A Simple Vector Proof of Feuerbach’s Theorem

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Abstract. The celebrated theorem of Feuerbach states that the nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles. In this note, we give a simple proof of Feuerbach’s Theorem using straightforward vector computations. All required preliminaries are proven here for the sake of completeness.

1. Notation and background

Let $ABC$ be a nonequilateral triangle. We denote its side-lengths by $a, b, c$, its semiperimeter by $s = \frac{1}{2}(a + b + c)$, and its area by $\Delta$. Its classical centers are the circumcenter $O$, the incenter $I$, the centroid $G$, and the orthocenter $H$ (Figure 1). The nine-point center $N$ is the midpoint of $OH$ and the center of the nine-point circle, which passes through the side-midpoints $A', B', C'$ and the feet of the three altitudes. The Euler Line Theorem states that $G$ lies on $OH$ with $OG : GH = 1 : 2$.

![Figure 1. The classical centers and the Euler division $OG : GH = 1 : 2.$](image)

We write $I_a, I_b, I_c$ for the excenters opposite $A, B, C$, respectively; these are points where one internal angle bisector meets two external angle bisectors. Like $I$, the points $I_a, I_b, I_c$ are equidistant from the lines $AB, BC, CA$, and thus are centers of three circles each tangent to the three lines. These are the excircles. The classical radii are the circumradius $R (= OA = OB = OC)$, the inradius $r$, and
the exradii \( r_a, r_b, r_c \). The following area formulas are well known (see, e.g., [1] and [2]):

\[
\Delta = \frac{abc}{4R} = rs = r_a(s - a) = \sqrt{s(s - a)(s - b)(s - c)}.
\]

Feuerbach’s Theorem states that the nine-point circle is tangent internally to the incircle, and externally to each of the excircles [3]. Two of the four points of tangency are shown in Figure 2.

2. Vector formalism

We view the plane as \( \mathbb{R}^2 \) with its standard vector space structure. Given triangle \( ABC \), the vectors \( A - C \) and \( B - C \) are linearly independent. Thus for any point \( X \), we may write \( X - C = \alpha(A - C) + \beta(B - C) \) for unique \( \alpha, \beta \in \mathbb{R} \). Defining \( \gamma = 1 - \alpha - \beta \), we find that

\[
X = \alpha A + \beta B + \gamma C, \quad \alpha + \beta + \gamma = 1.
\]

This expression for \( X \) is unique. One says that \( X \) has \textit{barycentric coordinates} \((\alpha, \beta, \gamma)\) with respect to triangle \( ABC \) (see, e.g., [1]). The barycentric coordinates are particularly simple when \( X \) lies on a side of triangle \( ABC \):

**Lemma 1.** Let \( X \) lie on the sideline \( BC \) of triangle \( ABC \). Then, with respect to triangle \( ABC \), \( X \) has barycentric coordinates \( \left(0, \frac{XC}{a}, \frac{BX}{a}\right) \).

**Proof.** Since \( X \) lies on line \( BC \) between \( B \) and \( C \), there is a unique scalar \( t \) such that \( X - B = t(C - B) \). In fact, the length of the directed segment \( BX = t \cdot BC = ta \), i.e., \( t = \frac{BX}{a} \). Rearranging, \( X = 0A + (1 - t)B + tC \), in which the coefficients sum to 1. Finally, \( 1 - t = \frac{a - BX}{a} = \frac{XC}{a} \). \( \square \)

The next theorem reduces the computation of a distance \( XY \) to the simpler distances \( AY, BY, \) and \( CY \), when \( X \) has known barycentric coordinates.
Theorem 2. Let $X$ have barycentric coordinates $(\alpha, \beta, \gamma)$ with respect to triangle $ABC$. Then for any point $Y$,
\[ XY^2 = \alpha AY^2 + \beta BY^2 + \gamma CY^2 - (\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2). \]

Proof. Using the well known identity $|V|^2 = V \cdot V$, we compute first that
\[
XY^2 = |Y - X|^2 \\
= |Y - \alpha A - \beta B - \gamma C|^2 \\
= |\alpha(Y - A) + \beta(Y - B) + \gamma(Y - C)|^2 \\
= \alpha^2 AY^2 + \beta^2 BY^2 + \gamma^2 CY^2 + 2\alpha\beta(Y - A) \cdot (Y - B) \\
+ 2\alpha\gamma(Y - A) \cdot (Y - C) + 2\beta\gamma(Y - B) \cdot (Y - C).
\]

