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More on the Extension of Fermat's Problem

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Abstract. We give an elementary and complete solution to the Fermat problem with arbitrary nonzero weights.

1. Introduction

At the end of his famous 1643 essay on maxima and minima, Pierre de Fermat (1601–1665) threw out a challenge: "Let he who does not approve of my method attempt the solution of the following problem: given three points in a plane, find a fourth point such that the sum of its distances to the three given points is a minimum!" Our Problem 1 is the most interesting case of his problem.

Problem 1 (Fermat's problem)**.** *Let ABC be a triangle. Find a point P such that $PA + PB + PC$ is minimum.*

The first published solution came from Evangelist Torricelli, published posthumously in 1659. Numerous subsequent solutions are readily to be found in books and journals. Problem 1 has been generalized in different ways. The generalization below is presumably the most natural.

Problem 2. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

Various approaches to the solution of Problem 2 can be found in [5, 6, 7, 8, 9, 11].

In 1941, R. Courant and H. Robbins [1] posed another problem inspired by Problem 1, replacing the weights $1, 1, 1$ with $-1, 1, 1$. Unfortunately, their claimed solution is flawed. Afterwards, other problems were suggested in the same spirit. The following problem is among the most natural [2, 3, 4, 10].

Problem 3. *Let ABC be a triangle and $x,\ y,\ z$ be non-zero real numbers. Find a point P such that $x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

The solution of problem 3 requires the solution of Problem 2 as well as solutions to Problems 4 and 5 below.

Problem 4. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $-x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

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Problem 5. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $-x \cdot PA - y \cdot PB + z \cdot PC$ is minimum.*

Here is an easy solution of Problem 5. If $z < x+y$, then $-x\cdot PA-y\cdot PB+z\cdot PC$ decreases without bound as PC goes to ∞, whence there exists no point P for which the minimum is attained; otherwise, when $z \geq x+y$ the minimum is attained when P coincides with C. The verification of our claim is straightforward.

Problem 4, on the other hand, is very tricky. A correct solution was first introduced in 1980 by L. N Tellier and B.Polanski employing trigonometry [10]. In 1998 J. Krarup gave a solution to problem 4 which was more specific in its conclusion [4]. In 2003 G. Jalal and J. Krarup [3] devised a different solution to problem 4 based on a geometric approach. Four years later, another solution was introduced by G. Ganchev and N. Nikolov using the concept of isogonal conjugacy [2]. However, these solutions are somewhat complicated and not elementary. In this article, we aim to deliver a synthetic and elementary solution to Problem 4.

2. Solution to Problem 4

Let a, b, c be the lengths of BC, CA, and AB of triangle ABC, respectively. Without loss of generality we assume $x = a$. Let $f(P) = -a\cdot PA+y\cdot PB+z\cdot PC$. The following is a summary of the main results.

(1) When a, y, z are the side-lengths of a triangle, construct triangle UBC such that $UC = y$, $UB = z$ and U, A are on the same side of line BC. There are three possibilities:

- **(1.1):** U is inside triangle ABC. $f(P)$ attains its minimum when P is the intersection point, other than U, of line AU and the circumcircle of triangle UBC.
- **(1.2):** $U = A$. $f(P)$ attains its minimum when P lies on the arc BC not containing A of the circumcircle of triangle ABC.
- **(1.3):** U is not inside triangle ABC and U is distinct from A. The minimum of $f(P)$ occurs when $P = B$ or C or both, according as $f(B) < f(C)$ or $f(B) > f(C)$ or $f(B) = f(C)$.

(2) When a, y, z are not the side-lengths of a triangle, we consider two possibilities.

- **(2.1):** $a \geq y + z$. There is no P such that $f(P)$ attains its minimum.
- **(2.2):** $a < y + z$. The minimum of $f(P)$ occurs when $P = B$ or $P = C$ according as $f(B) < f(C)$ or $f(B) > f(C)$.

We shall make use of the following four easy lemmas in the solution to the Problem 4.

