Forum Geometricorum
Volume 11 (2011) 201–203.

FORUM GEOM
ISSN 1534-1178

A Spatial View of the Second Lemoine Circle

Floor van Lamoen

Abstract. We consider circles in the plane as orthogonal projections of spheres in three dimensional space, and give a spatial characterization of the second Lemoine circle.

1. Introduction

In some cases in triangle geometry our knowledge of the plane is supported by a spatial view. A well known example is the way that David Eppstein found the Eppstein points, associated with the Soddy circles [2]. We will act in a similar way.

In the plane of a triangle ABC consider the tangential triangle $A'B'C'$. The three circles $A'(B)$, $B'(C)$, and $C'(A)$ form the A-, B-, and C-Soddy circles of the tangential triangle. We will, however, regard these as spheres T_A, T_B, and T_C in the three dimensional space. Let their radii be ρ_a, ρ_b, and ρ_c respectively. We consider the spheres that are tritangent to the triple of spheres T_A, T_B, and T_C externally. There are two such congruent spheres, symmetric with respect to the plane of ABC. We denote one of these by $T(\rho_t)$.

If we project $T(\rho_t)$ orthogonally onto the plane of ABC, then its center T is projected to the radical center of the three circles $A'(\rho_a + \rho_t)$, $B'(\rho_b + \rho_t)$ and $C'(\rho_c + \rho_t)$. In general, when a parameter t is added to the radii of three circles, their radical center depends linearly on t. Clearly the incenter of $A'B'C'$ (circumcenter of ABC) as well as the inner Soddy center of $A'B'C'$ are radical centers of $A'(\rho_a + t)$, $B'(\rho_b + t)$ and $C'(\rho_c + t)$ for particular values of t.

Proposition 1. *The orthogonal projection to the plane of ABC of the centers of spheres T tritangent externally to S_A, S_B, and S_C lie on the Soddy-line of $A'B'C'$, which is the Brocard axis of ABC.*

2. The second Lemoine circle as a sphere

Recall that the antiparallels through the symmedian point K meet the sides in six concyclic points (P, Q, R, S, U, and V in Figure 1), and that the circle through

Publication Date: September 28, 2011. Communicating Editor: Paul Yiu.

