Forum Geometricorum
Volume 11 (2011) 217–221.

FORUM GEOM
ISSN 1534-1178

Collinearity of the First Trisection Points of Cevian Segments

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Abstract. We show that the first trisection points of the cevian segments of a finite point P are collinear if and only if P lies on the Steiner circum-ellipse. Some further results are obtained concerning the line containing these first trisection points.

This note answers a question of Paul Yiu [3]: *In the plane of a given triangle ABC, what is the locus of P for which the "first" trisection points of the three cevian segments (the ones nearer to the vertices) are collinear* ? We identify this locus and establish some further results.

Let $P = (u : v : w)$ be in homogeneous barycentric coordinates with respect to triangle ABC. Its cevian triangle $A'B'C'$ have vertices

$$A' = (0 : v : w), \qquad B = (u : 0 : w), \qquad C' = (u : v : 0).$$

The first trisection points of the cevian segments AA', BB', CC' are the points dividing these segments in the ratio $1 : 2$. These are the points

$$X = (2(v+w) : v : w), \qquad Y = (u : 2(w+u) : w), \qquad Z = (u : v : 2(u+v)).$$

These three points are collinear if and only if

$$0 = \begin{vmatrix} 2(v+w) & v & w \\ u & 2(w+u) & w \\ u & v & 2(u+v) \end{vmatrix} = 6(u+v+w)(uv+vw+wu).$$

It follows that, for a finite point P, the first trisection points are collinear if and only if P lies on the Steiner circum-ellipse

$$uv + vw + wu = 0.$$

Proposition 1. *If $P = (u : v : w)$ lies on the Steiner circum-ellipse, the line $\mathcal{L}_P$ containing the three "first" trisection points of the three cevian segments AA', BB', CC' has equation*

$$\left(\frac{1}{v} - \frac{1}{w}\right) x + \left(\frac{1}{w} - \frac{1}{u}\right) y + \left(\frac{1}{u} - \frac{1}{v}\right) z = 0.$$

Publication Date: November 10, 2011. Communicating Editor: Paul Yiu.

Proof. The line $\mathcal{L}_P$ contains the point X since

$$\begin{aligned}&\left(\frac{1}{v}-\frac{1}{w}\right)\cdot 2(v+w)+\left(\frac{1}{w}-\frac{1}{u}\right)v+\left(\frac{1}{u}-\frac{1}{v}\right)w\\ =&-\frac{(v-w)(uv+vw+wu)}{uvw}\\ =&\,0;\end{aligned}$$

similarly for Y and Z. $\square$

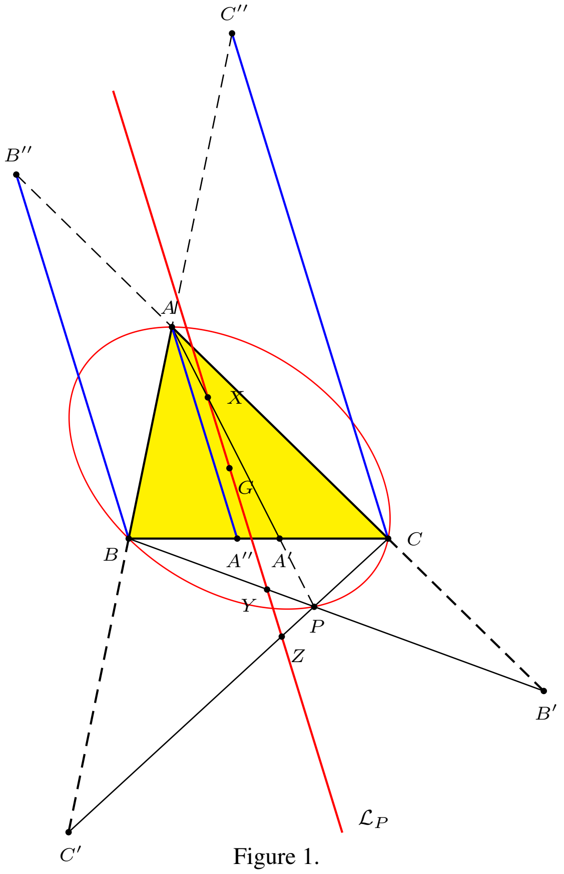

Figure 1.