Lemma 1 (Ptolemy's inequality)**.** *Let P be a point in the plane of triangle ABC.*
(a) *Unless P lies on the circumcircle of triangle ABC, the three numbers $BC\cdot PA$, $CA \cdot PB$, $AB \cdot PC$ are side lengths of some triangle.*
(b) *If P lies on the circumcircle of triangle ABC, then one of three numbers $BC \cdot PA$, $CA \cdot PB$, $AB \cdot PC$ is the sum of the other two numbers. Specifically,*
(i) *If P lies on the arc BC not containing A then $BC\cdot PA = CA\cdot PB+AB\cdot PC$.*

(ii) *If P lies on the arc CA not containing B, then $CA \cdot PB = AB \cdot PC + BC \cdot PA$.*
(iii) *If P lies on the arc AB not containing C, then $AB \cdot PC = BC \cdot PA + CA \cdot PB$.*

Lemma 2 (Euclid I.21)**.** *For any point P in the interior of triangle ABC, $PB + PC < AB + AC$.*

Lemma 3. *If $ABCD$ is a convex quadrilateral, then $AB + CD < AC + BD$.*

Lemma 4. . *Give an isosceles triangle ABC with $AB = AC$, Ax is the opposite ray of the ray AB. For every P lying inside xAC, we have $PB \geq PC$, with equality when P coincides with A.*

Lemma 3 is just the triangle inequality applied to the pair of triangles formed by the intersection point of the two diagonals together with the endpoints of the edges AB and CD. Lemma 4 holds because every point to the side of the perpendicular bisector of BC is closer to C than to B.

We now turn to the solution of Problem 4. There are two cases to consider.

Case 1. a, y, z are sides of some triangles. Construct triangle UBC such that $UC = y$, $UB = z$ and points U, A lie on the same side of line BC. There are three possibilities.
(1.1) U is inside triangle ABC (see Figure 1).

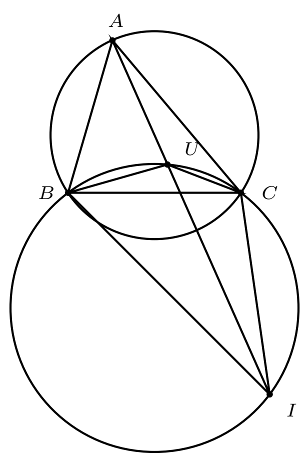

Figure 1.

Denote by I the intersection point, other than U, of line AU and the circumcircle of triangle UBC, which exists because U lies inside triangle ABC so that AU meets the interior of segment BC, which lies inside the circumcircle. Applying Lemma 1 to triangle UBC and an arbitrary point P, we have

$$\begin{aligned} f(P) &= -a \cdot PA + CU \cdot PB + UB \cdot PC \\ &\geq -a \cdot PA + BC \cdot PU = -a(PA - PU) \\ &\geq -a \cdot AU \\ &= f(I). \end{aligned}$$

The equality occurs when P lies on the arc BC not containing U of the circumcircle of triangle UBC and P lies on the opposite ray of the ray UA, that is, $P = I$. In conclusion, $f(P)$ attains its minimum value when $P = I$.

(1.2) $U = A$ (see Figure 2).

For every point P, applying Lemma 1 to triangle UBC and point P, we have

$$f(P) = -a \cdot PA + CU \cdot PB + UB \cdot PC \geq -a \cdot PA + BC \cdot PU = 0.$$

The equality occurs when P lies on the arc BC not containing A of the circumcircle of triangle ABC. From this, we conclude that $f(P)$ attains its minimum value when P lies on the arc BC not containing A of the circumcircle of triangle ABC.

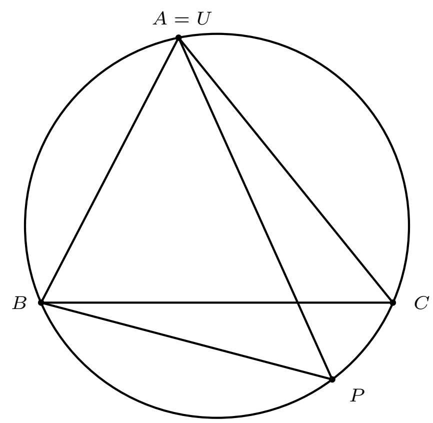

Figure 2.

(1.3) U is not inside triangle ABC and U is distinct from A. There are three situations to consider.

(a) $z - y = c - b$. Denote by Am and An, respectively the opposite rays of the rays AB and AC.

(i) If U lies inside angle mAC (see Figure 3a), then quadrilateral $AUCB$ is convex. By Lemma 3, we have $UB + AC > UC + AB$, which implies that $z - y = UB - UC > AB - AC = c - b$, a contradiction.

(ii) If U lies inside angle nAB (see Figure 3b), we apply Lemma 3 to convex quadrilateral $AUBC$: $UB + AC < UC + AB$. Hence, $z - y = UB - UC < AB - AC = c - b$, a contradiction.

