Forum Geometricorum
Volume 11 (2011) 205–210.

FORUM GEOM
ISSN 1534-1178

A Simple Vector Proof of Feuerbach's Theorem

Michael J. G. Scheer

Abstract. The celebrated theorem of Feuerbach states that the nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles. In this note, we give a simple proof of Feuerbach's Theorem using straightforward vector computations. All required preliminaries are proven here for the sake of completeness.

1. Notation and background

Let ABC be a nonequilateral triangle. We denote its side-lengths by a, b, c, its semiperimeter by $s = \frac{1}{2}(a + b + c)$, and its area by Δ. Its *classical centers* are the circumcenter O, the incenter I, the centroid G, and the orthocenter H (Figure 1). The nine-point center N is the midpoint of OH and the center of the nine-point circle, which passes through the side-midpoints A', B', C' and the feet of the three altitudes. The Euler Line Theorem states that G lies on OH with $OG : GH = 1 : 2$.

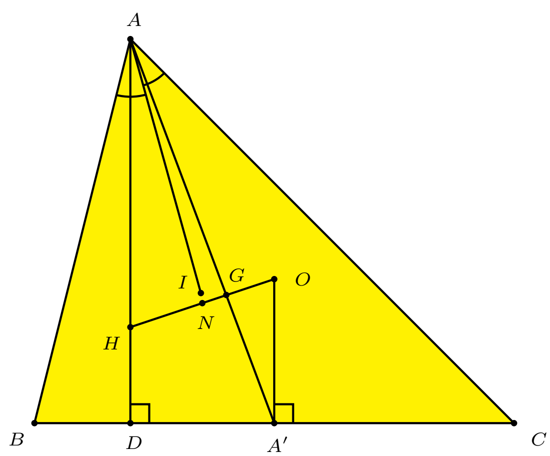

Figure 1. The classical centers and the Euler division $OG : GH = 1 : 2$.

We write I_a, I_b, I_c for the excenters opposite A, B, C, respectively; these are points where one internal angle bisector meets two external angle bisectors. Like I, the points I_a, I_b, I_c are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circumradius R ($= OA = OB = OC$), the inradius r, and

Publication Date: October 21, 2011. Communicating Editor: Paul Yiu.
The author wishes to thank his teacher, Mr. Joseph Stern, for offering numerous helpful suggestions and comments during the writing of this note.

the exradii r_a, r_b, r_c. The following area formulas are well known (see, e.g., [1] and [2]):

$$\Delta = \frac{abc}{4R} = rs = r_a(s-a) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Feuerbach's Theorem states that *the nine-point circle is tangent internally to the incircle, and externally to each of the excircles* [3]. Two of the four points of tangency are shown in Figure 2.

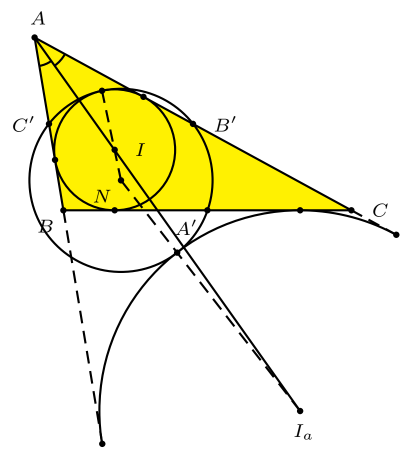

Figure 2. The excenter I_a and A-excircle; Feuerbach's theorem.

2. Vector formalism

We view the plane as $\mathbb{R}^2$ with its standard vector space structure. Given triangle ABC, the vectors $A-C$ and $B-C$ are linearly independent. Thus for any point X, we may write $X-C = \alpha(A-C)+\beta(B-C)$ for unique $\alpha, \beta \in \mathbb{R}$. Defining $\gamma = 1-\alpha-\beta$, we find that

$$X = \alpha A + \beta B + \gamma C, \qquad \alpha+\beta+\gamma = 1.$$

This expression for X is unique. One says that X has *barycentric coordinates* (α, β, γ) with respect to triangle ABC (see, e.g., [1]). The barycentric coordinates are particularly simple when X lies on a side of triangle ABC:

Lemma 1. *Let X lie on the sideline BC of triangle ABC. Then, with respect to triangle ABC, X has barycentric coordinates* $\big(0, \frac{XC}{a}, \frac{BX}{a}\big)$.

