Forum Geometricorum
Volume 11 (2011) 145–154.

FORUM GEOM
ISSN 1534-1178

On a Triad of Circles Tangent to the Circumcircle and the Sides at Their Midpoints

Luis González

Abstract. With synthetic methdos, we study, for a given triangle ABC, a triad of circles tangent to the arcs BC, CA, AB of its circumcircle and the sides BC, CA, AB at their midpoints.

1. Introduction

Given a triangle ABC and an interior point T with cevian triangle $A_0B_0C_0$, consider the triad of circles each tangent to a sideline and the circumcircle internally at a point on the opposite side of the corresponding vertex. Lev Emelyanov [1] has shown that the inner Apollonius circle of the triad is also tangent to the incircle (see Figure 1).

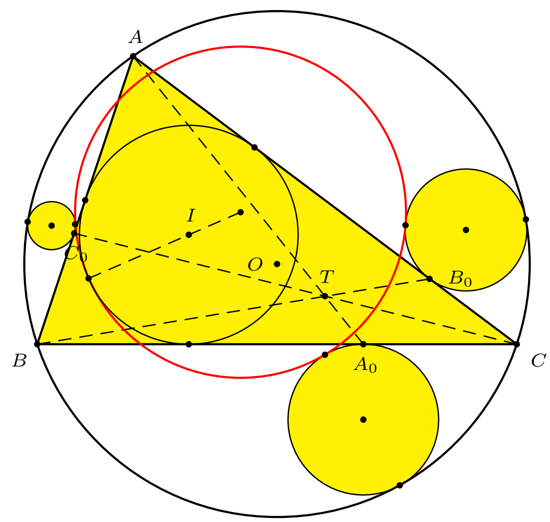

Figure 1.

Yiu [5] has studied this configuration in more details. The points of tangency with the circumcircle form the circumcevian triangle of the barycentric product $I \cdot T$, where I is the incenter. Let X, Y, Z be the intersections of the sidelines EF, FD, EF of the intouch triangle and the corresponding sidelines of the cevian triangle $A_0B_0C_0$. Then XYZ is perspective with DEF, and the perspector is the point F_T on the incircle tangent to the Emelyanov circle, *i.e.*, the inner Apollonius

Publication Date: July 13, 2011. Communicating Editor: Paul Yiu.

circle of the triad ([5, Proposition 12]). In particular, for $T = G$, the centroid of triangle ABC, this point of tangency is the Feuerbach point F_e, which is famously the point of tangency of the incircle with the nine-point circle. Also, in this case,
(i) the radical center of the triad of circles is the triangle center X_{1001} which divides GX_{55} in the ratio $GX_{1001} : X_{1001}X_{55} = R + r : 3R$, where X_{55} is the internal center of similitude of the circumcircle and incircle,
(ii) the center of the Emelyanov circle is the point which divides IN in the ratio 2 : 1 (see Figure 2).

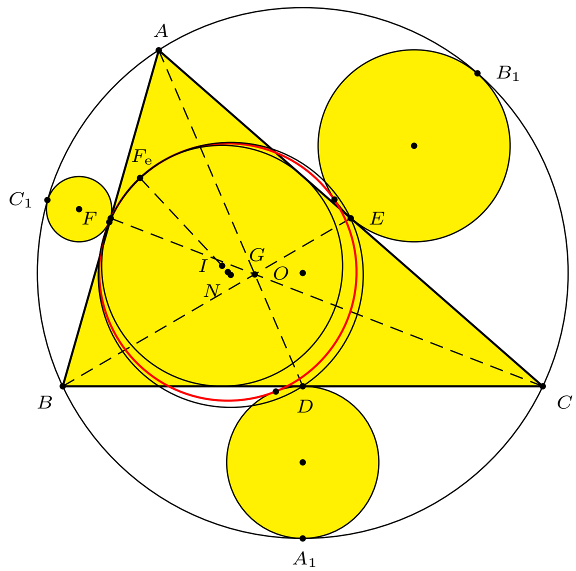

Figure 2.

Yiu obtained these conclusions by computation with barycentric coordinates. In this paper, we revisit the triad of circles $\Gamma(G)$ by synthetic methods.

