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The Area of a Bicentric Quadrilateral

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Abstract. We review and prove a total of ten different formulas for the area of a bicentric quadrilateral. Our main result is that this area is given by

$$K = \left| \frac{m^2 - n^2}{k^2 - l^2} \right| kl$$

where m, n are the bimedians and k, l the tangency chords.

1. The formula $K = \sqrt{abcd}$

A bicentric quadrilateral is a convex quadrilateral with both an incircle and a circumcircle, so it is both tangential and cyclic. It is well known that the square root of the product of the sides gives the area of a bicentric quadrilateral. In [12, pp.127–128] we reviewed four derivations of that formula and gave a fifth proof. Here we shall give a sixth proof, which is probably as simple as it can get if we use trigonometry and the two fundamental properties of a bicentric quadrilateral.

Theorem 1. *A bicentric quadrilateral with sides a, b, c, d has the area*

$$K = \sqrt{abcd}.$$

Proof. The diagonal AC divide a convex quadrilateral $ABCD$ into two triangles ABC and ADC. Using the law of cosines in these, we have

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D. \tag{1}$$

The quadrilateral has an incircle. By the Pitot theorem $a + c = b + d$ [4, pp.65–67] we get $(a - b)^2 = (d - c)^2$, so

$$a^2 - 2ab + b^2 = d^2 - 2cd + c^2. \tag{2}$$

Subtracting (2) from (1) and dividing by 2 yields

$$ab(1 - \cos B) = cd(1 - \cos D). \tag{3}$$

In a cyclic quadrilateral opposite angles are supplementary, so that $\cos D = -\cos B$. We rewrite (3) as

$$(ab + cd) \cos B = ab - cd. \tag{4}$$

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The area K of a convex quadrilateral satisfies $2K = ab\sin B + cd\sin D$. Since $\sin D = \sin B$, this yields

$$2K = (ab + cd)\sin B. \tag{5}$$

Now using (4), (5) and the identity $\sin^2 B + \cos^2 B = 1$, we have for the area K of a bicentric quadrilateral

$$(2K)^2 = (ab + cd)^2(1 - \cos^2 B) = (ab + cd)^2 - (ab - cd)^2 = 4abcd.$$

Hence $K = \sqrt{abcd}$. □

Corollary 2. *A bicentric quadrilateral with sides a, b, c, d has the area*

$$K = ac\tan\frac{\theta}{2} = bd\cot\frac{\theta}{2}$$

where θ is the angle between the diagonals.

Proof. The angle θ between the diagonals in a bicentric quadrilateral is given by

$$\tan^2\frac{\theta}{2} = \frac{bd}{ac}$$

according to [8, p.30]. Hence we get

$$K^2 = (ac)(bd) = (ac)^2\tan^2\frac{\theta}{2}$$

and similar for the second formula. □

Corollary 3. *In a bicentric quadrilateral $ABCD$ with sides a, b, c, d we have*

$$\tan\frac{A}{2} = \sqrt{\frac{bc}{ad}} = \cot\frac{C}{2},$$
$$\tan\frac{B}{2} = \sqrt{\frac{cd}{ab}} = \cot\frac{D}{2}.$$

Proof. A well known trigonometric formula and (3) yields

$$\tan\frac{B}{2} = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{cd}{ab}} \tag{6}$$

where we also used $\cos D = -\cos B$ in (3). The formula for D follows from $B = \pi - D$. By symmetry in a bicentric quadrilateral, we get the formula for A by the change $b \leftrightarrow d$ in (6). Then we use $A = \pi - C$ to complete the proof. □

Therefore, not only the area but also the angles have simple expressions in terms of the sides.

The area of a bicentric quadrilateral also gives a condition when a tangential quadrilateral is cyclic. Even though we did not express it in those terms, we have already proved the following characterization in the proof of Theorem 9 in [12]. Here we give another short proof.