Proof. Since X lies on line BC between B and C, there is a unique scalar t such that $X-B = t(C-B)$. In fact, the length of the directed segment $BX = t\cdot BC = ta$, *i.e.*, $t = \frac{BX}{a}$. Rearranging, $X = 0A + (1-t)B + tC$, in which the coefficients sum to 1. Finally, $1-t = \frac{a-BX}{a} = \frac{XC}{a}$. □

The next theorem reduces the computation of a distance XY to the simpler distances AY, BY, and CY, when X has known barycentric coordinates.

Theorem 2. *Let X have barycentric coordinates (α, β, γ) with respect to triangle ABC. Then for any point Y,*

$$XY^2 = \alpha AY^2 + \beta BY^2 + \gamma CY^2 - (\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2).$$

Proof. Using the well known identity $|V|^2 = V \cdot V$, we compute first that

$$\begin{aligned} XY^2 &= |Y - X|^2 \\ &= |Y - \alpha A - \beta B - \gamma C|^2 \\ &= |\alpha(Y - A) + \beta(Y - B) + \gamma(Y - C)|^2 \\ &= \alpha^2 AY^2 + \beta^2 BY^2 + \gamma^2 CY^2 + 2\alpha\beta(Y - A) \cdot (Y - B) \\ &\quad + 2\alpha\gamma(Y - A) \cdot (Y - C) + 2\beta\gamma(Y - B) \cdot (Y - C). \end{aligned}$$

On the other hand,

$$c^2 = |B - A|^2 = |(Y - A) - (Y - B)|^2 = AY^2 + BY^2 - 2(Y - A) \cdot (Y - B).$$

Thus,

$$2\alpha\beta(Y - A) \cdot (Y - B) = \alpha\beta(AY^2 + BY^2 - c^2).$$

Substituting this and its analogues into the preceding calculation, the total coefficient of AY^2 becomes $\alpha^2 + \alpha\beta + \alpha\gamma = \alpha(\alpha + \beta + \gamma) = \alpha$, for example. The result is the displayed formula. □

3. Distances from N to the vertices

Lemma 3. *The centroid G has barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

Proof. Let G' be the point with barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and we will prove $G = G'$. By Lemma 1, $A' = \frac{1}{2}B + \frac{1}{2}C$. We calculate

$$\frac{1}{3}A + \frac{2}{3}A' = \frac{1}{3}A + \frac{2}{3}\left(\frac{1}{2}B + \frac{1}{2}C\right) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C = G',$$

which implies that G' is on segment AA'. Similarly, G' is on the other two medians of triangle ABC. However the intersection of the medians is G, and so $G = G'$. □

Lemma 4 (Euler Line Theorem)**.** $H - O = 3(G - O)$.

Proof. Let $H' = O + 3(G - O)$ and we will prove $H = H'$. By Lemma 3,

$$H' - O = 3(G - O) = A + B + C - 3O = (A - O) + (B - O) + (C - O).$$

And so,

$$\begin{aligned} (H' - A) \cdot (B - C) &= \{(H' - O) - (A - O)\} \cdot \{(B - O) - (C - O)\} \\ &= \{(B - O) + (C - O)\} \cdot \{(B - O) - (C - O)\} \\ &= (B - O) \cdot (B - O) - (C - O) \cdot (C - O) \\ &= |OB|^2 - |OC|^2 \\ &= 0, \end{aligned}$$

which implies H' is on the altitude from A to BC. Similarly, H' is on the other two altitudes of triangle ABC. Since H is defined to be the intersection of the altitudes, it follows that $H = H'$. □

Lemma 5. $(A - O) \cdot (B - O) = R^2 - \frac{1}{2}c^2$.

