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The Distance from the Incenter to the Euler Line

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Abstract. It is well known that the incenter of a triangle lies on the Euler line if and only if the triangle is isosceles. A natural question to ask is how far the incenter can be from the Euler line. We find least upper bounds, across all triangles, for that distance relative to several scales. Those bounds are found relative to the semi-perimeter of the triangle, the length of the Euler line and the circumradius, as well as the length of the longest side and the length of the longest median.

1. Introduction

A quiet thread of interest in the relationship of the incenter to the Euler line has persisted to this day. Given a triangle, the Euler line joins the circumcenter, O, to the orthocenter, H. The centroid, G, trisects this line (being closer to O) and the center of the nine-point circle, N, bisects it. It is known that the incenter, I, of a triangle lies on the Euler line if and only if the triangle is isosceles (although proofs of this fact are thin on the ground). But you can't just choose any point, on or off the Euler line, to be the incenter of a triangle. The points you can choose are known, as will be seen. An obvious question asks how far can the incenter be from the Euler line. For isosceles triangles the distance is 0. Clearly this question can only be answered relative to some scale, we will consider three scales: the length of the Euler line, $\mathcal{E}$, the circumradius, R, and the semiperimeter, s. Along the way we will see that the answer for the semiperimeter also gives us the answer relative to the longest side, μ, and the longest median, ν. To complete the list of lengths, let d be the distance of the incenter from the Euler line.

Time spent playing with triangles using any reasonable computer geometry package will convince you that the following are reasonable conjectures.

$$\frac{d}{\mathcal{E}} \leq \frac{1}{3}, \qquad \frac{d}{R} \leq \frac{1}{2} \qquad \text{and} \qquad \frac{d}{s} \leq \frac{1}{3}$$

Maybe with strict inequalities, but then again the limits might be attained.

A large collection of relationships between the centers of a triangle is known, for example, if R is the radius of the circumcircle and r the radius of the incircle,

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then we have

$$\begin{aligned} OI^2 &= R(R-2r) \\ IN &= \frac{1}{2}(R-2r) \end{aligned}$$

Before moving on, it is worth noting that the second of the above gives an immediate upper bound for the distance relative to the circumradius. As the inradius of a non-degenerate triangle must be positive we have $d \leq IN = \frac{R}{2} - r < \frac{R}{2}$, and hence

$$\frac{d}{R} < \frac{1}{2}.$$

2. Relative to the Euler line

The relationships given above, and others, can be used to show that for any triangle the incenter, I, must lie within the *orthocentroidal circle* punctured at the center of the nine-point circle, N, namely, the disk with diameter GH except for the circumference and the point N.

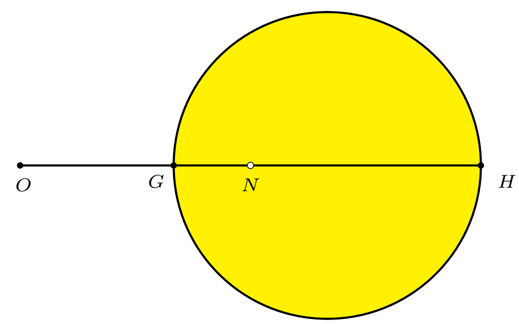

Figure 1

In 1984 Guinand [1] showed that every such point gives rise to a triangle which has the nominated points as its centers. Guinand shows that if $OI = \rho$, $IN = \sigma$ and $OH = \kappa$ then the cosines of the angles of the triangle we seek are the zeros of the following cubic.

$$p(c) = c^3 + \frac{3}{2}\left(\frac{4\sigma^2}{3\rho^2} - 1\right)c^2 + \frac{3}{4}\left(-\frac{2\kappa^2\sigma^2}{3\rho^4} + \frac{8\sigma^4}{3\rho^4} - \frac{4\sigma^2}{\rho^2} + 1\right)c + \frac{1}{8}\left(\frac{4\kappa^2\sigma^2}{\rho^4} - 1\right).$$

Stern [2] approached the problem using complex numbers and provides a simpler derivation of a cubic, and explicitly demonstrates that the triangle found is unique. His approach also provided the vertices directly, as complex numbers.

