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Solving Euler's Triangle Problems with Poncelet's Pencil

Roger C. Alperin

Abstract. We determine the unique triangle given its orthocenter, circumcenter and another particular triangle point. The main technique is to realize the triangle as special intersection points of the circumcircle and a rectangular hyperbola in Poncelet's pencil.

1. Introduction

Euler's triangle problem asks one to determine a triangle when given its orthocenter H, circumcenter O and incenter I. This problem has received some recent attention. Some notable papers are those of Scimemi [10], Smith [11], and Yiu [12]. It is known that the solution to the problem is not (ruler-compass) constructible in general so other methods are necessary. For example, Yiu solves the Euler triangle problem with the auxiliary construction of a cubic curve and then realizes the solution as the intersection points of a rectangular hyperbola and the circumcircle.

From the work of Guinand [5] we know that a necessary and sufficient condition for a solution is that I lies inside the circle with diameter GH (G is the centroid) but different from N, the center of Euler's nine-point circle.

We approach these triangle determination problems by realizing the triangle vertices as the intersections of the circumcircle and a rectangular hyperbola in the Poncelet pencil. We can solve Euler's triangle problem using either Feuerbach's hyperbola or Jerabek's hyperbola. We establish some further properties of the Poncelet pencil in order to prove that the triangle is uniquely determined. For the solution using Jerabek's hyperbola we use some methods suggested by the work of Scimemi. As an aid we develop some of the relations between Wallace-Simson lines and the Poncelet pencil.

Also we use Kiepert's hyperbola to solve (uniquely) the triangle determination problem when given O, H and any one of following: the symmedian point K, the first Fermat point F_+, the Steiner point S_{t} or the Tarry point T_{a}.

2. Data

Suppose that the three points O, H, I are given. As Euler and Feuerbach showed, $OI^2 = R(R-2r)$ and $2NI = R-2r$, where R is the circumradius and r is the inradius, and we have $R = \frac{OI^2}{2NI}$. The nine-point circle has center N, the midpoint of OH and radius $\frac{R}{2}$. Thus the circumcircle $\mathcal{C}$ and the Euler circle can be constructed. The Feuerbach point F_{e} can now also be constructed as the intersection point of the nine-point circle and the extension of the ray NI beginning at the center N of the nine-point circle and passing through the incenter I ([12]).

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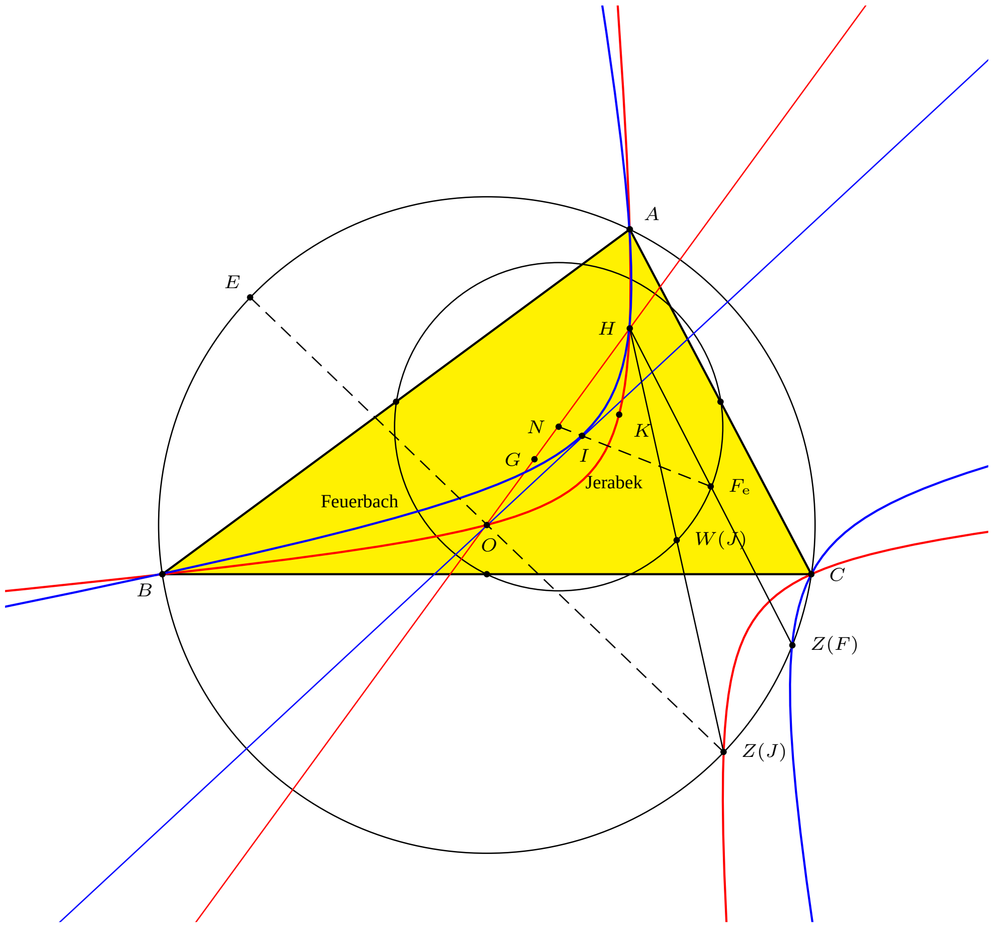