2. Some preliminary results

Proposition 1. *Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are tangent to a circle $\Gamma(R)$ through A, B, respectively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is given by*

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)},$$

where the sign is positive if the tangency is external, and negative if the tangency is internal.

Proof. Without loss of generality assume that $r_1 \geq r_2$. Let ε_1 (respectively ε_2 be $+1$ or -1 according as the tangency of (O) and (O_1) (respectively (O_2) is external or internal. Figure 3 shows the case when (O) is both tangent internally to (O_1) and (O_2). Let $\angle O_1OO_2 = \theta$. By the law of cosines,

$$O_1O_2^2 = (R + \varepsilon_1 r_1)^2 + (R + \varepsilon_2 r_2)^2 - 2(R + \varepsilon_1 r_1)(R + \varepsilon_2 r_2)\cos\theta. \quad (1)$$

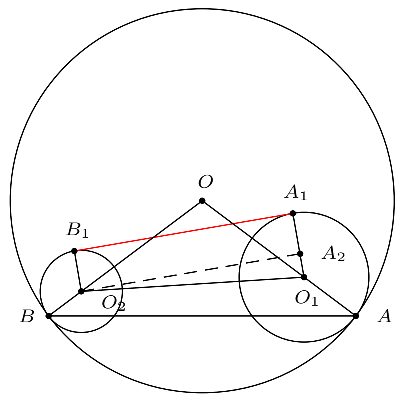

Figure 3.

From the isosceles triangle OAB, we have

$$AB^2 = 2R^2(1 - \cos\theta). \tag{2}$$

Let A_1B_1 be a common tangent of (O_1) and (O_2), external or internal according as $\varepsilon_1\varepsilon_2 = +1$ or -1. If A_2 is the orthogonal projection of O_2 on the line O_1A_1, applying the Pythagorean theorem to the right triangle $O_1O_2A_2$, we have

$$\delta_{12}{}^2 = A_1B_1^2 = O_1O_2^2 - (\varepsilon_1 r_1 - \varepsilon_2 r_2)^2.$$

Eliminating $\cos\theta$ and O_1O_2 from (1) and (2), we have

$$\begin{aligned}\delta_{12}^2 = &\ (R+\varepsilon_1 r_1)^2 + (R+\varepsilon_2 r_2)^2 - (\varepsilon_1 r_1 - \varepsilon_2 r_2)^2 - 2(R+\varepsilon_1 r_1)(R+\varepsilon_2 r_2)\left(1 - \frac{AB^2}{2R^2}\right)\\ = &\ \frac{AB^2}{R^2}\cdot(R+\varepsilon_1 r_1)(R+\varepsilon_2 r_2).\end{aligned}$$

From this the result follows. □

We shall also make use of the following famous theorem.

Proposition 2 (Casey's theorem [2, §172])**.** *Given four circles* $\Gamma_i, i = 1, 2, 3, 4$, *let* δ_{ij} *denote the length of a common tangent* (either internal or external) *between* Γ_i *and* Γ_j. *The four circles are tangent to a fifth circle* Γ *(or line) if and only if for appropriate choice of signs,*

$$\delta_{12}\cdot\delta_{34} \pm \delta_{13}\cdot\delta_{42} \pm \delta_{14}\cdot\delta_{23} = 0$$

3. A triad of circles

Consider a triangle ABC with D, E, F the midpoints of the sides BC, CA, AB respectively. The circle ω_1 tangent to BC at D, and to the arc of the circumcircle on the opposite side of A touches the circumcircle at A_1, the second intersection with the line AI. The circles ω_b and ω_c are similarly defined.

Lemma 3. *The lines through the incenter* I *parallel to* AB *and* AC *are tangent to the circle* $A_1(D)$.

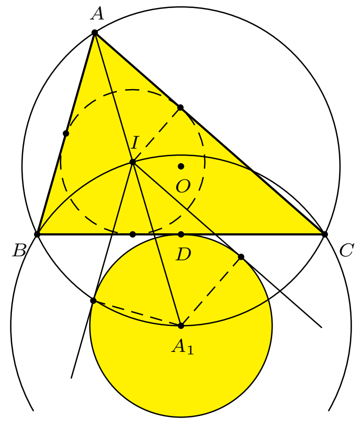

Figure 4.