Theorem 4. *A tangential quadrilateral with sides a, b, c, d is also cyclic if and only if it has the area $K = \sqrt{abcd}$.*

Proof. The area of a tangential quadrilateral is according to [8, p.28] given by

$$K = \sqrt{abcd}\sin\frac{B+D}{2}.$$

It's also cyclic if and only if $B + D = \pi$; hence a tangential quadrilateral is cyclic if and only if it's area is $K = \sqrt{abcd}$. □

This is not a new characterization of bicentric quadrilaterals. One quite long trigonometric proof of it was given by Joseph Shin in [15] and more or less the same proof of the converse can be found in the solutions to Problem B-6 in the 1970 William Lowell Putnam Mathematical Competition [1, p.69].

In this characterization the formulation of the theorem is important. The tangential and cyclic quadrilaterals cannot change roles in the formulation, nor can the formulation be that it's a bicentric quadrilateral if and only if the area is given by the formula in the theorem. This can be seen with an example. A rectangel is cyclic but not tangential. Its area satisfy the formula $K = \sqrt{abcd}$ since opposite sides are equal. Thus it's important that it must be a tangential quadrilateral that is also cyclic if and only if the area is $K = \sqrt{abcd}$, otherwise the conclution would be that a rectangle also has an incircle, which is obviously false.

2. Other formulas for the area of a bicentric quadrilateral

In this section we will prove three more formulas for the area of a bicentric quadrilateral, where the area is given in terms of other quantities than the sides. Let us first review a few other formulas and one double inequality for the area that can be found elsewhere.

In [12], Theorem 10, we proved that a bicentric quadrilateral has the area

$$K = \sqrt[4]{efgh}(e+f+g+h)$$

where e, f, g, h are the tangent lengths, that is, the distances from the vertices to the point where the incircle is tangent to the sides.

According to Juan Carlos Salazar [14], a bicentric quadrilateral has the area

$$K = 2\,MN\sqrt{EQ \cdot FQ}$$

where M, N are the midpoints of the diagonals; E, F are the intersection points of the extensions of opposite sides, and Q is the foot of the normal to EF through the incenter I (see Figure 1). This is a remarkable formula since the area is given in terms of only three distances. A short proof is given by "pestich" at [14]. He first proved that a bicentric quadrilateral has the area

$$K = 2\,MN \cdot IQ$$

which is even more extraordinary, since here the area is given in terms of only two distances!

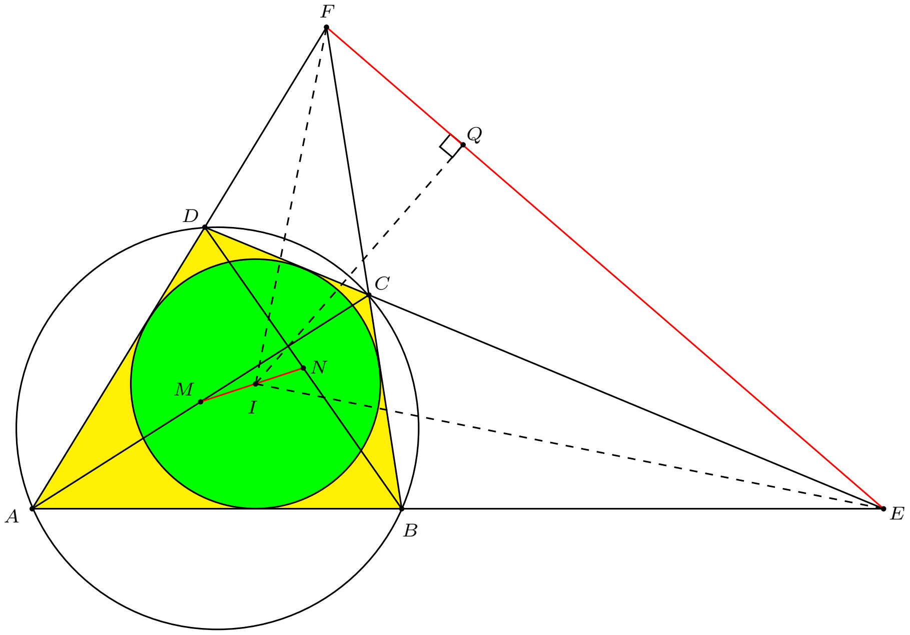

Figure 1. The configuration of Salazar's formula

The angle EIF (see Figure 1) is a right angle in a bicentric quadrilateral,[1] so we also get that the area of a bicentric quadrilateral is given by

$$K = \frac{2\,MN \cdot EI \cdot FI}{EF}$$

where we used the well known property that the product of the legs is equal to the product of the hypotenuse and the altitude in a right triangle. The last three formulas are not valid in a square since there we have $MN = 0$.

