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More on Twin Circles of the Skewed Arbelos

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Abstract. Two pairs of congruent circles related to the skewed arbelos are given.

1. Introduction

We showed several twin circles related to the skewed arbelos in [2]. In this article we give two more pairs, by generalizing two Archimedean circle pairs in [1]. We begin with a brief review of these circles. Let O be a point on a segment AB, and α, β, γ be the circles with diameters OA, OB, AB respectively. We denote one of the intersections of γ and the radical axis of α and β by I. The intersection of IA and α coincides with the tangency point of α and one of the external common tangents of α and β. The circle passing through this point and touching the line IO is Archimedean; so is the one tangent to IO and passing through the intersection of β with the external common tangent. These are the circles W_9 and W_{10} in [1] (see Figure 1). The circle touching α internally and touching the tangents of β from A is also Archimedean; so is the one tangent to β and the the tangents of α from B. These are the circles W_6 and W_7 in [1] (see Figure 2).

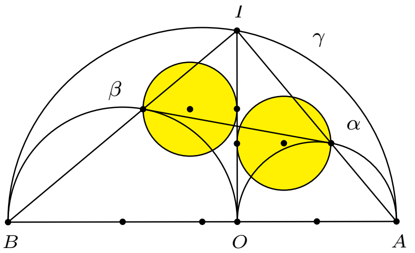

Figure 1

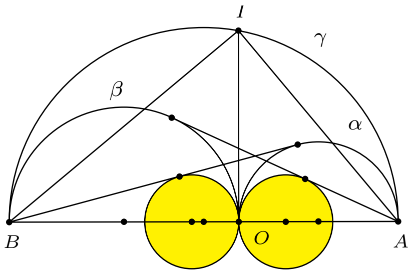

Figure 2

Suppose the circles α and β have radii a and b respectively. We set up rectangular coordinate system with origin O, so that A and B have coordinates $(2a, 0)$ and $(-2b,\ 0)$ respectively. Consider a variable circle touching α and β at points different from O. Such a circle is expressed by the equation

$$\left(x - \frac{b-a}{t^2-1}\right)^2 + \left(y - \frac{2t\sqrt{ab}}{t^2-1}\right)^2 = \left(\frac{a+b}{t^2-1}\right)^2 \tag{1}$$

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for a real number $t \neq \pm 1$ [2]. We denote this circle by γ_t. It is tangent to α at

$$A_t = \left(\frac{2ab}{at^2+b},\ \frac{2at\sqrt{ab}}{at^2+b} \right),$$

and to β at

$$B_t = \left(-\frac{2ab}{a+bt^2},\ \frac{2bt\sqrt{ab}}{a+bt^2} \right).$$

The tangency (in both cases) is internal if $|t| < 1$ and external if $|t| > 1$. The configuration $(\alpha,\ \beta,\ \gamma_t)$ is called a skewed arbelos.

2. Archimedean twin circles in the skewed arbelos

The circle γ_t intersects the y-axis at the points

$$I_t^+ = \left(0,\ \frac{2\sqrt{ab}}{t+1}\right) \quad \text{and} \quad I_t^- = \left(0,\ \frac{2\sqrt{ab}}{t-1}\right)$$

respectively.

The lines $I_t^+A_t$ and $I_t^-A_t$ have equations

$$\begin{aligned}(at-b)x-(t+1)\sqrt{ab}y+2ab =&\ 0,\\ (at+b)x+(t-1)\sqrt{ab}y-2ab =&\ 0.\end{aligned}$$

These lines intersect the circle α again at the points

$$\left(\frac{2ab}{a+b},\ \frac{2a\sqrt{ab}}{a+b}\right) \quad \text{and} \quad \left(\frac{2ab}{a+b},\ -\frac{2a\sqrt{ab}}{a+b}\right).$$

Now, it is well known that the common radius of the twin Archimedean circles tangent to α, β, γ is $r_{\rm A} = \frac{ab}{a+b}$.

These intersections are the fixed points

$$\left(2r_{\rm A}, 2r_{\rm A}\sqrt{\frac{a}{b}}\right) \quad \text{and} \quad \left(2r_{\rm A}, -2r_{\rm A}\sqrt{\frac{a}{b}}\right).$$

Similarly, with B_t, the lines $I_t^+B_t$ and $I_t^-B_t$ intersect the circle β at the points

$$\left(-2r_{\rm A}, 2r_{\rm A}\sqrt{\frac{b}{a}}\right) \quad \text{and} \quad \left(-2r_{\rm A}, -2r_{\rm A}\sqrt{\frac{b}{a}}\right).$$

From these we obtain the following result.

Theorem 1. *For $t \neq \pm 1$, if I_t is one of the intersections of the circle γ_t and the line IO, then the circle touching IO and passing through the remaining intersection of the line I_tA_t* (respectively I_tB_t) *and the circle α* (respectively β) *is an Archimedean circle of the arbelos* (formed by α, β and γ; see Figure 3).

Two of these circles are W_9 and W_{10}, independent of the value of t. If $t = \pm 1$, A_t and the remaining intersection coincide.