Figure 1. Feuerbach and Jerabek Hyperbolas

The Poncelet pencil is a pencil of rectangular hyperbolas determined by a triangle, namely, the conics are the isogonal transforms of the lines through O ([1]). It can be characterized as the pencil of conics through the vertices of the given triangle where each conic is a rectangular hyperbola. The Feuerbach hyperbola $\mathcal{F}$ is the isogonal transform of the line OI; this has the Feuerbach point $F_{\rm e}$ as its center. It is tangent to the line OI at the point I. Thus the five linear conditions: rectangular, duality of $F_{\rm e}$ and the line at infinity, duality of I and the line OI determine the equation for the rectangular hyperbola $\mathcal{F}$ explicitly, without knowing the vertices of the triangle.

Generally, two conics intersect in four points. The four intersections of the Feuerbach hyperbola and the circumcircle consists of the three triangle vertices together with a fourth point, called the circumcircle point of the hyperbola ([1]). The point Z on the line through $H, F_{\rm e}$ with $HF_{\rm e} = F_{\rm e}Z$ is a point on the circumcircle since the central similarity at H with scale 2 takes the nine-point circle to the circumcircle. We show (Proposition 3) that this point Z is the circumcircle point of Feuerbach's hyperbola, that is, Z is also on Feuerbach's hyperbola.

The solution to the Euler triangle problem for O, I, H is now given by the following.

Theorem 1. *Suppose that $I \neq N$ is interior to the open disk with diameter GH. Let $\mathcal{F}$ be the rectangular hyperbola with center $F_{\rm e}$ and duality of I with line OI. The intersection of $\mathcal{F}$ with the circumcircle $\mathcal{C}$ consists of the circumcircle point Z and the vertices of the unique triangle ABC with incenter I, orthocenter H and circumcenter O.*

The solution to Euler's problem is unique, when it exists, since there is no ambiguity in determining which three of the four points of intersection of the circumcircle and hyperbola are the triangle vertices.

However, if the circumcircle point Z is a triangle vertex, then Feuerbach's hyperbola is tangent to the circumcircle at that vertex. In this case we can construct a vertex and so the triangle is actually constructible by ruler-compass methods. This situation arises if and only if the line $HF_{\rm e}$ is perpendicular to OI as we show in Proposition 4.

Corollary 2. *We can solve the triangle problem when given O, H and either the Nagel point $N_{\rm a}$ or the Spieker center $S_{\rm p}$.*

Proof. We use the fact that the four points $I, G, S_{\rm p}, N_{\rm a}$ lie on a line with ratio $IG : GS_{\rm p} : S_{\rm p}N_{\rm a} = 2 : 1 : 3$. Given O, H we can construct G, and then given either $S_{\rm p}$ or $N_{\rm a}$, we can determine I. Thus we can solve the triangle problem with the hyperbola $\mathcal{F}$ as constructed in Theorem 1. □

3. Poncelet Pencil

In this section we develop the results about the Poncelet pencil used in the proof of Theorem 1.

