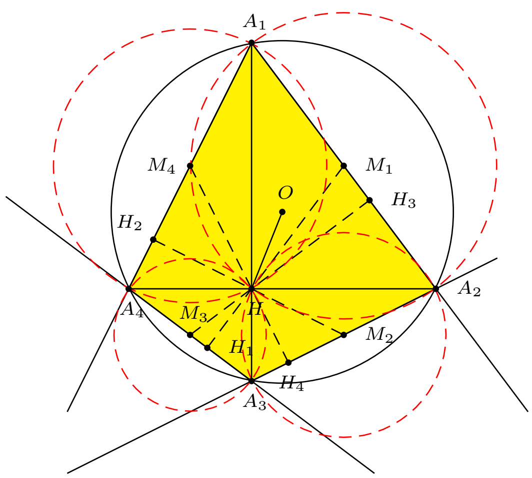

Figure 8.

If $\mathbf{Q}$ is cyclic and orthodiagonal, we call the circle containing the eight points Y_i', Y_i', $i = 1, 2, 3, 4$, the second Droz-Farny circle of $\mathbf{Q}$. But observe that the circle with diameter a side of $\mathbf{Q}$ passes through H, because the diagonals of $\mathbf{Q}$ are perpendicular. The points Y_i, Y_i' are simply the vertices A_i of $\mathbf{Q}$, each counted twice. The second Droz-Farny circle coincides with the circumcircle of $\mathbf{Q}$ (see Figure 8).

5. An ellipse through eight points

Suppose that **Q** is any convex quadrilateral and let K be the common point of the diagonals of **Q**. Consider the Varignon parallelogram $M_1M_2M_3M_4$ of **Q**. Let us fix a segment of length r, greater than the distance of A_i from the line $M_{i-1}M_i$, $i = 1, 2, 3, 4$. Let Z_i and Z_i' be the intersections of the circle with center A_i and radius r with the line $M_{i-1}M_i$. We obtain altogether eight points.

Let p_i be the perpendicular drawn from A_i to the line $M_{i-1}M_i$, and let C_i be the common point of p_i and p_{i+1} (see Figure 9). Since p_i and p_{i+1} are the perpendicular bisectors of the segments Z_iZ_i' and $Z_{i+1}Z_{i+1}'$ respectively, we have

Theorem 8. *The points Z_i, Z_i', Z_{i+1}, Z_{i+1}' lie on a circle with center C_i.*

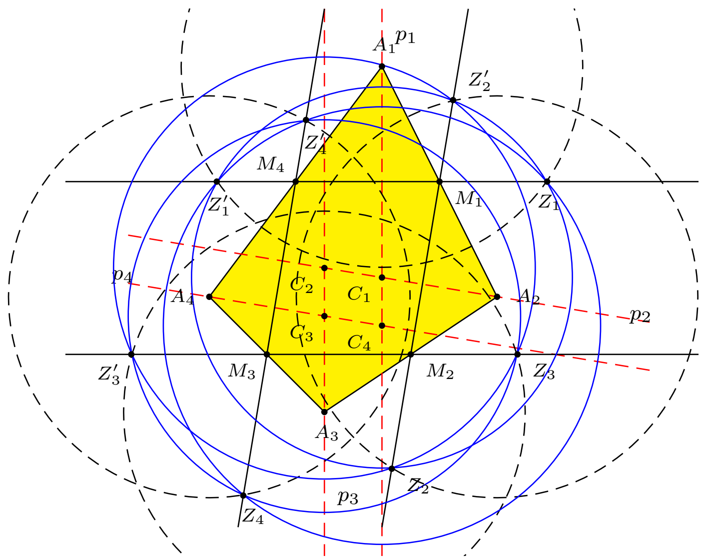

Figure 9.

Theorem 8 states that there are four circles, each passing through the points Z_i, Z_i', that belong to the lines containing two consecutive sides of the Varignon parallelogram of **Q** (see Figure 10).