Let P be a point on the Steiner circum-ellipse, with cevian triangle $A'B'C'$. Construct points A'', B'', C'' on the lines BC, CA, AB respectively such that

$$BA' = A''C, \qquad CB' = B''A, \qquad AC' = C''B.$$

The lines AA'', BB'', CC'' are parallel, and their common infinite point is the isotomic conjugate of P. It is clear from Proposition 1 that the line $\mathcal{L}_P$ contains the isotomic conjugate of P:

$$\left(\frac{1}{v}-\frac{1}{w}\right)\cdot\frac{1}{u}+\left(\frac{1}{w}-\frac{1}{u}\right)\cdot\frac{1}{v}+\left(\frac{1}{u}-\frac{1}{v}\right)\cdot\frac{1}{w}=0.$$

It follows that $\mathcal{L}_P$ is parallel to AA'', BB'', and CC''; see Figure 1.

Corollary 2. *Given a line ℓ containing the centroid G, there is a unique point P on the Steiner circum-ellipse such that $\mathcal{L}_P = \ell$, namely, P is the isotomic conjugate of the infinite point of the line ℓ.*

Lemma 3. *Let $\ell : px+qy+rz = 0$ be a line through the centroid G. The conjugate diameter in the Steiner circum-ellipse is the line $(q-r)x+(r-p)y+(p-q)z = 0$.*

Proof. The parallel of ℓ through A intersects the ellipse again at the point

$$N = \left(\frac{1}{q-r} : \frac{1}{p-q} : \frac{1}{r-p}\right).$$

The midpoint of AN is the point

$$(2(r-p)(p-q) - (q-r)^2 : (q-r)(r-p) : (q-r)(p-q)).$$

This point lies on the line

$$(q-r)x + (r-p)y + (p-q)z = 0,$$

which also contains the centroid. This line is therefore the conjugate diameter of ℓ. $\square$

Corollary 4. *Let $\ell : px + qy + rz = 0$ be a line through the centroid G, so that $P_\ell := \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ is a point on the Steiner circum-ellipse. The conjugate diameter of ℓ in the ellipse is the line $\mathcal{L}_{P_\ell}$.*

Proposition 5. *Let P and Q be points on the Steiner circum-ellipse. The lines $\mathcal{L}_P$ and $\mathcal{L}_Q$ are conjugate diameters if and only if P and Q are antipodal.*

Proof. Suppose $\mathcal{L}_P$ has equation $px+qy+rz = 0$ with $p+q+r = 0$. The line $\mathcal{L}_Q$ is the conjugate diameter of $\mathcal{L}_P$ if and only if its has equation $(q-r)x+(r-p)y+(p-q)z = 0$. Thus, P and Q are the points $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ and $\left(\frac{1}{q-r} : \frac{1}{r-p} : \frac{1}{p-q}\right)$. These are antipodal points since the line joining them, namely,

$$p(q-r)x + q(r-p)y + r(p-q)z = 0,$$

contains the centroid G. $\square$

It is easy to determine the line through G whose intersections with the Steiner circum-ellipse are two points P and Q such that $\mathcal{L}_P : px + qy + rz = 0$ and $\mathcal{L}_Q : (q-r)x + (r-p)y + (p-q)z = 0$ are the axes of the ellipse. This is the case if the two conjugate axes are perpendicular. Now, the infinite points of the lines $\mathcal{L}_P$ and $\mathcal{L}_Q$ are respectively $q-r : r-p : p-q$ and $p : q : r$. They are perpendicular, according to [4, §4.5, Theorem] if and only if

$$(b^2+c^2-a^2)p(q-r) + (b^2+c^2-a^2)q(r-p) + (a^2+b^2-c^2)r(p-q) = 0.$$

Equivalently,

$$(b^2+c^2-a^2)\left(\frac{1}{r}-\frac{1}{q}\right)+(c^2+a^2-b^2)\left(\frac{1}{p}-\frac{1}{r}\right)+(a^2+b^2-c^2)\left(\frac{1}{q}-\frac{1}{p}\right) = 0,$$

or

$$\frac{b^2-c^2}{p} + \frac{c^2-a^2}{q} + \frac{a^2-b^2}{r} = 0.$$

Therefore, $P = \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ is an intersection of the Steiner circum-ellipse and the line

$$(b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z = 0,$$

which contains the centroid G and the symmedian point $K = (a^2 : b^2 : c^2)$.