Proof. Let O be the circumcenter, and R the circumradius. Since $OD = R\cos A$, it is enough to show that

$$R(1-\cos A) + r = AA_1 \sin\frac{A}{2}. \tag{3}$$

This follows from $r = IA\sin\frac{A}{2}$ and $1-\cos A = 2\sin^2\frac{A}{2}$. (3) is equivalent to $2R\sin\frac{A}{2} + IA = AA_1$, which follows from $A_1B = A_1I = 2R\sin\frac{A}{2}$. □

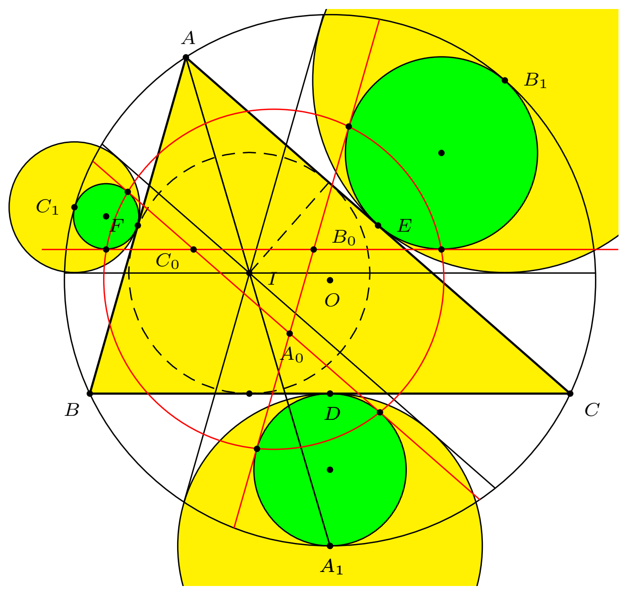

Figure 5.

Note that the length of the tangent from I to $A_1(D)$ is

$$A_1I\cos\frac{A}{2} = A_1B\cos\frac{A}{2} = 2R\sin\frac{A}{2}\cos\frac{A}{2} = R\sin A = \frac{a}{2}.$$

The homothety $\mathsf{h}\left(D, \frac{1}{2}\right)$ takes the two tangents through I to the circle $A'(D)$ to two tangents of ω_a through the midpoint A_2 of ID. These have lengths $\frac{a}{4}$.

Similarly, let B_2 and C_2 be the midpoints of IE and IF respectively. The two tangents from B_2 (respectively C_2 to ω_b (respectively ω_c) have lengths $\frac{b}{4}$ (respectively $\frac{c}{4}$). Now, since DE is parallel to BC, so is the line B_2C_2. These six tangents fall on three lines bounding a triangle $A_2B_2C_2$ homothetic to ABC with factor $-\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$ and homothetic center J dividing IG in the ratio $IJ : JG = 3 : 2$. This leads to the configuration of three (pairwise) common tangents of the triad $(\omega_a,\ \omega_b,\ \omega_c)$ parallel to the sidelines of triangle ABC (see Figure 5). These tangents all have length $\frac{s}{2} = \frac{a+b+c}{4}$. It also follows that the 6 points of tangency lie on a circle, whose center is the incenter of triangle $A_2B_2C_2$, and radius $\frac{1}{4}\sqrt{r^2+s^2}$. This latter fact follows from the proposition below, applied to triangle $A_2B_2C_2$.

Proposition 4. *The sides of a triangle ABC are extended to points X_b, X_c, Y_c, Y_a, Z_a, Z_b such that*

$$AY_a = AZ_a = a, \quad BZ_b = BX_b = b, \quad CX_c = CY_c = c.$$

The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a circle concentric with the incircle and with radius $\sqrt{r^2+s^2}$, where r is the inradius and s the semiperimeter of the triangle.