Suppose Δ is a triangle with circumcircle $\mathcal{C}$. For Δ the isogonal transform of the lines through the circumcenter O gives the Poncelet pencil of rectangular hyperbolas discussed in [1]. The centers of these hyperbolas lie on the nine-point circle. The orthocenter H lies on every hyperbola of this pencil.

Let $\mathcal{P}$ be a hyperbola of the Poncelet pencil. The conic $\mathcal{P}$ and $\mathcal{C}$ meet at the vertices of Δ and a fourth point.

Thus we have the following result.

Proposition 3. *Let $\mathcal{P}$ be the hyperbola of the Poncelet pencil whose center is W. The point Z so that $HW = WZ$ on the line HW is on the circumcircle of Δ and the hyperbola $\mathcal{P}$.*

Proof. A rectangular hyperbola is symmetric about its center. Hence, any line through the center meets the hyperbola in two points of equal distance from the center. Thus the line HW meets $\mathcal{P}$ at another point Z so that $WZ = HW$.

Since the central similarity at H with scale factor 2 takes the Euler circle to the circumcircle then the point Z is also on the circumcircle. □

The Z is called the circumcircle point of the hyperbola $\mathcal{P}$. It may happen that this point Z is one of the vertices of Δ. The center of the hyperbola is denoted W.

We analyze that situation. A point Y on $\mathcal{K}$ is a vertex if and only if HY is an altitude. So if Z is a vertex then HZ is an altitude and hence W also lies on an altitude since Z lies on HW. Conversely, if W lies on the altitude then Z also lies on that altitude and hence is a vertex.

Proposition 4. *A hyperbola $\mathcal{P}$ of the Poncelet pencil with center W is tangent to the circumcircle if and only if HW is perpendicular to $\mathcal{L} = \mathcal{P}^*$ if and only if the circumcircle point Z is a vertex of the triangle.*

Proof. If Z is a vertex then its isogonal transform is at infinity on BC and on the transform $\mathcal{L} = \mathcal{P}^*$. Hence $\mathcal{L}$ is parallel to BC. Thus from the remarks above, HW is an altitude if and only if the circumcircle point is a vertex. Then tangency of the circumcircle and hyperbola occurs if and only if HW is perpendicular to $\mathcal{L}$. □