In [2, p.64] Alsina and Nelsen proved that the area of a bicentric quadrilateral satisfy the inequalities

$$4r^2 \leq K \leq 2R^2$$

where r, R are the radii in the incircle and circumcircle respectively. We have equality on either side if and only if it is a square.

Problem 1 on Quiz 2 at the China Team Selection Test 2003 [5] was to prove that in a tangential quadrilateral $ABCD$ with incenter I,

$$AI \cdot CI + BI \cdot DI = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

The right hand side gives the area of a bicentric quadrilateral, so from this we get another formula for this area. It is easier to prove the following theorem than solving the problem from China,[2] since in a bicentric quadrilateral we can also use that opposite angles are supplementary angles.

[1]This is proved in Theorem 5 in [13], where the notations are different from here.

[2]One solution is given by Darij Grinberg in [9, pp.16–19].

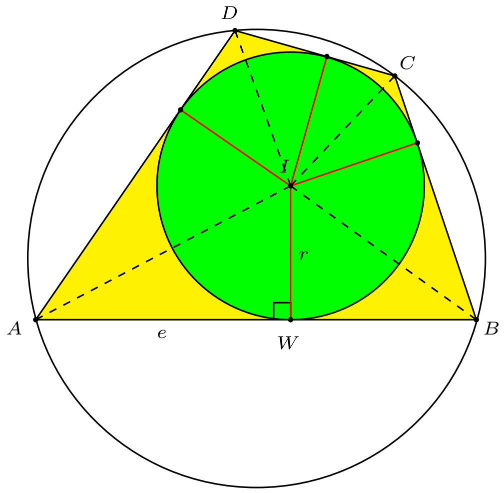

Figure 2. Partition of a bicentric quadrilateral into kites

Theorem 5. *A bicentric quadrilateral $ABCD$ with incenter I has the area*

$$K = AI \cdot CI + BI \cdot DI.$$

Proof. The quadrilateral has an incircle, so $\tan \frac{A}{2} = \frac{r}{e}$ where r is the inradius (see Figure 2). It also has a circumcircle, so $A+C = B+D = \pi$. Thus $\cot \frac{C}{2} = \tan \frac{A}{2}$ and $\sin \frac{C}{2} = \cos \frac{A}{2}$. A bicentric quadrilateral can be partitioned into four right kites by four inradii, see Figure 2.

Triangle AIW has the area $\frac{er}{2} = \frac{r^2}{2\tan \frac{A}{2}}$. Thus the bicentric quadrilateral has the area

$$K = r^2 \left(\frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} + \frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{D}{2}} \right).$$

Hence we get

$$\begin{aligned} K &= r^2 \left(\frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{D}{2}} + \frac{1}{\tan \frac{B}{2}} \right) \\ &= r^2 \left(\left(\tan \frac{A}{2} + \cot \frac{A}{2} \right) + \left(\tan \frac{B}{2} + \cot \frac{B}{2} \right) \right) \\ &= r^2 \left(\frac{1}{\sin \frac{A}{2} \cos \frac{A}{2}} + \frac{1}{\sin \frac{B}{2} \cos \frac{B}{2}} \right) \\ &= \frac{r^2}{\sin \frac{A}{2} \sin \frac{C}{2}} + \frac{r^2}{\sin \frac{B}{2} \sin \frac{D}{2}} \\ &= AI \cdot CI + BI \cdot DI \end{aligned}$$

where we used that $\sin \frac{A}{2} = \frac{r}{AI}$ and similar for the other angles. $\square$

Corollary 6. *A bicentric quadrilateral $ABCD$ has the area*

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right)$$

where r is the inradius.