(iii) If U lies inside angle mAn (see Figure 3c), then A lies inside triangle UBC. By Lemma 2, we have $UB + UC > AB + AC$, which implies that $(UB - AB) + (UC - AC) > 0$. Thus, by virtue of $UB - AB = z - c = y - b = UC - AC$, we have $UB - AB = UC - AC > 0$.

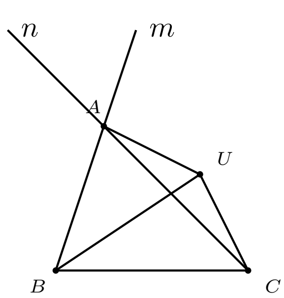

Figure 3a

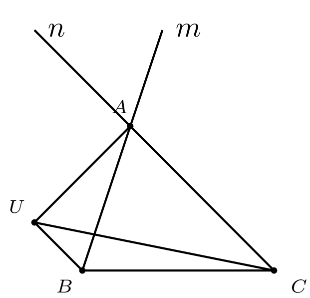

Figure 3b

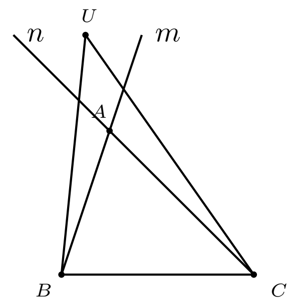

Figure 3c

Now, for any point P, applying Lemma 1 to triangle ABC and point P, taking into account that $PB + PC \geq BC$ we have

$$\begin{aligned} f(P) &= -a \cdot PA + UC \cdot PB + UB \cdot PC \\ &= -BC \cdot PA + AC \cdot PB + AB \cdot PC + (UC - AC)PB + (UB - AB)PC \\ &\geq (UB - AB)(PB + PC) \\ &\geq (UB - AB)BC \\ &= a(UB - AB) \\ &= f(B) = f(C). \end{aligned}$$

The equality occurs if and only if P lies on the arc BC not containing A of circumcircle of triangle ABC and P belongs to the segment BC, *i.e.*, $P = B$ or $P = C$. In conclusion, $f(P)$ attains its minimum value when $P = B$ or $P = C$.

(b) $z - y < c - b$.

(i) $UB - AB > 0$. Similar to (a), note that $UC - AC = y - b > z - c = UB - AB > 0$. Therefore, for any point P,

$$\begin{aligned} f(P) &\geq (UC - AC)PB + (UB - AB)PC \\ &\geq (UB - AB)(PB + PC) \\ &\geq (UB - AB)BC \\ &= a(UB - AB) \\ &= f(B). \end{aligned}$$

Equality holds if and only if P lies on the arc BC not containing A of the circumcircle of triangle ABC, P coincides with B, and P belongs to segment BC. This means that $P = B$.

(ii) $UB - AB \leq 0$. As before, denote by Am and An respectively the opposite rays of the rays AB and AC. If U lies inside angle mAC (see Figure 4a), then quadrilateral $AUCB$ is convex. By Lemma 3, we have $UB + AC > UC + AB$, implying that $z - y = UB - UC > AB - AC = c - b$, a contradiction.

Thus, U and C are on different sides of line AB (Figure 4b). Since $UB \leq AB$, there is a point T on the segment AB such that $TB = UB$. Hence, applying Lemma 4 to isosceles triangle BUT and point C, we have $UC \geq TC$.

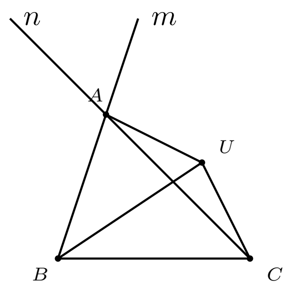

Figure 4a

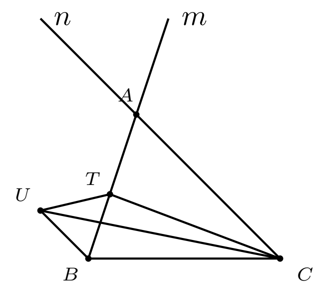

Figure 4b

From this, applying Lemma 1 to triangle TBC and an arbitrary point P, we have

$$\begin{aligned} f(P) &\geq -a\cdot PA + TC\cdot PB + TB\cdot PC && \text{(i)}\\ &\geq -a\cdot PA + BC\cdot TP && \text{(ii)}\\ &= a(-PA+TP)\\ &\geq -a\cdot AT && \text{(iii)}\\ &= a(UB-AB)\\ &= f(B). \end{aligned}$$

Equality occurs if and only if (i) $P=B$, (ii) P lies on arc BC not containing T of the circumcircle of triangle TBC, and (iii) P lies on the opposite ray of the ray TA. Together these mean that $P=B$.