On the other hand,
\[
c^2 = |B - A|^2 = |(Y - A) - (Y - B)|^2 = AY^2 + BY^2 - 2(Y - A) \cdot (Y - B).
\]
Thus,
\[
2\alpha\beta(Y - A) \cdot (Y - B) = \alpha\beta(AY^2 + BY^2 - c^2).
\]
Substituting this and its analogues into the preceding calculation, the total coefficient of $AY^2$ becomes $\alpha^2 + \alpha\beta + \alpha\gamma = \alpha(\alpha + \beta + \gamma) = \alpha$, for example. The result is the displayed formula. \hfill \Box

3. Distances from $N$ to the vertices

Lemma 3. The centroid $G$ has barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof. Let $G'$ be the point with barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and we will prove $G = G'$. By Lemma 1, $A' = \frac{1}{2}B + \frac{1}{2}C$. We calculate
\[
\frac{1}{3}A + \frac{2}{3}A' = \frac{1}{3}A + \frac{2}{3}\left(\frac{1}{2}B + \frac{1}{2}C\right) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C = G',
\]
which implies that $G'$ is on segment $AA'$. Similarly, $G'$ is on the other two medians of triangle $ABC$. However the intersection of the medians is $G$, and so $G = G'$. \hfill \Box

Lemma 4 (Euler Line Theorem). $H - O = 3(G - O)$.

Proof. Let $H' = O + 3(G - O)$ and we will prove $H = H'$. By Lemma 3,
\[
H' - O = 3(G - O) = A + B + C - 3O = (A - O) + (B - O) + (C - O).
\]
And so,
\[
(H' - A) \cdot (B - C) = ((H' - O) - (A - O)) \cdot ((B - O) - (C - O)) \\
= ((B - O) + (C - O)) \cdot ((B - O) - (C - O)) \\
= (B - O) \cdot (B - O) - (C - O) \cdot (C - O) \\
= |OB|^2 - |OC|^2 \\
= 0,
\]
which implies $H'$ is on the altitude from $A$ to $BC$. Similarly, $H'$ is on the other two altitudes of triangle $ABC$. Since $H$ is defined to be the intersection of the altitudes, it follows that $H = H'$.

**Lemma 5.** $(A - O) \cdot (B - O) = R^2 - \frac{1}{2}c^2$.

*Proof.* One has
\[
c^2 = |A - B|^2 \\
= |(A - O) - (B - O)|^2 \\
= OA^2 + OB^2 - 2(A - O) \cdot (B - O) \\
= 2R^2 - 2(A - O) \cdot (B - O).
\]

We now find $AN, BN, CN$, which are needed in Theorem 2 for $Y = N$.

**Theorem 6.** $4AN^2 = R^2 - a^2 + b^2 + c^2$.

*Proof.* Since $N$ is the midpoint of $OH$, we have $H - O = 2(N - O)$. Combining this observation with Theorem 2, and using Lemma 5, we obtain
\[
4AN^2 = |2(A - O) - 2(N - O)|^2 \\
= |(A - O) - (B - O) - (C - O)|^2 \\
= AO^2 + BO^2 + CO^2 \\
- 2(A - O) \cdot (B - O) - 2(A - O) \cdot (C - O) + 2(B - O) \cdot (C - O) \\
= 3R^2 - 2\left(R^2 - \frac{1}{2}c^2\right) - 2\left(R^2 - \frac{1}{2}b^2\right) + 2\left(R^2 - \frac{1}{2}a^2\right) \\
= R^2 - a^2 + b^2 + c^2.
\]

**4. Proof of Feuerbach’s Theorem**

**Theorem 7.** The incenter $I$ has barycentric coordinates $(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s})$.

*Proof.* Let $I'$ be the point with barycentric coordinates $(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s})$, and we will prove $I = I'$. Let $F$ be the foot of the bisector of angle $A$ on side $BC$. Applying the law of sines to triangles $ABF$ and $ACF$, and using $\sin(\pi - x) = \sin x$, we find that
\[
\frac{BF}{c} = \frac{\sin BAF}{\sin BFA} = \frac{\sin CAF}{\sin CFA} = \frac{FC}{b}.
\]
The equations $b \cdot BF = c \cdot FC$ and $BF + FC = a$ jointly imply that $BF = \frac{ac}{a + c}$. By Lemma 1, $F = (1 - t)B + tC$, where $t = \frac{BF}{a} = \frac{c}{b + c}$. Now,
\[
\frac{b + c}{2s} \cdot F + \frac{a}{2s} \cdot A = \frac{b + c}{2s} \left(\frac{b}{b + c} \cdot B + \frac{c}{b + c} \cdot C\right) + \frac{a}{2s} \cdot A \\
= \frac{a}{2s} \cdot A + \frac{b}{2s} \cdot B + \frac{c}{2s} \cdot C \\
= I'.
\]
which implies that $I'$ is on the angle bisector of angle $A$. Similarly, $I'$ is on the other two angle bisectors of triangle $ABC$. Since $I$ is the intersection of the angle bisectors, this implies $I = I'$.