Theorem 9. *The eight points Z_i, Z_i', $i = 1, 2, 3, 4$, all lie on a circle if and only if* **Q** *is orthodiagonal.*

Proof. Suppose first that the eight points Z_i, Z_i', $i = 1, 2, 3, 4$, all lie on a circle. If C is the center of the circle, then each C_i coincides with C. Since the lines A_1C_1 and A_3C_3 both are perpendicular to A_2A_4, the point C must lie on A_1A_3 and then **Q** is orthodiagonal.

Conversely, let **Q** be orthodiagonal. Since A_1A_3 is perpendicular to M_1M_4, the point C_1 lies on A_1A_3. Since A_2A_4 is perpendicular to M_1M_2, C_1 also lies on A_2A_4. It follows that C_1 coincides with K. Analogously, each of C_2, C_3, C_4 also coincides with K, and the four circles coincide each other in one circle with center K. □

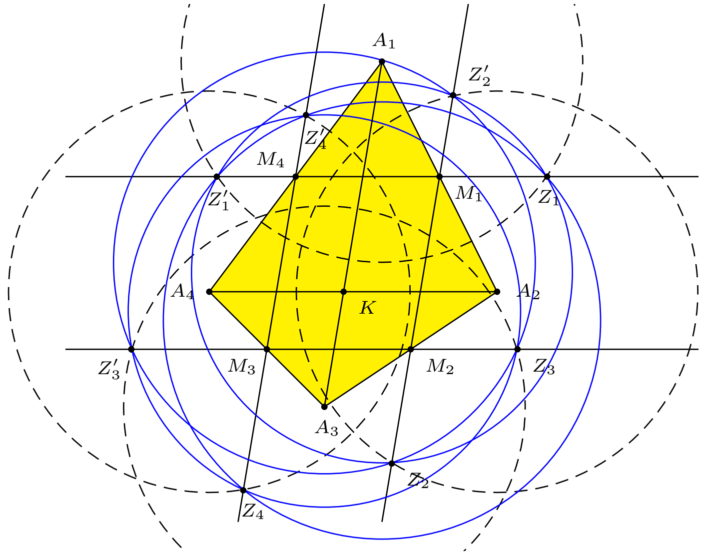

Figure 10.

Because of Theorem 9 we can state that if **Q** is orthodiagonal, the eight points Z_i, Z'_i, $i = 1, 2, 3, 4$, all lie on a circle with center K (see Figure 11).

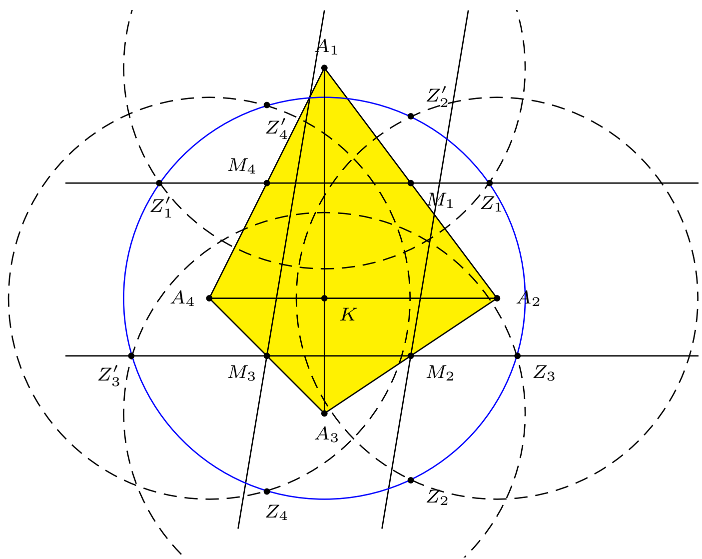

Figure 11.

Corollary 10 below follows from Theorem 5 and from the fact that in a cyclic and orthodiagonal quadrilateral **Q** the common point of the diagonals is the anticenter of **Q**.

Corollary 10. *If* **Q** *is cyclic and orthodiagonal, the circle containing the eight points* Z_i, Z'_i, $i = 1, 2, 3, 4$, *obtained by getting the circumradius of* **Q** *as r, coincides with the first Droz-Farny circle of* **Q**.

We conclude the paper with the following general result.