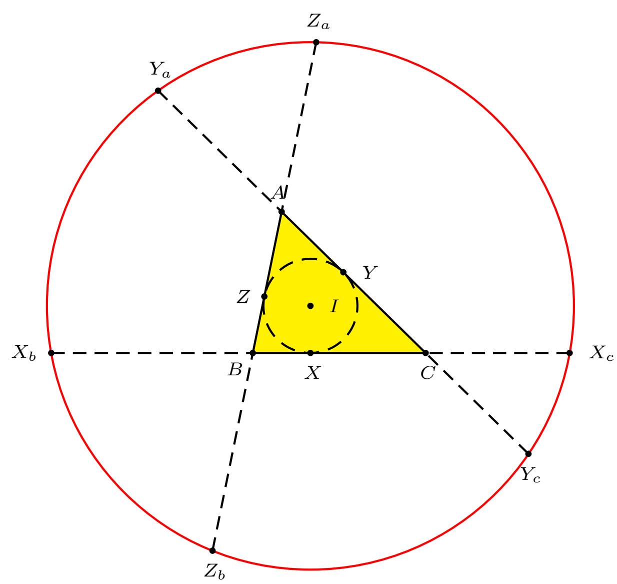

Figure 6.

Proof. If the incircle touches BC at X, then $BX = s - b$. It follows that $X_bX = b + (s-b) = s$, and $IX_b = \sqrt{r^2+s^2}$. The same result holds for the other five points. □

4. On the radical center

Proposition 5. *The radical center of the circles $\omega_a, \omega_b, \omega_c$ is the midpoint between the incenter I and Mittenpunkt of $\triangle ABC$.*

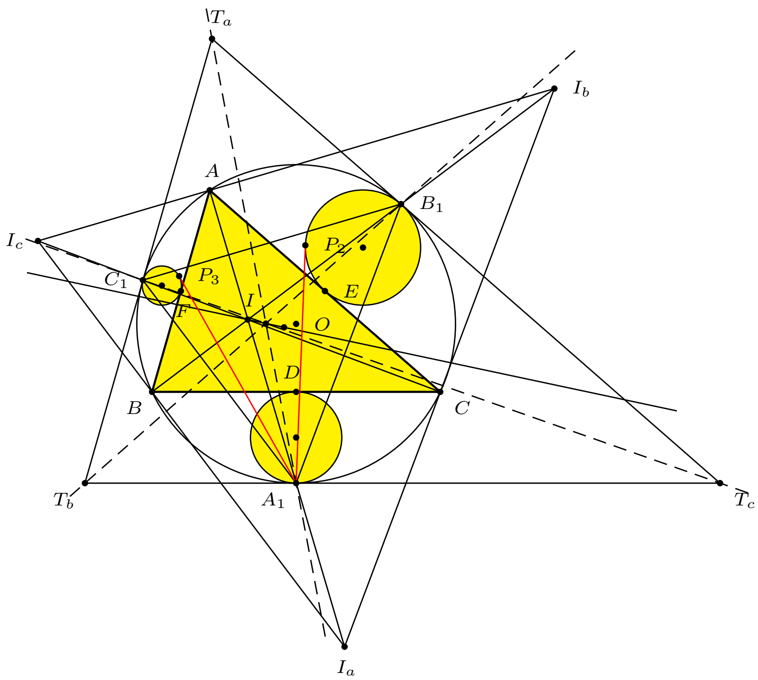

Figure 7.

Proof. Let A_1P_2 and A_1P_3 be the tangent segments from A_1 to ω_b and ω_c ($P_2 \in \omega_b$ and $P_3 \in \omega_c$) (See Figure 7). By Casey's theorem for $(A_1), (A), (C), \omega_b$ and $(A_1), (A), (B), \omega_c$, all tangent to the circumcircle, we have

$$A_1P_2 \cdot AC = A_1A \cdot CE + A_1C \cdot AE \Longrightarrow A_1P_2 = \frac{1}{2}(A_1A + A_1C) \qquad (4)$$

$$A_1P_3 \cdot AB = A_1A \cdot BF + A_1B \cdot AF \Longrightarrow A_1P_3 = \frac{1}{2}(A_1A + A_1B) \qquad (5)$$

Since $A_1B = A_1C$, from (4) and (5) we have $A_1P_2 = A_1P_3$, *i.e.*, A_1 has equal powers with respect to the circles ω_b and ω_c. If $T_aT_bT_c$ is the tangential triangle of $A_1B_1C_1$, then $T_aB_1 = T_aC_1$ implies that T_a has also equal powers with respect to ω_b and ω_c. Therefore, the line A_1T_a is the radical axis of ω_b and ω_c. Likewise, B_1T_b and C_1T_c are the radical axes of ω_c, ω_a and ω_a, ω_b respectively. Hence, the radical center L of $\omega_a, \omega_b, \omega_c$, being the intersection of A_1T_a, B_qT_b, C_1T_c, is the symmedian point of triangle $A_1B_1C_1$. Now, since the homothety $\mathsf{h}(I, 2)$ takes triangle $A_1B_1C_1$ into the excentral triangle $I_aI_bI_c$, and the latter has symmedian point X_9, the Mittenpunkt of triangle ABC, the symmedian point of $A_1B_1C_1$ is the midpoint of IX_9. According to [3], this is the triangle center X_{1001}. $\square$