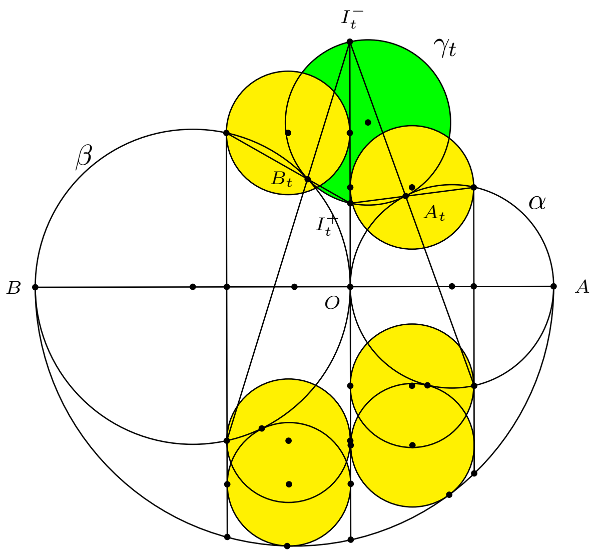

Figure 3.

3. Non-Archimedean twin circles

In this section we generalize the Archimedean circles W_6 and W_7.

We denote the lines $x = 2r_{\mathrm{A}}$ and $x = -2r_{\mathrm{A}}$ by $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ respectively. Let δ_t^α be the circle touching the line $\mathcal{L}_\alpha$ from the side opposite to A_t and the tangents of β from A_t. The circle δ_t^β is defined similarly. The circle δ_t^α lies in the region $x < 2r_{\mathrm{A}}$ if $|t| < 1$ (see Figure 4), and in the region $x > 2r_{\mathrm{A}}$ if $|t| > 1$ (see Figure 5).

If $t = \pm 1$, γ_t reduces to an external common tangent of α and β, and has equation

$$(a - b)x \mp 2\sqrt{ab}y + 2ab = 0,$$

which are obtained from (1) by letting t approach to ± 1. In these cases, we regard δ_t^α as the tangency point of α with an external common tangent of α and β.

Lemma 2. *The circles δ_t^α and δ_t^β are congruent with common radii $|1 - t^2|r_{\mathrm{A}}$.*

Proof. Let s be the radius of the circle δ_t^α. If γ_t touches α and β internally, then we get

$$\left(\frac{2ab}{at^2 + b} - 2r_{\mathrm{A}}\right) : s = \frac{2ab}{at^2 + b} : b.$$

Solving the equation we get $s = (1 - t^2)r_{\mathrm{A}}$. Similarly we get $s = (t^2 - 1)r_{\mathrm{A}}$ in the case γ_t touching α and β externally. $\square$

The point A_t divides the segment joining the centers of δ_t^α and β in the ratio $|1-t^2|r_{\mathrm{A}} : b$ externally (respectively internally) if γ_t touches α and β internally (respectively externally). Therefore the center of δ_t^α is the point $\left(r_{\mathrm{A}}(1 + t^2), 2tr_{\mathrm{A}}\sqrt{\frac{a}{b}}\right)$.

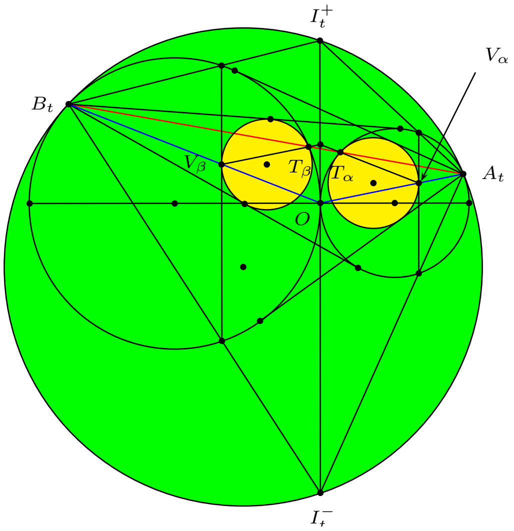

Figure 4.

The circles α and δ_t^α are tangent to each other, since they have only one point in common, namely,

$$T_\alpha = \left(\frac{2abt^2}{a+bt^2}, \frac{2at\sqrt{ab}}{a+bt^2} \right).$$

This point T_α lies on the line A_tB_t since

$$T_\alpha = \frac{at^2+b}{a+b} \cdot A_t + \frac{a(1-t^2)}{a+b} \cdot B_t.$$

Similarly, the circles δ_t^β is tangent to β at

$$T_\beta = \left(-\frac{2abt^2}{at^2+b}, \frac{2bt\sqrt{ab}}{at^2+b} \right),$$

which also lies on the line A_tB_t. The point T_α lies on the line A_tB_t, since

$$T_\beta = \frac{b(1-t^2)}{a+b} \cdot A_t + \frac{a+bt^2}{a+b} \cdot B_t.$$

We summarize the results (see Figures 4 and 5).