Theorem 11. *If* **Q** *is a convex quadrilateral, the eight points* Z_i, Z'_i, $i = 1, 2, 3, 4$, *all lie on an ellipse whose axes are the bisectors of the angles between the diagonals of* **Q**. *Moreover, the area of the ellipse is equal to the area of a circle with radius* r.

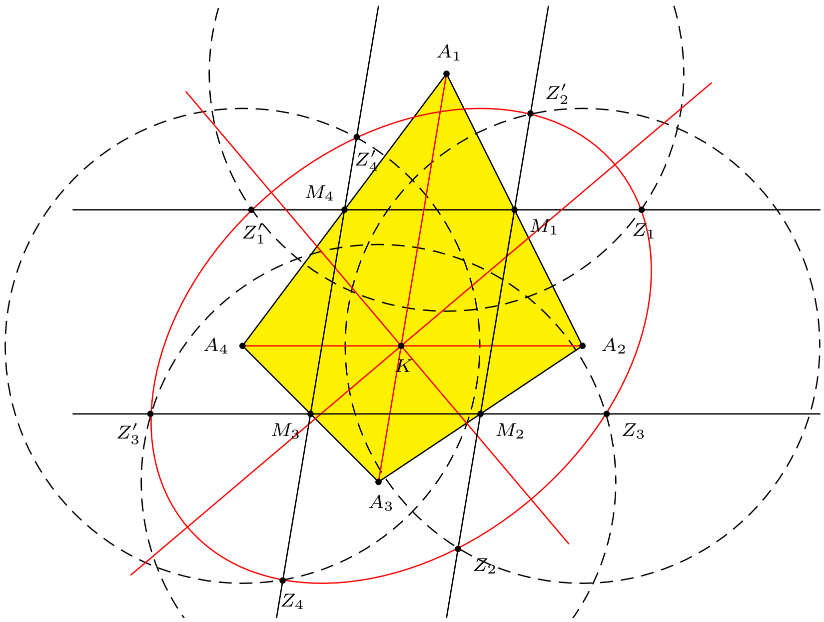

Figure 12.

Proof. We set up a Cartesian coordinate system with axes the bisectors of the angles between the diagonals of **Q**. The equations of the diagonals are of the form $y = mx$ and $y = -mx$, with $m > 0$. The vertices of **Q** have coordinates $A_1(a_1,\ ma_1)$, $A_2(a_2,\ -ma_2)$, $A_3(a_3,\ ma_3)$, $A_4(a_4,\ -ma_4)$, with $a_1, a_2 > 0$ and $a_3, a_4 < 0$. By calculations, the coordinates of the points Z_i and Z'_i are

$$\left(\frac{a_i \pm \sqrt{(m^2+1)r^2 - m^2 a_i^2}}{m^2+1},\ \frac{m^3 a_i \mp \sqrt{(m^2+1)r^2 - m^2 a_i^2}}{m^2+1} \right).$$

These eight points Z_i, Z'_i, $i = 1, 2, 3, 4$, lie on the ellipse

$$m^4 x^2 + y^2 = m^2 r^2. \tag{3}$$

Moreover, since the lengths of the semi axes of the ellipse are $\frac{r}{m}$ and mr, the area enclosed by the ellipse is equal to πr^2. □

Note that (3) is the equation of a circle if and only if $m = 1$. In other words, the ellipse is a circle if and only if $\mathbf{Q}$ is orthodiagonal.

References

[1] L. Brand, The eight-point circle and the nine-point circle, *Amer. Math. Monthly*, 51 (1944) 84–85.

[2] A. Droz-Farny, Notes sur un théoréme de Steiner, *Mathesis*, 21 (1901) 268–271.

[3] R. Honsberger, *Mathematical Gems II*, Math. Assoc. America, 1976.

[4] R. Honsberger, *Episodes in nineteenth and twentieth century Euclidean geometry*, Math. Assoc. America, 1995.

[5] M.F. Mammana, B. Micale and M. Pennisi, Orthic quadrilaterals of a Convex quadrilateral, *Forum Geom.*, 10 (2010) 79–91.

[6] B. Micale and M. Pennisi, On the altitudes of quadrilaterals, *Int. J. Math. Educ. Sci. Technol.*, 36 (2005) 15–24.

[7] P. Yiu, *Notes on Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

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