5. On the Inner Apollonius circle

Emelyanov [1] has shown that the inner Apollonius circle of the triad $(\omega_a,\ \omega_b,\ \omega_c)$ is tangent to the incircle at the Feuerbach point; see also Yiu [5, §5]. The center of the inner Apollonius circle divides IN in the ratio $2:1$.

Proposition 6. *The Apollonius circle ω externally tangent to $\omega_a,\ \omega_b,\ \omega_c$ is also tangent to the incircle of triangle ABC through its Feuerbach point $F_{\rm e}$.*

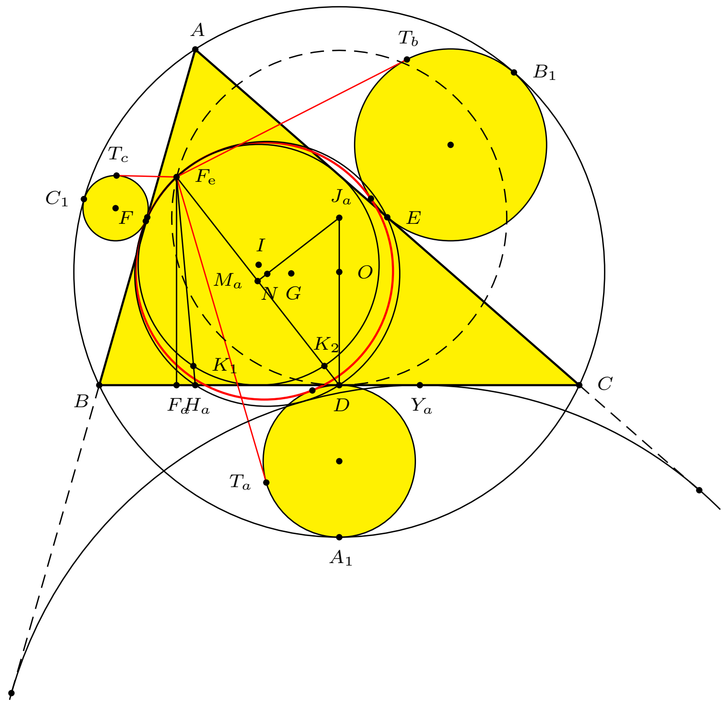

Figure 8.

Proof. Without loss of generality we assume that $b \geq a \geq c$. Let the incircle (I) touch BC, CA, AB at X_a, X_b, X_c respectively. By Casey's theorem there exists a circle ω tangent to $\omega_a, \omega_a, \omega_c$ externally and tangent to (I) internally if and only if

$$\delta_{bc} \cdot DX_a - \delta_{ca} \cdot EX_b - \delta_{ab} \cdot FX_c = 0. \tag{6}$$

Since $\delta_{bc} = \delta_{ca} = \delta_{ab} = \frac{s}{2}$ by Proposition 2, then (6) is an obvious identity because of $DX_a = \frac{1}{2}(b-c)$, $EX_b = \frac{1}{2}(a-c)$ and $FX_c = \frac{1}{2}(b-a)$. (In fact, this tangency is still true if we consider D, E, F as the feet of three concurring cevians. For a proof with similar arguments see [1].) Now, it remains to show that $\omega \cap (I)$ is the Feuerbach point $F_{\rm e}$ of triangle ABC.