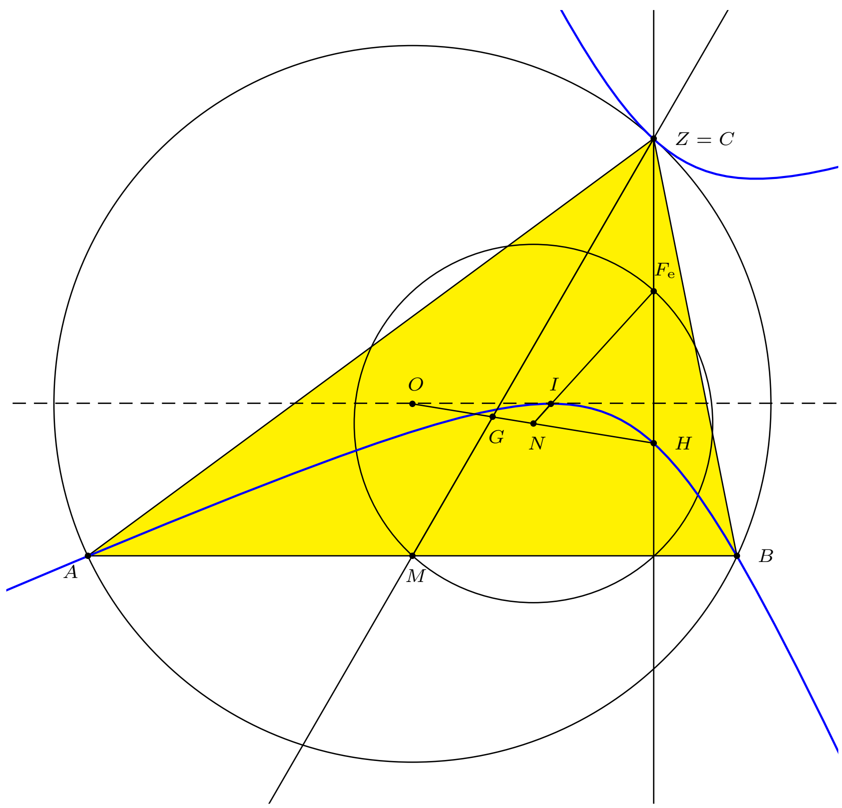

Figure 2. Triangle is constructible when OI is perpendicular to HF_e. Relation to Feuerbach hyperbola.

In this tangent case we can solve the Euler triangle problem easily with ruler and compass if we have the hyperbola's center W since then we can construct the vertex $C = Z$. The other vertices are then also easy to obtain: on the line CG, construct the point M with ratio $CG : GM = 2 : 1$. The line through M, perpendicular to HC meets the circumcircle at two other vertices of the triangle.

For the case of Feuerbach hyperbola, $W = F_{\mathrm{e}}$; the point Z is a vertex if and only if OI is perpendicular to HF_{e}.

4. Solution with Jerabek's Hyperbola

We now develop some of the useful relations between the Poncelet pencil, circumcircle points and Simson lines. These allow us to prove the relation of Scimemi's Euler point to the circumcircle point of Jerabek's hyperbola.

4.1. *Simson Lines and Orthopole.* See [6, §327-338, 408]. Denote the isogonal conjugate of X by X^*.

Theorem 5. *For S on the circumcircle of triangle ABC, SS^* is perpendicular to the Wallace-Simson line of S, i.e., S^* lies on the Wallace-Simson line of the antipode S' of S.*

Proof. From [4] the Wallace-Simson line of S passes through the isogonal conjugate T^* of its antipodal point $T = S'$. Thus it is perpendicular to SS^* since the angle between T^* and S^* is 90 degrees, the angle being halved by the isogonal transformation. □

Theorem 6. *Consider the line $\mathcal{L}$ though the circumcenter of triangle ABC, meeting the circumcircle at U, U'. Let $\mathcal{K} = \mathcal{L}^*$.*
(i) *The Wallace-Simson lines of U, U' are asymptotes of $\mathcal{K}$ and meet at the center $W(\mathcal{K})$ of $\mathcal{K}$ on the nine-point circle of triangle ABC.*
(ii) *The center $W(\mathcal{K})$ is the orthopole of $\mathcal{L}$.*
(iii) *The Wallace-Simson line of $C(\mathcal{K})$ is perpendicular to $\mathcal{L}$. This line bisects the segment from H to $C(\mathcal{K})$ at $W(\mathcal{K})$.*

Proof. The asymptotes of $\mathcal{K}$ are the Wallace-Simson lines of the isogonal conjugates of the points at infinity of $\mathcal{K}$, [4, p. 196]. Hence the center of $\mathcal{K}$ is the orthopole of $\mathcal{K}^* = \mathcal{L}$ (see [6, §406]).

A dilation at H by $\frac{1}{2}$ takes the the circumcircle to the nine-point circle. The Wallace-Simson line of any point S on the circumcircle bisects the segment HS and passes through a point of the nine-point circle [6, §327]. Thus midpoints of U, U' with H are antipodal points on the nine-point circle and lie on Wallace-Simson lines (asymptotes of $\mathcal{K}$).