Proof. Using one of the equalities in the proof of Theorem 5, we get

$$K = r^2\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}\cos\frac{B}{2}}\right) = r^2\left(\frac{1}{\frac{1}{2}\sin A} + \frac{1}{\frac{1}{2}\sin B}\right)$$

and the result follows. □

Here is an alternative, direct proof of Corollary 6:

In a tangential quadrilateral with sides a, b, c, d and semiperimeter s we have $K = rs = r(a + c) = r(b + d)$. Hence

$$\begin{aligned} K^2 &= r^2(a + c)(b + d) \\ &= r^2(ad + bc + ab + cd) \\ &= r^2\left(\frac{2K}{\sin A} + \frac{2K}{\sin B}\right) \end{aligned}$$

since in a cyclic quadrilateral $ABCD$, the area satisfies $2K = (ad + bc)\sin A = (ab + cd)\sin B$. Now factor the right hand side and then divide both sides by K. This completes the proof.

From Corollary 6 we get another proof of the inequality $4r^2 \leq K$, different form the one given in [2, p.64]. We have

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) \geq 2r^2(1 + 1) = 4r^2$$

for $0 < A < \pi$ and $0 < B < \pi$.

In [12], Theorem 11, we proved that a bicentric quadrilateral with diagonals p, q and tangency chords [3] k, l has the area

$$K = \frac{klpq}{k^2 + l^2}. \tag{7}$$

We shall use this to derive another beautiful formula for the area of a bicentric quadrilateral. In the proof we will also need the following formula for the area of a convex quadrilateral, which we have not found any reference to.

Theorem 7. *A convex quadrilateral with diagonals p, q and bimedians m, n has the area*

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (m^2 - n^2)^2}.$$

[3] A tangency chord is a line segment connecting the points on two opposite sides where the incircle is tangent to those sides in a tangential quadrilateral.

Proof. A convex quadrilateral with sides a, b, c, d and diagonals p, q has the area

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2} \tag{8}$$

according to [6, p.243], [11] and [16]. The length of the bimedians [4] m, n in a convex quadrilateral are given by

$$m^2 = \tfrac{1}{4}(p^2 + q^2 - a^2 + b^2 - c^2 + d^2), \tag{9}$$

$$n^2 = \tfrac{1}{4}(p^2 + q^2 + a^2 - b^2 + c^2 - d^2). \tag{10}$$

according to [6, p.231] and post no 2 at [10] (both with other notations). From (9) and (10) we get

$$4(m^2 - n^2) = -2(a^2 - b^2 + c^2 - d^2)$$

so

$$(a^2 - b^2 + c^2 - d^2)^2 = 4(m^2 - n^2)^2.$$

Using this in (8), the formula follows. □

The next theorem is our main result and gives the area of a bicentric quadrilateral in terms of the bimedians and tangency chords (see Figure 3).

Theorem 8. *A bicentric quadrilateral with bimedians m, n and tangency chords k, l has the area*

$$K = \left|\frac{m^2 - n^2}{k^2 - l^2}\right| kl$$

if it is not a kite.

Proof. From Theorem 7 we get that in a convex quadrilateral

$$(m^2 - n^2)^2 = (pq)^2 - 4K^2. \tag{11}$$

Rewriting (7), we have in a bicentric quadrilateral

$$pq = \frac{k^2 + l^2}{kl}K.$$

Inserting this into (11) yields

$$\begin{aligned}(m^2 - n^2)^2 &= \frac{(k^2 + l^2)^2}{k^2l^2}K^2 - 4K^2\\ &= K^2\left(\frac{(k^2 + l^2)^2 - 4k^2l^2}{k^2l^2}\right)\\ &= K^2\left(\frac{(k^2 - l^2)^2}{k^2l^2}\right).\end{aligned}$$

Hence

$$\left|m^2 - n^2\right| = K\frac{\left|k^2 - l^2\right|}{kl}$$

and the formula follows.