Let $F_{\rm e}T_a$, $F_{\rm e}T_b$, $F_{\rm e}T_c$ be the tangent segments from $F_{\rm e}$ to $\omega_a, \omega_b, \omega_c$ respectively. If F_a is the orthogonal projection of $F_{\rm e}$ onto BC and $J_a(\rho_a)$ is the circle passing through $F_{\rm e}$ tangent to BC at D (see Figure 6). Let R_a denote the radius of ω_a, then using Proposititon 1 for ω_a and (F_e) (with zero radius) externally tangent to $J_a(\rho_a)$, we get

$$F_\mathrm{e}D = F_\mathrm{e}T_a\sqrt{\frac{\rho_a}{\rho_a + R_a}}. \tag{7}$$

If M_a denotes the midpoint of DF_e, then the right triangles DJ_aM_a and $F_\mathrm{e}DF_a$ are similar. This gives $2\rho_a \cdot F_\mathrm{e}F_a = F_\mathrm{e}D^2$. Thus, substituting ρ_a from this expression into (7) gives

$$F_\mathrm{e}D = \sqrt{\frac{F_\mathrm{e}D^2}{F_\mathrm{e}D^2 + 2F_\mathrm{e}F_a \cdot R_a}} \cdot F_\mathrm{e}T_a. \tag{8}$$

If H_a denotes the foot of the A-altitude of triangle ABC, then $F_\mathrm{e}F_a$ and the nine-point circle $N\left(\frac{R}{2}\right)$ become the F_e-altitude and circumcircle of triangle $F_\mathrm{e}H_aD$. Consequently, $F_\mathrm{e}D \cdot F_\mathrm{e}H_a = R \cdot F_\mathrm{e}F_a$. Substituting $F_\mathrm{e}F_a$ from this expression into (8) and rearranging, we get

$$F_\mathrm{e}T_a = \sqrt{F_\mathrm{e}D\left(F_\mathrm{e}D + \frac{2F_\mathrm{e}H_a \cdot R_a}{R}\right)}. \tag{9}$$

Let $F_\mathrm{e}H_a$ and $F_\mathrm{e}D$ intersect the incircle (I) again at K_1 and K_2. Since F_e is the exsimilicenter of (I) and (N), K_1K_2 is parallel to DH_a. Therefore, the arcs X_aK_1 and X_aK_2 of (I) are equal, and $F_\mathrm{e}X_a$ bisects angle $H_aF_\mathrm{e}D$. Hence, by the angle bisector theorem, we have $\frac{F_\mathrm{e}H_a}{F_\mathrm{e}D} = \frac{X_aH_a}{X_aD}$. Substituting $F_\mathrm{e}H_a$ from this expression into (9) gives

$$F_\mathrm{e}T_a = \sqrt{1 + 2 \cdot \frac{X_aH_a}{X_aD} \cdot \frac{R_a}{R}} \cdot F_\mathrm{e}D. \tag{10}$$

Since the incenter I and the A-excenter I_a divide harmonically A and the trace V_a of the A-angle bisector, the points of tangency X_a and Y_a of the line BC with the incircle (I) and the A-excircle divide harmonically H_a and V_a. Since D is also the midpoint of X_aY_a, $DX_a{}^2 = DY_a{}^2 = DH_a \cdot DV_a$. Equivalently,

$$\frac{H_aD}{X_aD} = \frac{X_aD}{V_aD} \Longrightarrow \frac{H_aD}{X_aD} - 1 = \frac{X_aH_a}{X_aD} = \frac{X_aD}{V_aD} - 1 = \frac{V_aX_a}{V_aD}.$$

Since V_a is the insimilicenter of (I) and the circle $A_1(D)$, it follows that

$$\frac{r}{2R_a} = \frac{V_aX_a}{V_aD} = \frac{X_aH_a}{X_aD}. \tag{11}$$

Substituting the ratio $\frac{X_aH_a}{X_aD}$ from (11) into (10), we have

$$F_\mathrm{e}T_a = \sqrt{\frac{R+r}{R}} \cdot F_\mathrm{e}D. \tag{12}$$

On the other hand, using Proposition 1 for the incircle (I) and the point circle (D), both internally tangent to the nine-point circle (N), we have

$$F_\mathrm{e}D = \sqrt{\frac{R}{R-2r}} \cdot DX_a = \sqrt{\frac{R}{R-2r}} \cdot \frac{b-c}{2}. \tag{13}$$

Combining equations (12) and (13), we have

$$F_{\mathrm{e}}T_a = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{b-c}{2}. \tag{14}$$

By similar reasoning, we have the expressions

$$F_{\mathrm{e}}T_b = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{a-c}{2}, \tag{15}$$

$$F_{\mathrm{e}}T_c = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{b-a}{2}. \tag{16}$$