The isogonal transform of the circumcircle point $C(K)$ lies on the line $\mathcal{L} = \mathcal{K}^*$ and the line at infinity. Thus the Wallace Simson line of $C(\mathcal{K})$ is perpendicular to $\mathcal{L}$ by Theorem 5. In [6, §406] a point W is constructed from UU^* so that its Wallace-Simson line is perpendicular to UU^*. Thus by uniqueness of the directions of Wallace-Simson lines this point W is $C(\mathcal{K})$. As shown there the Wallace-Simson line of W is also coincident with the Wallace-Simson lines of both U and U'. Thus

$W = C(\mathcal{K})$ is the dilation by 2 of the center of the right hyperbola and HW is bisected by the Wallace-Simson line of W at the center of $\mathcal{K}$. □

4.2. Scimemi has introduced the Euler point E in [10]. He shows that it is constructible from given O, H, I. Following Scimemi's Theorem 2 ([10]) and the previous theorems we obtain the following.

Corollary 7. *Let $\mathcal{L}$ be a line through the circumcenter of triangle ABC, $\mathcal{K} = \mathcal{L}^*$, $W = C(\mathcal{K})$ the circumcircle point of $\mathcal{K}$. Then the reflection of W', the antipodal of W, in the sides of ABC lie on a line passing through H, which is parallel to $\mathcal{L}$ and perpendicular to the Wallace-Simson line of W.*

Scimemi's Euler point E is the point of coincidence of the reflections of the Euler line in the sides of the triangle. Thus it is antipodal to the circumcircle point of Jerabek's hyperbola defined by $\mathcal{K} = \mathcal{L}^*$, where $\mathcal{L}$ is the Euler line.

Hence we can construct the center of Jerabek's hyperbola since it is the midpoint of HE. Thus Jerabek's hyperbola is determined linearly from the data: it is a rectangular hyperbola; it passes through H and O; there is a duality of its center with the line at infinity.

4.3. *Construction with Feuerbach and Jerabek Hyperbolas.* We can use both Feuerbach's and Jerabek's hyperbolas to determine the triangle. The common points are the triangle vertices and the orthocenter H.

5. Construction with Kiepert's hyperbola

In [2] it is shown that the symmedian point K ranges over the open disk with diameter GH punctured at its center.

Theorem 8. *We can uniquely determine the triangle when given O, H and K.*

Proof. It is known that the center W of Kiepert's hyperbola $\mathcal{K}$ is the inverse of K in the orthocentroidal circle with diameter GH and center J [7]. Thus the center W is constructible given O, H, K. Since this point is the intersection of the ray JK with the Euler circle, we also can construct the radius of the Euler circle and hence also the radius of the circumcircle. Hence we may construct the circumcircle since the center O is given.

We can provide linear conditions to determine Kiepert's hyperbola $\mathcal{K}$: rectangular hyperbola, duality of W and the line at infinity, passing through G and H.

The circumcircle point Z is the intersection of HW with the circumcircle. The four intersections of Kiepert's hyperbola and the circumcircle are the points of the triangle and the point Z. □

Thus also the triangle is uniquely determined and ruler-compass constructible if Kiepert's hyperbola is tangent to the circumcircle. This last condition is equivalent to HW is perpendicular to the line OK, where W is the center of Kiepert's hyperbola.

5.1. *Kiepert's Hyperbola and Fermat's Points.* Given O, H and the first Fermat point F_+, we can construct the second Fermat point F_- since it the inverse of F_+ in the orthocentroidal circle [2, p.63]. The center Z of Kiepert's hyperbola is the midpoint of these two Fermat points [4, p.195]. These conditions determine Kiepert's hyperbola: rectangular, center W, passing through G, H.

Now also we can construct the circumcircle point W of Kiepert's hyperbola since it is the symmetry about Z of the orthocenter point H. Now we have the center O and W a point of the circumcircle. Hence we can now determine the triangle uniquely.