Proof. One has

$$\begin{aligned} c^2 &= |A - B|^2 \\ &= |(A - O) - (B - O)|^2 \\ &= OA^2 + OB^2 - 2\,(A - O) \cdot (B - O) \\ &= 2R^2 - 2\,(A - O) \cdot (B - O). \end{aligned}$$ □

We now find AN, BN, CN, which are needed in Theorem 2 for $Y = N$.

Theorem 6. $4AN^2 = R^2 - a^2 + b^2 + c^2$.

Proof. Since N is the midpoint of OH, we have $H - O = 2(N - O)$. Combining this observation with Theorem 2, and using Lemma 5, we obtain

$$\begin{aligned} 4AN^2 =&\ |2(A - O) - 2(N - O)|^2 \\ =&\ |(A - O) - (B - O) - (C - O)|^2 \\ =&\ AO^2 + BO^2 + CO^2 \\ &\ - 2(A - O) \cdot (B - O) - 2(A - O) \cdot (C - O) + 2(B - O) \cdot (C - O) \\ =&\ 3R^2 - 2\left(R^2 - \frac{1}{2}c^2\right) - 2\left(R^2 - \frac{1}{2}b^2\right) + 2\left(R^2 - \frac{1}{2}a^2\right) \\ =&\ R^2 - a^2 + b^2 + c^2. \end{aligned}$$ □

4. Proof of Feuerbach's Theorem

Theorem 7. *The incenter I has barycentric coordinates $\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$.*

Proof. Let I' be the point with barycentric coordinates $(a/2s, b/2s, c/2s)$, and we will prove $I = I'$. Let F be the foot of the bisector of angle A on side BC. Applying the law of sines to triangles ABF and ACF, and using $\sin(\pi - x) = \sin x$, we find that

$$\frac{BF}{c} = \frac{\sin BAF}{\sin BFA} = \frac{\sin CAF}{\sin CFA} = \frac{FC}{b}.$$

The equations $b \cdot BF = c \cdot FC$ and $BF + FC = a$ jointly imply that $BF = \frac{ac}{b+c}$. By Lemma 1, $F = (1 - t)B + tC$, where $t = \frac{BF}{a} = \frac{c}{b+c}$. Now,

$$\begin{aligned} \frac{b+c}{2s} \cdot F + \frac{a}{2s} \cdot A =&\ \frac{b+c}{2s}\left(\frac{b}{b+c} \cdot B + \frac{c}{b+c} \cdot C\right) + \frac{a}{2s} \cdot A \\ =&\ \frac{a}{2s} \cdot A + \frac{b}{2s} \cdot B + \frac{c}{2s} \cdot C \\ =&\ I', \end{aligned}$$

which implies that I' is on the angle bisector of angle A. Similarly, I' is on the other two angle bisectors of triangle ABC. Since I is the intersection of the angle bisectors, this implies $I = I'$. □

Theorem 8 (Euler). $OI^2 = R^2 - 2Rr$.

Proof. We use $X = I$ and $Y = O$ in Theorem 2 to obtain

$$\begin{aligned}
OI^2 &= \frac{a}{2s}R^2 + \frac{b}{2s}R^2 + \frac{c}{2s}R^2 - \left(\frac{bc}{(2s)^2}a^2 + \frac{ca}{(2s)^2}b^2 + \frac{ab}{(2s)^2}c^2\right) \\
&= R^2 - \frac{a^2bc + b^2ac + c^2ab}{(2s)^2} \\
&= R^2 - \frac{abc}{2s} \\
&= R^2 - \left(\frac{abc}{2\Delta}\right)\left(\frac{\Delta}{s}\right) \\
&= R^2 - 2Rr.
\end{aligned}$$

The last step here uses the area formulas of §1—in particular $\Delta = rs = abc/4R$. □

