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# A New Formula Concerning the Diagonals and Sides of a Quadrilateral

Larry Hoehn

**Abstract**. We derive a formula relating the sides and diagonal sections of a general convex quadrilateral along with the special case of a quadrilateral with an incircle.

Let $ABCD$ be a convex quadrilateral whose diagonals intersect at $E$. We use the notation $a = AB$, $b = BC$, $c = CD$, $d = DA$, $e = AE$, $f = BE$, $g = CE$, and $h = DE$ as seen in the figure below.

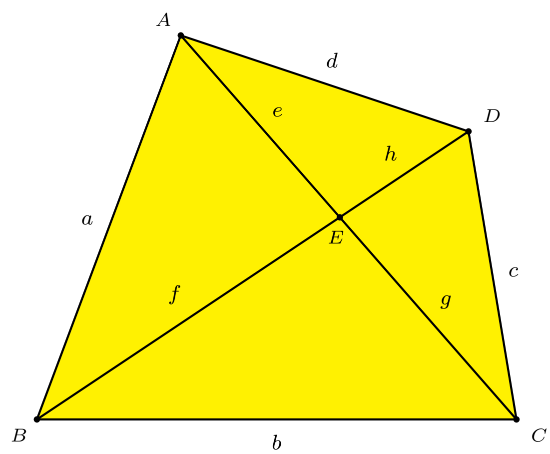


Figure 1. Convex quadrilateral

By the law of cosines in triangles $ABE$ and $CDE$ we obtain

$$\frac{e^2+f^2-a^2}{2ef} = \cos AEB = \cos CED = \frac{g^2+h^2-c^2}{2gh}.$$

This can be rewritten as

$$gh(e^2+f^2-a^2) = ef(g^2+h^2-c^2),$$

rewritten further as

$$fh(fg-eh) - eg(fg-eh) = a^2gh - c^2ef,$$

and still further as

$$fh - eg = \frac{a^2gh - c^2ef}{fg-eh}.$$

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In the same manner for triangles $BCE$ and $DAE$ we obtain

$$fh - eg = \frac{b^2eh - d^2fg}{ef - gh}.$$

By setting the right sides of these two equations equal to each other and some computation, we obtain

$$(ef - gh)(a^2gh - c^2ef) = (fg - eh)(b^2eh - d^2fg),$$

which can be rewritten as

$$efgh(a^2 + c^2 - b^2 - d^2) = a^2g^2h^2 + c^2e^2f^2 - b^2e^2h^2 - d^2f^2g^2.$$

By adding the same quantity to both sides we obtain

$$\begin{aligned} &efgh(a^2 + c^2 - b^2 - d^2) + efgh(2ac - 2bd) \\ &= a^2g^2h^2 + c^2e^2f^2 - b^2e^2h^2 - d^2f^2g^2 + 2aghcef - 2behdfg, \end{aligned}$$

which can be factored into

$$efgh((a + c)^2 - (b + d)^2) = (agh + cef)^2 - (beh + dfg)^2,$$

or

$$efgh(a+c+b+d)(a+c-b-d) = (agh+cef+beh+dfg)(agh+cef-beh-dfg).$$

This is a formula for an arbitrary quadrilateral. If the quadrilateral has an incircle, then the sums of the opposite sides are equal with $a + c = b + d$. By making this substitution in the last equation above we get the result that

$$agh + cef = beh + dfg.$$

This is a nice companion result to Ptolemy's theorem for a quadrilateral inscribed in a circle. In the notation of this paper Ptolemy's theorem states that

$$(e + g)(f + h) = ac + bd.$$

Larry Hoehn: Austin Peay State University, Clarksville, Tennessee 37044, USA
*E-mail address*: hoehnl@apsu.edu