Now we can conclude that $f(P)$ attains its minimum value when $P=B$.

(c) $z-y>c-b$. Similar to (b), interchanging y with z and b with c we conclude that $f(P)$ attains its minimum value when $P=C$.

Case 2. $a,\ y,\ z$ are not side lengths of any triangle. There are two possibilities

(2.1) $a\geq y+z$.

(a) $a=y+z$. Point U is taken on the segment BC such that $y=UC$ and $z=UB$ (see Figure 5).

For every point P we have

$$\overrightarrow{PU}=\frac{UC}{BC}\cdot\overrightarrow{PB}+\frac{UB}{BC}\cdot\overrightarrow{PC}=\frac{y}{a}\cdot\overrightarrow{PB}+\frac{z}{a}\cdot\overrightarrow{PC}.$$

It follows that

$$a\cdot PU=|a\cdot\overrightarrow{PU}|=|y\cdot\overrightarrow{PB}+z\cdot\overrightarrow{PC}|\leq y\cdot PB+z\cdot PC.$$

Therefore,

$$f(P)=-a\cdot PA+y\cdot PB+z\cdot PC\geq -a\cdot PA+a\cdot PU=-a(PA-PU)\geq -a\cdot AU.$$

The equality occurs if and only if the vectors $\overrightarrow{PB},\ \overrightarrow{PC}$ have the same direction and point P belongs to the opposite ray of the ray UA. However, these conditions

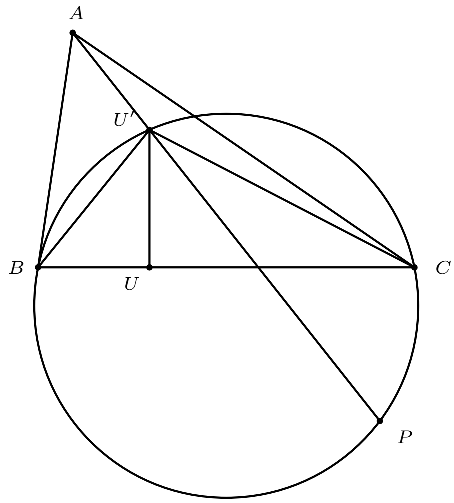

Figure 5.

are incompatible, whence the equality can not be attained. Thus

$$f(P) > -a \cdot AU. \tag{1}$$

For a sufficiently small positive value of ε, take U' inside triangle ABC such that $UU' \perp BC$ and $UU' = \frac{\varepsilon}{a}$. Let P be the intersection, distinct from U', of the line $U'A$ and the circumcircle of $U'BC$ (which exists because U' is inside triangle ABC as shown in Figure 5).

From (1), applying Lemma 1 to triangle $U'BC$ and P, we have

$$\begin{aligned} -a \cdot AU &< f(P) \\ &= -a \cdot PA + CU \cdot PB + UB \cdot PC \\ &< -a \cdot PA + CU' \cdot PB + U'B \cdot PC \\ &= -a \cdot PA + BC \cdot PU' \\ &= -a \cdot AU'. \end{aligned}$$

It follows that

$$|f(P) - (-a \cdot AU)| < |-a \cdot AU' - (-a \cdot AU)| = a|AU - AU'| < a \cdot UU' = a \cdot \frac{\varepsilon}{a} = \varepsilon. \tag{2}$$

From (1), (2), we can affirm that there does not exist a point P such that $f(P)$ attains its minimum value.

(b) $a > y + z$. For every point P, we have

$$\begin{aligned} f(P) &= -a \cdot PA + y \cdot PB + z \cdot PC \\ &= (-a + y + z)PA + y(PB - PA) + z(PC - PA) \\ &\leq (-a + y + z)PA + y \cdot AB + z \cdot AC. \end{aligned}$$

However, as PA tends to $+\infty$, $f(P)$ tends to $-\infty$. This implies that there does not exist a point P such that $f(P)$ attains its minimum value.

(2.2) $a < y + z$.
(a) $y \geq z + a$. For every point P we have

$$\begin{aligned} f(P) &= -a \cdot PA + y \cdot PB + z \cdot PC \\ &= a(PB - PA) + (y - z - a)PB + z(PB + PC) \\ &\geq -a \cdot BA + z \cdot BC \\ &= f(B). \end{aligned}$$

The equality holds if and only if $P = B$. Hence $f(P)$ attains its minimum value when $P = B$.
(b) $z \geq y + a$. Similarly, interchanging y with z and b with c, we conclude that $f(P)$ attains its minimum value when $P = C$.

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