Now, by Casey's theorem there exists a circle externally tangent to $\omega_a, \omega_b, \omega_c$ and (F_{e}) (with zero radius), if and only if

$$\delta_{bc} \cdot F_{\mathrm{e}}T_a - \delta_{ca} \cdot F_{\mathrm{e}}T_b - \delta_{ab} \cdot F_{\mathrm{e}}T_c = 0.$$

Since $\delta_{bc} = \delta_{ca} = \delta_{ab} = \frac{s}{2}$ by Proposition 2, the latter condition becomes $F_{\mathrm{e}}T_a - F_{\mathrm{e}}T_b - F_{\mathrm{e}}T_c = 0$, which is easily verified by (14), (15), (16). Hence, we conclude that ω is tangent to (I) through F_{e}, as desired. □

Proposition 7. *The center I_0 of the inner Apollonius circle ω of $\omega_a, \omega_b, \omega_c$ is the intersection of the lines X_3X_{1001}, X_1X_{11} and its radius ρ equals a third of the sum of the inradius and circumradius of triangle ABC.*

Proof. By Proposition 5, the radical center L of $\omega_a, \omega_b, \omega_c$ is the midpoint X_{1001} between the incenter I and the Mittenpunkt X_9 of triangle ABC. Hence, the inversion with center X_{1001} and power equal to the power of X_{1001} to $\omega_a, \omega_b, \omega_c$, carries these circles into themselves and swaps ω and the circumcircle of $\triangle ABC$ due to conformity. Since the center of the inversion is also a similitude center between the circle at its inverse, it follows that I_0 lies on the line connecting the circumcenter X_3 and X_{1001}. But, from Proposition 6, we deduce that I_0 lies on the line connecting I and the Feuerbach point F_{e}. Therefore $I_0 = OX_{1001} \cap IF_{\mathrm{e}}$.

Let D_a be the tangency point of ω with ω_a. Applying Proposition 1 to the two triads of circles $(F_{\mathrm{e}}), \omega_a, \omega$ and $(I), \omega_a, \omega$, respectively, we obtain

$$F_{\mathrm{e}}T_a^2 = \frac{\rho + R_a}{\rho} \cdot F_{\mathrm{e}}{D_a}^2,$$

$$X_aD^2 = \frac{(\rho + R_a)(\rho - r)}{\rho^2} \cdot F_{\mathrm{e}}{D_a}^2.$$

Eliminating $(\rho + R_a)F_{\mathrm{e}}{D_a}^2$ from these two latter expressions and using (14), we have

$$\left(\frac{F_{\mathrm{e}}T_a}{X_aD}\right)^2 = \frac{\rho}{\rho - r} \Longrightarrow \frac{R+r}{R-2r} = \frac{\rho}{\rho - r} \Longrightarrow \rho = \frac{R+r}{3}.$$

□

Remark. Since F_{e} is the insimilicenter of (I) and ω,

$$\frac{F_{\mathrm{e}}I_0}{F_{\mathrm{e}}I} = \frac{\rho}{r} = \frac{R+r}{3r}.$$

Consequently, in absolute barycentric coordinates,

$$I_0 = \frac{R+r}{3r} \cdot I - \frac{R-2r}{3r} \cdot F_{\mathrm{e}} = \frac{R+r}{3r} \cdot I - \frac{R-2r}{3r} \cdot \frac{R \cdot I - 2r \cdot N}{R-2r} = \frac{I+2N}{3},$$

where N is the nine-point center. It does not appear in the current edition of [3], though its homogeneous barycentric coordinates are recorded in [5].

References

[1] L. Emelyanov, A Feuerbach type theorem on six circles, *Forum Geom.*, 1 (2001) 173–175; correction, *ibid.*, 177.
[2] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.
[3] C. Kimberling, Encyclopedia of Triangle Centers, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] I. Shariguin, *Problemas de Geometrié (Planimetrié)*, Ed. Mir, Moscú, 1989.
[5] P. Yiu, On Emelyanov’s circle theorem, *Journal for Geometry and Graphics*, 9 (2005) 155–167.

Luis González: 5 de Julio Avenue, Maracaibo, Venezuela
E-mail address: alain.levelut@neuf.fr