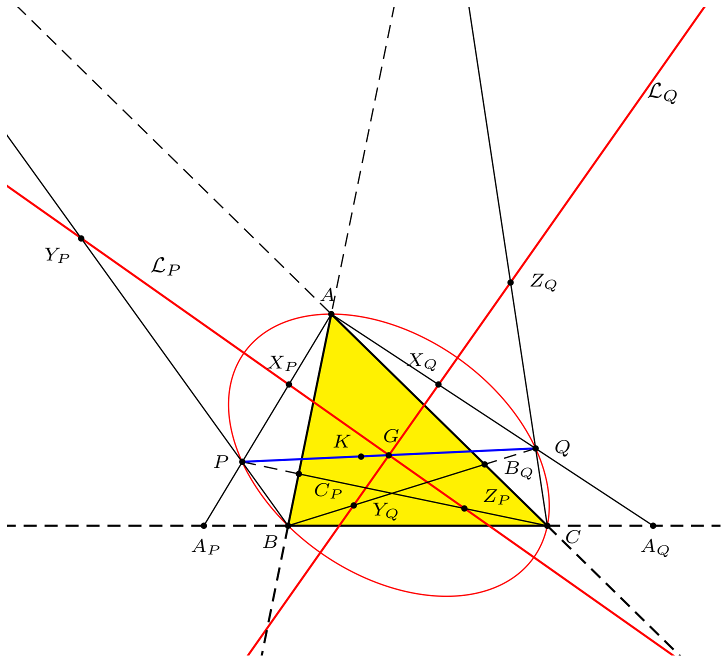

Figure 2.

A point on the line GK has homogeneous barycentric coordinates $(a^2 + t : b^2 + t : c^2 + t)$ for some t. If this point lies on the Steiner circum-ellipse, then

$$(b^2 + t)(c^2 + t) + (c^2 + t)(a^2 + t) + (a^2 + t)(b^2 + t) = 0.$$

From this,

$$t = -\frac{1}{3}(a^2 + b^2 + c^2 \pm \sqrt{D}),$$

where

$$D = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$

We obtain, with $\varepsilon = \pm 1$, the two points P and Q

$$(b^2 + c^2 - 2a^2 + \varepsilon\sqrt{D} :\ c^2 + a^2 - 2b^2 + \varepsilon\sqrt{D} :\ a^2 + b^2 - 2c^2 + \varepsilon\sqrt{D})$$

for which $\mathcal{L}_P$ and $\mathcal{L}_Q$ are the axes of the Steiner circum-ellipse. Their isotomic conjugates are the points X_{3413} and X_{3414} in [2], which are the infinite points of the axes of the ellipse.

We conclude the present note with two remarks.

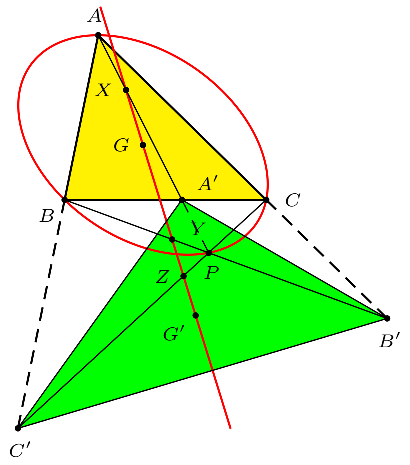

Figure 3.

Remarks. (1) The line $\mathcal{L}_P$ contains the centroids of triangle ABC and the cevian triangle $A'B'C'$ of P.

Proof. Clearly $\mathcal{L}_P$ contains the centroid $G = (1 : 1 : 1)$. Since X, Y, Z are on the line $\mathcal{L}_P$, so is the centroid $G' = \frac{1}{3}(X + Y + Z)$ of the degenerate triangle XYZ. Since

$$X = \frac{2A + A'}{3}, \qquad Y = \frac{2B + B'}{3}, \qquad Z = \frac{2C + C'}{3},$$

it follows that $\mathcal{L}_P$ also contains the point $\frac{1}{3}(A' + B' + C')$, the centroid of the cevian triangle of P.

(2) More generally, if we consider X, Y, Z dividing the cevian segments AA', BB', CC' in the ratio $m : n$, these points are collinear if and only if

$$n(n - m)(u + v + w)(uv + vw + wu) + (2m - n)(m + n)uvw = 0.$$

In particular, for the "second" trisection points, we take $m = 2,\ n = 1$ and obtain for the locus of P the cubic

$$u(v^2 + w^2) + v(w^2 + u^2) + w(u^2 + v^2) - 6uvw = 0,$$

which is the Tucker nodal cubic K015 in Gibert's catalogue of cubic curves [1].

References

[1] B. Gibert, *Cubics in the Triangle Plane*, available from `http://bernard.gibert.pagesperso-orange.fr`.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[3] P. Yiu, Hyacinthos message 2273, January 1, 2001.

[4] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

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