**Theorem 8** (Euler). $OI^2 = R^2 - 2Rr$.

*Proof.* We use $X = I$ and $Y = O$ in Theorem 2 to obtain

$$OI^2 = \frac{a}{2s}R^2 + \frac{b}{2s}R^2 + \frac{c}{2s}R^2 - \left( \frac{bc}{(2s)^2}a^2 + \frac{ca}{(2s)^2}b^2 + \frac{ab}{(2s)^2}c^2 \right)$$

$$= R^2 - \frac{abc}{2s} + b^2ac + c^2ab$$

$$= R^2 - (\frac{abc}{2\Delta})(\frac{\Delta}{s})$$

$$= R^2 - 2Rr.$$  

The last step here uses the area formulas of §1—in particular $\Delta = rs = abc/4R$.  

**Theorem 9.** $IN = \frac{1}{2}R - r$ and $IaN = \frac{1}{2}R + r_a$

*Proof.* To find the distance $IN$, we set $X = I$ and $Y = N$ in Theorem 2, with Theorems 6 and 7 supplying the distances $AN$, $BN$, $CN$, and the barycentric coordinates of $I$. For brevity in our computation, we use cyclic sums, in which the displayed term is transformed under the permutations $(a, b, c)$, $(b, c, a)$, and $(c, a, b)$, and the results are summed (thus, symmetric functions of $a, b, c$ may be factored through the summation sign, and $\sum_a a = a + b + c = 2s$). The following computation results:

$$IN^2 = \sum_a \left( \frac{a}{2s} \right) R^2 - a^2 + b^2 + c^2 - \sum \left( \frac{b}{2s} \cdot \frac{c}{2s} \right) a^2$$

$$= \frac{R^2}{8s} \left( \sum_a a \right) + \frac{1}{8s} \left( \sum a^3 + ab^2 + ac^2 \right) - \frac{abc}{(2s)^2} \left( \sum a \right)$$

$$= \frac{R^2}{4} + \frac{(-a + b + c)(a - b + c)(a + b - c) + 2abc}{8s} - \frac{abc}{2s}$$

$$= \frac{R^2}{4} + \frac{(2s - 2a)(2s - 2b)(2s - 2c)}{8s} \frac{abc}{4s}$$

$$= \frac{R^2}{4} + \frac{(\Delta^2/s)}{s} - \frac{4R\Delta}{4s}$$

$$= \left( \frac{1}{2}R \right)^2 + r^2 - Rr$$

$$= \left( \frac{1}{2}R - r \right)^2.$$
The two penultimate steps again use the area formulas of §1. Theorem 8 tells us that $OJ^2 = 2R(\frac{1}{2}R - r)$, and so $\frac{1}{2}R - r$ is nonnegative. Thus we conclude $IN = \frac{1}{2}R - r$. A similar calculation applies to the $A$-excircle, with two modifications: (i) $I_a$ has barycentric coordinates

$$\left(\frac{-a}{2(s-a)}, \frac{b}{2(s-a)}, \frac{c}{2(s-a)}\right),$$

and (ii) in lieu of $\Delta = rs$, one uses $\Delta = r_a(s-a)$. The result is $I_aN = \frac{1}{2}R + r_a$.  

We are now in a position to prove Feuerbach’s Theorem.

**Theorem 10** (Feuerbach, 1822). In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.

**Proof.** Suppose first that the nine-point circle and the incircle are nonconcentric. Two nonconcentric circles are internally tangent if and only if the distance between their centers is equal to the positive difference of their radii. Since the nine-point circle is the circumcircle of the medial triangle $A'B'C'$, its radius is $\frac{1}{2}R$. Thus the positive difference between the radii of the nine-point circle and the incircle is $\frac{1}{2}R - r$ which is $IN$ by Theorem 9. This implies that the nine-point circle and the incircle are internally tangent. Also, since the sum of the radii of the $A$-excircle and the nine-point circle is $I_aN$, by Theorem 9, the nine-point circle is externally tangent to the $A$-excircle. Suppose now that the nine-point circle and the incircle are concentric, that is $I = N$. Then $0 = IN = \frac{1}{2}R - r = OJ^2/2R$ and so $I = O$. This clearly implies triangle $ABC$ is equilateral.  

For historical details, see [3] and [4].

**References**