[4] A bimedian is a line segment connecting the midpoints of two opposite sides in a quadrilateral.

It is not valid in two cases, when $m = n$ or $k = l$. In the first case we have according to (9) and (10) that

$$-a^2 + b^2 - c^2 + d^2 = a^2 - b^2 + c^2 - d^2 \quad \Leftrightarrow \quad a^2 + c^2 = b^2 + d^2$$

which is a well known condition for when a convex quadrilateral has perpendicular diagonals. The second case is equivalent to that the quadrilateral is a kite according to Corollary 3 in [12]. Since the only tangential quadrilateral with perpendicular diagonals is the kite (see the proof of Corollary 3 in [12]), this is the only quadrilateral where the formula is not valid. [5] □

In view of the expressions in the quotient in the last theorem, we conclude with the following theorem concerning the signs of those expressions in a tangential quadrilateral. Let $m = EG$ and $n = FH$ be the bimedians, and $k = WY$ and $l = XZ$ be the tangency chords in a tangential quadrilateral, see Figure 3.

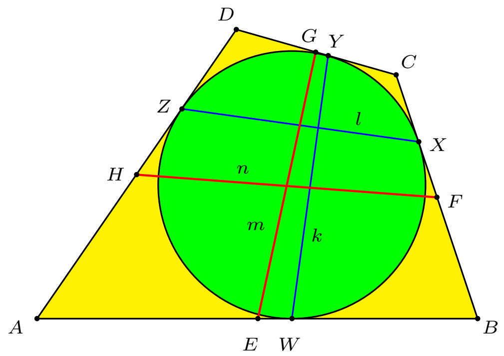

Figure 3. The bimedians m, n and the tangency chords k, l

Theorem 9. *Let a tangential quadrilateral have bimedians m, n and tangency chords k, l. Then*

$$m < n \quad \Leftrightarrow \quad k > l$$

where m and k connect the same pair of opposite sides.

Proof. Eulers extension of the parallelogram law to a convex quadrilateral with sides a, b, c, d states that

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$

where v is the distance between the midpoints of the diagonals p, q (this is proved in [7, p.107] and [3, p.126]). Using this in (9) and (10) we get that the length of the bimedians in a convex quadrilateral can also be expressed as

$$m = \tfrac{1}{2}\sqrt{2(b^2 + d^2) - 4v^2},$$
$$n = \tfrac{1}{2}\sqrt{2(a^2 + c^2) - 4v^2}.$$

[5] This also means it is not valid in a square since a square is a special case of a kite.

Thus in a tangential quadrilateral we have

$$\begin{aligned} & m < n \\ \Leftrightarrow \quad & b^2 + d^2 < a^2 + c^2 \\ \Leftrightarrow \quad & (f+g)^2 + (h+e)^2 < (e+f)^2 + (g+h)^2 \\ \Leftrightarrow \quad & fg + he < ef + gh \\ \Leftrightarrow \quad & (e-g)(h-f) < 0 \end{aligned}$$

where $e = AW, f = BX, g = CY$ and $h = DZ$ are the tangent lengths.

In [12], Theorem 1, we proved that the lengths of the tangency chords in a tangential quadrilateral are

$$k = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+f)(f+h)(h+g)(g+e)}},$$
$$l = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+h)(h+f)(f+g)(g+e)}}.$$

Thus

$$\begin{aligned} & k > l \\ \Leftrightarrow \quad & (e+f)(f+h)(h+g)(g+e) < (e+f)(f+h)(h+g)(g+e) \\ \Leftrightarrow \quad & eh + fg < ef + gh \\ \Leftrightarrow \quad & (e-g)(h-f) < 0. \end{aligned}$$

Hence in a tangential quadrilateral

$$m < n \quad \Leftrightarrow \quad (e-g)(h-f) < 0 \quad \Leftrightarrow \quad k > l$$

which proves the theorem. □

We also note that the bimedians are congruent if and only if the tangency chords are congruent. Such equivalences will be investigated further in a future paper.

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