Theorem 3. *The circles δ_t^α and δ_t^β are congruent with common radius $|1-t^2|r_{\rm A}$. The points of tangency A_t, B_t, T_α, and T_β are collinear.*

The circles δ_t^α and δ_t^β are generalizations of the Archimedean circles W_6 and W_7, which correspond to $t = 0$.

The circle δ_t^α touches the line $\mathcal{L}_\alpha$ at the point

$$V_\alpha = \left(2r_{\rm A}, 2tr_{\rm A}\sqrt{\frac{a}{b}} \right).$$

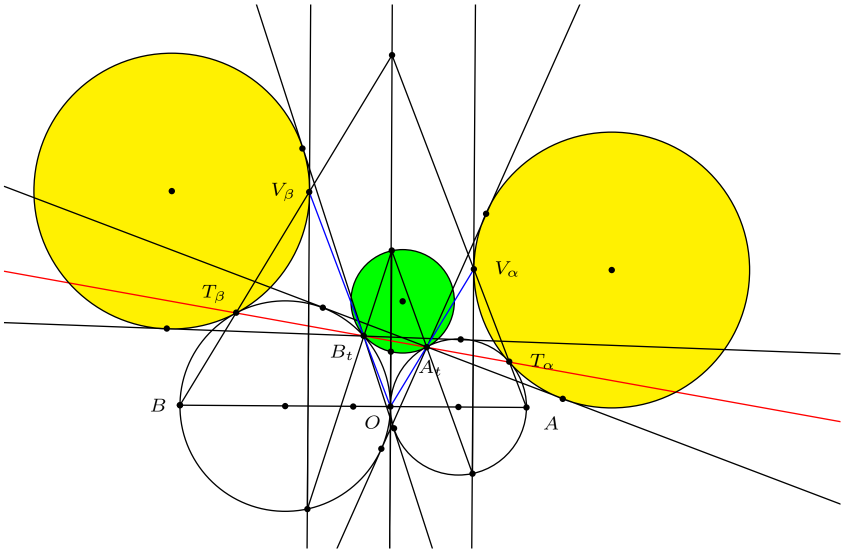

Figure 5.

From these coordinates it is clear that A_t, V_α and O are collinear (see Figures 4 and 5). Similarly, the circle δ_t^β touches the line $\mathcal{L}_\beta$ at

$$V_\beta = \left(-2r_\mathrm{A}, 2tr_\mathrm{A}\sqrt{\frac{b}{a}}\right)$$

collinear with B_t and O. Now, the line $T_\alpha V_\alpha$ is parallel to $B_t O$. Since the distances from V_α and V_β to the line IO (the radical axis of α and β) are equal, the lines $T_\alpha V_\alpha$ and $T_\beta V_\beta$ intersect on the radical axis. This intersection and O, V_α, V_β form the vertices of a parallelogram.

4. A special case

For real numbers t and w, the circles δ_t^α and δ_t^β and the circles δ_w^α and δ_w^β are congruent if and only if $|1-t^2| = |1-w^2|$, *i.e.*, $|t| = |w|$ or $t^2 + w^2 = 2$. From the equation of the circle γ_t, it is clear that this is the same condition for the congruence of the circles γ_t and γ_w. Therefore we get the following corollary.

Corollary 4. *For real numbers t and w, the circles δ_t^α and δ_t^β and the circles δ_w^α and δ_w^β are congruent if and only if γ_t and γ_w are congruent.*

In the case $t^2 + w^2 = 2$, one of the circles γ_t and γ_w touches α and β internally and the other externally. In particular, the circles δ_t^α and δ_t^β are Archimedean circles of the arbelos formed by α, β and γ if and only if γ_t is congruent to γ, *i.e.*, $t = \pm\sqrt{2}$ (see Figure 6).

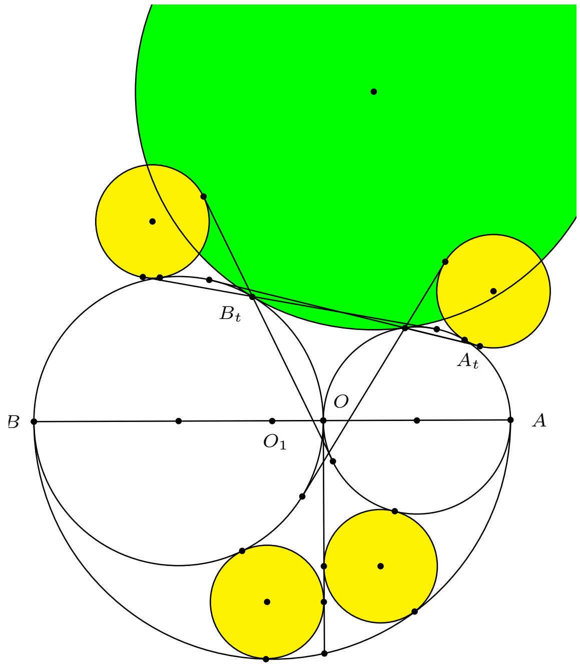

Figure 6.

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