Theorem 9. $IN = \frac{1}{2}R - r$ *and* $I_aN = \frac{1}{2}R + r_a$

Proof. To find the distance IN, we set $X = I$ and $Y = N$ in Theorem 2, with Theorems 6 and 7 supplying the distances AN, BN, CN, and the barycentric coordinates of I. For brevity in our computation, we use *cyclic sums*, in which the displayed term is transformed under the permutations (a, b, c), (b, c, a), and (c, a, b), and the results are summed (thus, symmetric functions of a, b, c may be factored through the summation sign, and $\sum_{\circlearrowleft} a = a + b + c = 2s$). The following computation results:

$$\begin{aligned}
IN^2 &= \sum_{\circlearrowleft}\left(\frac{a}{2s}\right)\frac{R^2 - a^2 + b^2 + c^2}{4} - \sum_{\circlearrowleft}\left(\frac{b}{2s}\cdot\frac{c}{2s}\right)a^2 \\
&= \frac{R^2}{8s}\left(\sum_{\circlearrowleft} a\right) + \frac{1}{8s}\left(\sum_{\circlearrowleft}(-a^3 + ab^2 + ac^2)\right) - \frac{abc}{(2s)^2}\left(\sum_{\circlearrowleft} a\right) \\
&= \frac{R^2}{4} + \frac{(-a+b+c)(a-b+c)(a+b-c) + 2abc}{8s} - \frac{abc}{2s} \\
&= \frac{R^2}{4} + \frac{(2s-2a)(2s-2b)(2s-2c)}{8s} - \frac{abc}{4s} \\
&= \frac{R^2}{4} + \frac{(\Delta^2/s)}{s} - \frac{4R\Delta}{4s} \\
&= \left(\frac{1}{2}R\right)^2 + r^2 - Rr \\
&= \left(\frac{1}{2}R - r\right)^2.
\end{aligned}$$

The two penultimate steps again use the area formulas of §1. Theorem 8 tells us that $OI^2 = 2R(\frac{1}{2}R - r)$, and so $\frac{1}{2}R - r$ is nonnegative. Thus we conclude $IN = \frac{1}{2}R - r$. A similar calculation applies to the A-excircle, with two modifications: (i) I_a has barycentric coordinates

$$\left(\frac{-a}{2(s-a)}, \frac{b}{2(s-a)}, \frac{c}{2(s-a)}\right),$$

and (ii) in lieu of $\Delta = rs$, one uses $\Delta = r_a(s-a)$. The result is $I_aN = \frac{1}{2}R + r_a$. □

We are now in a position to prove Feuerbach's Theorem.

Theorem 10 (Feuerbach, 1822)**.** *In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.*

Proof. Suppose first that the nine-point circle and the incircle are nonconcentric. Two nonconcentric circles are internally tangent if and only if the distance between their centers is equal to the positive difference of their radii. Since the nine-point circle is the circumcircle of the medial triangle $A'B'C'$, its radius is $\frac{1}{2}R$. Thus the positive difference between the radii of the nine-point circle and the incircle is $\frac{1}{2}R - r$ which is IN by Theorem 9. This implies that the nine-point circle and the incircle are internally tangent. Also, since the sum of the radii of the A-excircle and the nine-point circle is I_aN, by Theorem 9, the nine-point circle is externally tangent to the A-excircle. Suppose now that the nine-point circle and the incircle are concentric, that is $I = N$. Then $0 = IN = \frac{1}{2}R - r = OI^2/2R$ and so $I = O$. This clearly implies triangle ABC is equilateral. □

For historical details, see [3] and [4].

References

[1] H. S. M. Coxeter *Introduction to Geometry*, 2nd edition, John Wiley & Sons, Hoboken, N.J., 1969.

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, D.C., 1967.

[3] K. W. Feuerbach, *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks und mehrerer durch sie bestimmten Linien und Figuren: Eine analytisch-trigonometrische Abhandlung*, Riegel & Wiesner, Nürnberg, 1822.

[4] J. S. MacKay, History of the nine point circle, *Proc. Edinb. Math. Soc.*, 11 (1892) 19–61.

Michael J. G. Scheer: 210 West 101 Street Apt 2J, New York, NY 10025, USA
E-mail address: mgscheer@gmail.com