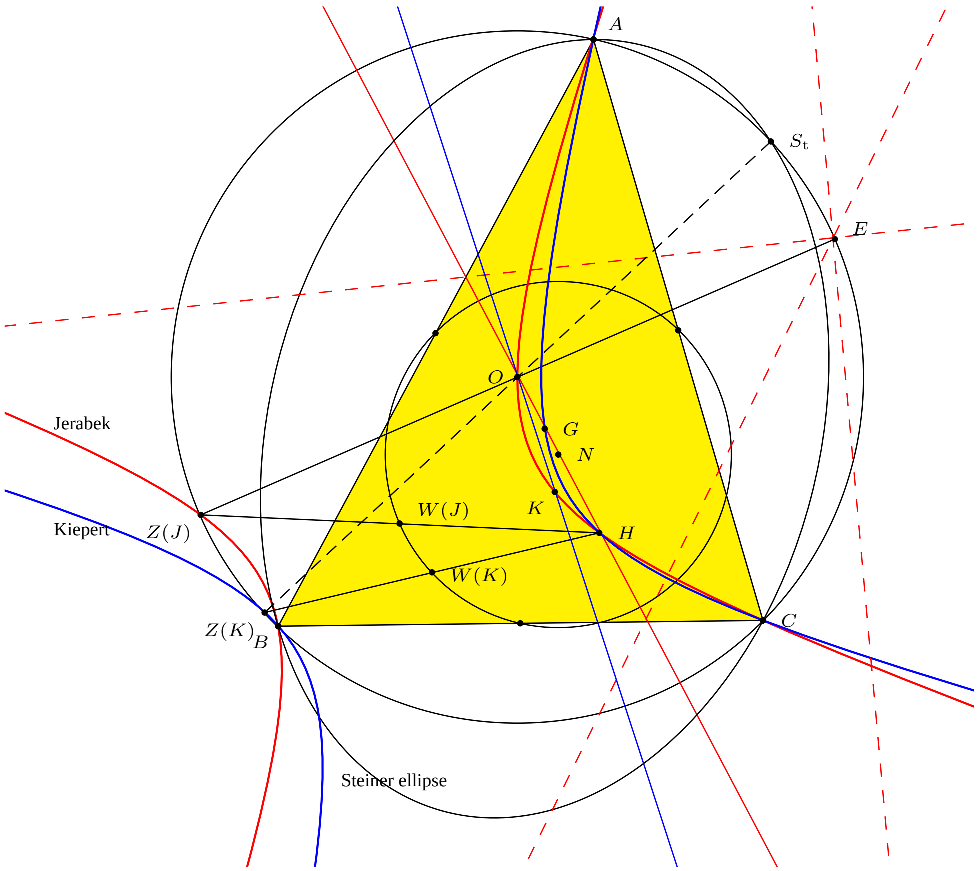

Figure 3. Kiepert Hyperbola, Jerabek Hyperbola, Steiner's Ellipse and Euler Point E

5.2. *Kiepert's Hyperbola and Steiner or Tarry points.* With O and either the Steiner or Tarry point we can construct the circumcircle of the desired triangle since each of these points lies on the circumcircle. Moreover, since the Tarry point is the circumcircle point of Kiepert's hyperbola [7] we may construct the center of the Kiepert hyperbola. Also Steiner's point is antipodal to the Tarry point on the circumcircle so we may use the Steiner point to determine the center. Thus using the duality of the line at infinity and the center of the Kiepert hyperbola, we may determine the rectangular hyperbola also passing through G and H. This is Kiepert's hyperbola, so the intersections of this with the circumcircle give the triangle and the Tarry point. Thus the triangle is uniquely determined.