these points is called the second Lemoine circle, [1] with center K and radius

$$r_L = \frac{abc}{a^2+b^2+c^2}.$$

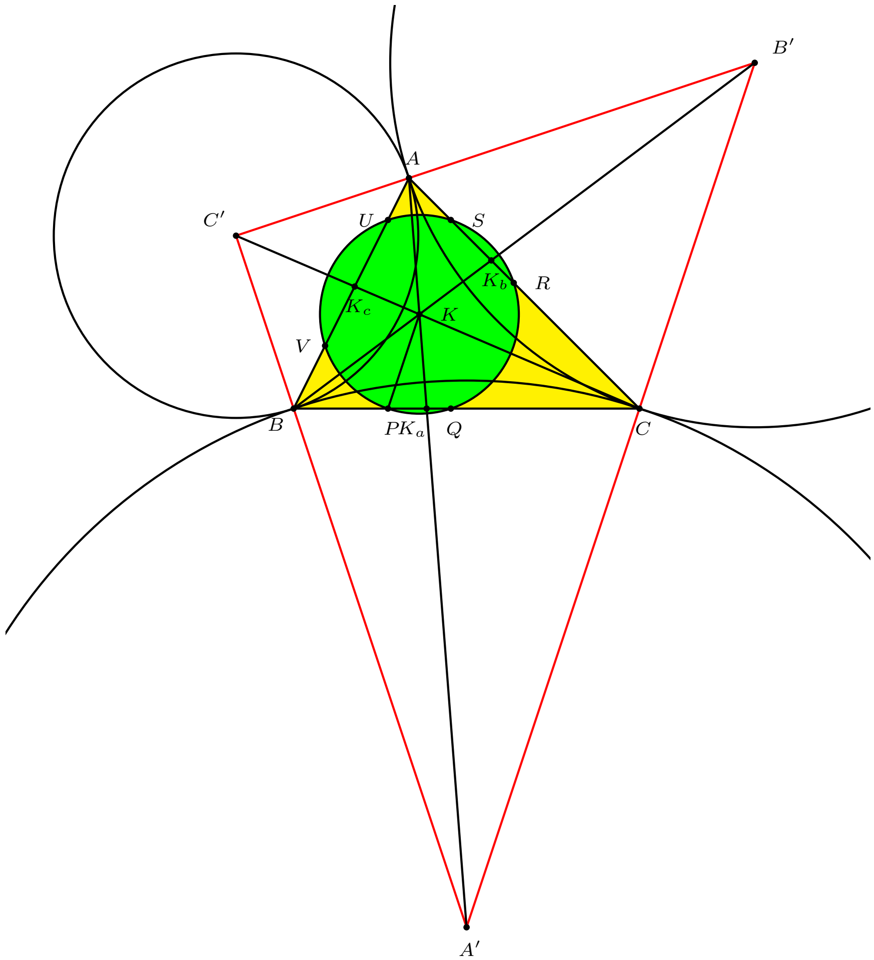

Figure 1. The second Lemoine circle and the circles $A'(B)$, $B'(C)$, $C'(A)$

Proposition 2. *The second Lemoine circle is the orthogonal projection on the plane of ABC of a sphere T tritangent externally to T_A, T_B, and T_C. Furthermore,*

(1) *the center T_K of T has a distance of $2r_L$ to the plane of ABC;*
(2) *the highest points of T, T_A, T_B, and T_C are coplanar.*

---

[1] An alternative name is the cosine circle, because the sides of ABC intercept chords of lengths are proportional to the cosines of the vertex angles. Since however there are infinitely many circles with this property, see [1], this name seems less appropriate.

Proof. Since A' is given in barycentrics by $(-a^2 : b^2 : c^2)$, we find with help of the distance formula (see for instance [3], where some helpful information on circles in barycentric coordinates is given as well):

$$\rho_a = d_{A',B} = \frac{abc}{b^2+c^2-a^2},$$
$$d^2_{A',K} = \frac{4a^4b^2c^2(2b^2+2c^2-a^2)}{(a^2+b^2+c^2)^2(b^2+c^2-a^2)^2}.$$

Now combining these, we see that the power K with respect to $A'(\rho_a + r_L)$ is equal to

$$\mathcal{P} = d^2_{A',K} - (\rho_a + r_L)^2 = -\frac{4a^2b^2c^2}{(a^2+b^2+c^2)^2} = -4r_L^2.$$

By symmetry, K is indeed the radical center of $A'(\rho_a + r_L)$, $B'(\rho_b + r_L)$, and $C'(\rho_c + r_L)$. Therefore, the second Lemoine circle is indeed the orthogonal projection of a sphere externally tritangent to T_A, T_B, and T_C. In addition $-\mathcal{P} = d^2_{K,T_K}$, which proves (1).

Now from

$$\rho_b \cdot A' - \rho_a \cdot B' \sim S_A \cdot A' - S_B \cdot B' = (-a^2 : b^2 : 0)$$

we see by symmetry that the plane through highest points of T_A, T_B, and T_C meets the plane of ABC in the Lemoine axis $\frac{x}{a^2} + \frac{y}{b^2} + \frac{x}{c^2} = 0$. The fact that $3r_L \cdot A' - \rho_a \cdot K \sim (-2a^2 : b^2 : c^2)$ lies on the Lemoine axis as well completes the proof of (2).

Note finally that from the similarity of triangles $A'CK_a$ and KPK_a in Figure 1 K_a divides $A'K$ in the ratio of the radii of T_A and T. This means that the vertices of the cevian triangle $K_aK_bK_c$ are the orthogonal projections of the points of contact of T with T_A, T_B, and T_C respectively. □

References

[1] J.-P. Ehrmann and F. M. van Lamoen, The Stammler circles, *Forum Geom.*, 2 (2002) 151–161.
[2] D. Eppstein, Tangent spheres and triangle centers, *Amer. Math. Monthly*, 108 (2001) 63–66.
[3] V. Volonec, Circles in barycentric coordinates, *Mathematical Communications*, 9 (2004) 79–89.

Floor van Lamoen: Ostrea Lyceum, Bergweg 4, 4461 NB Goes, The Netherlands.
E-mail address: fvanlamoen@planet.nl