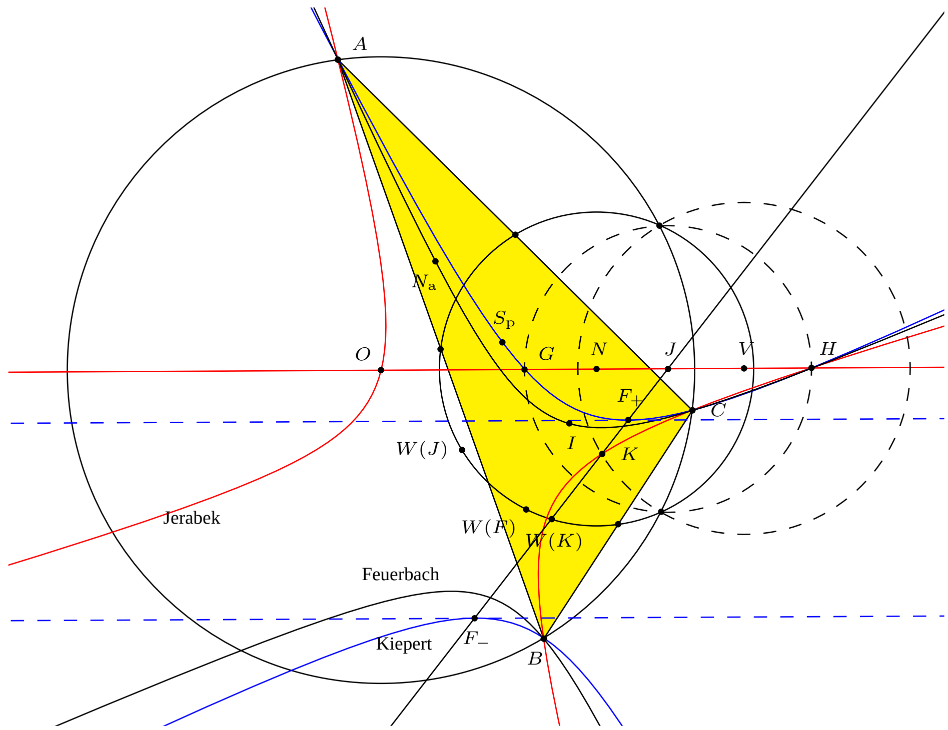

Figure 4. Relations to Orthocentroidal Disk

5.3. *Location of the Symmedian point.* Since the center of Kiepert's hyperbola lies on the nine-point circle, we can obtain the location of the symmedian point K by inversion in the orthocentroidal circle.

The familiar formula for inverting a circle $\mathcal{C}$ of radius c in a circle $\mathcal{K}$ of radius k gives a circle $\mathcal{C}'$ of radius c' with $c'^2(d^2-c^2)^2 = k^4c^2$, where d is the distance of the center of $\mathcal{C}$ from the center of $\mathcal{K}$.

In the case of inverting the nine-point circle in the orthocentroidal circle, we have $k = 2d = \frac{OH}{3}$, $c = \frac{R}{2}$ and the circle $\mathcal{C}'$ has center V on the Euler line. Thus we have that $X = K$ satisfies the equation $VX^2 = c'^2 = (\frac{k^2c}{d^2-c^2})^2 = (\frac{2R\cdot OH^2}{OH^2-R^2})^2$. As a comparison, one knows that the incenter $X = I$ satisfies the quartic equation $OX^4 = 4R^2 \cdot JX^2$ [5]. In addition, it is known that the symmedian point ([2]) and incenter I ([5]) lie inside the orthocentroidal circle with center J.

5.4. *Tangent Lines at Fermat Points.* See Figure 4. The Fermat points lie on the line through the center $W(K)$ of Kiepert's hyperbola; also the symmedian K,

$W(K)$, J lie on the same line since F_+ and F_- are inverses in the orthocentroidal circle with center J [7].

The Kiepert hyperbola is the isotomic transform of the line through H^s and G, where H^s is the isotomic transform of H [4]. Since G is a fixed point of the isotomic transformation this line is tangent to the hyperbola at G. As shown in [3], H^s is the symmedian of the anticomplementary triangle, so in fact K is also on that line and $H^sG : GL = 2 : 1$.

Let Y be the dual of line GH in Kiepert's hyperbola; then Y lies on the tangent line H^sG to G; hence the dual of J passes through Y. Since J is the midpoint of GH then the line JY passes through the center $W(K)$. But we have already shown that K is on the line $JW(K)$ and H^sG; thus $Y = K$. Consequently the dual of any point on GH passes through K; in particular the dual of the point at infinity on GH passes through K and the center $W(K)$. Consequently the two intersections of this line with the conic, F_+ and F_- have their tangents parallel to the Euler line GH.

Remark. The editors have pointed out the very recent reference [9].

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Roger C. Alperin: Department of Mathematics, San Jose State University, San Jose, California 95192, USA

E-mail address: `alperin@math.sjsu.edu`