Consideration of the orthocentroidal circle provides the answer to our question relative to $\mathcal{E}$, the length of the Euler line. The incenter must lie strictly within the orthocentroidal circle which has radius one third the length of the Euler line. Guinand has proved that all such points, except N, lead to a suitable triangle. Thus the least upper bound, over all non-degenerate triangles, of the ratio $\frac{d}{\mathcal{E}}$ is $\frac{1}{3}$, with triangles approaching this upper-bound being defined by having incenters close to the points on circumference of the orthocentroidal circle on a radius perpendicular to the Euler line. For any given non-degenerate triangle we obtain the strict inequality $\frac{d}{\mathcal{E}} < \frac{1}{3}$.

Consideration of Figure 2 gives us more information. Taking $OH = 3$ as our scale. For triangles with I close to the limit point above, the angle IGH is close to $\frac{\pi}{4}$. Moreover, with I near that point, a calculation using the inferred values of $OI \approx \sqrt{5}$ and $IN \approx \frac{\sqrt{5}}{2}$ shows that the circumradius will be close to $\sqrt{5}$, and the inradius will be close to 0.

We observed above that $IN < \frac{R}{2}$. This distance only becomes relevant for us if IN is perpendicular to the Euler line. Consideration of the orthocentroidal circle again allows us to see that this may happen, with the angle IGH being close to $\frac{\pi}{3}$. In this case the circumradius will be close to $\sqrt{3}$.

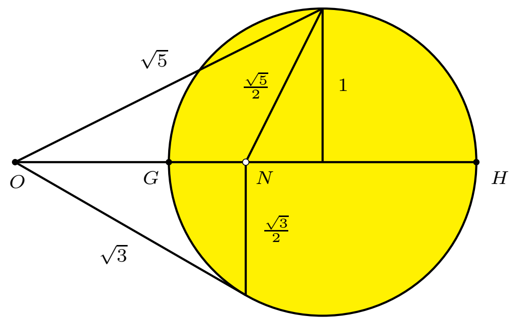

Figure 2

Remark. It is easy to see that the last case also gives the least upper bound of the angle IOH as $\frac{\pi}{6}$.

3. Relative to the triangle

We now wish to find the maximal distance relative to the dimensions of the triangle itself. The relevant dimensions will be the length of the longest median, ν, the length of the longest side, μ, and the semiperimeter, s. It is clear that $\nu < \mu < s$ (see Lemma 4 below).

The following are well-known, and show that the incenter and centroid lie within the medial triangle, the triangle formed by the three midpoints of the sides.

Lemma 1. *The incenter, I, lies in the medial triangle.*

Lemma 2. *The centroid of triangle ABC is the centroid of the medial triangle.*

Lemma 3. *The distance from the incenter to the centroid is less than one third the length of the longest median of the triangle.*

Proof. We have just shown that both the incenter and centroid lie inside the medial triangle. Therefore the distance from the incenter to the centroid is less than the largest distance from the centroid to a vertex of the medial triangle. (Consider the circle centered at O passing through the most distant vertex.)

Now the distance of the centroid from the vertices of the medial triangle is, by definition, the distance from the centroid to the mid-points of the side of triangle ABC. Those distances are equal to one third the lengths of the medians, and the result follows. □

Lemma 4. *The length of a median is less than μ. Hence, $\nu < \mu < s$.*

Proof. Consider the median from A. If we rotate the triangle through π about M_A, the mid-point of the side opposite A, we obtain the parallelogram $ABDC$. The diagonal AD has twice the length AM_A. As A, B and D form a non-degenerate triangle we have

$$2AM_A = AD < AB + BD = AB + AC \leq 2\mu,$$

where μ is the length of the longest side. Thus the median $AM_A < \mu$. This is also true for the other two medians. Thus, $\nu \leq \mu$. □

Proposition 5. *The distance, d, from the incenter to the Euler line satisfies*

$$\frac{d}{s} < \frac{d}{\mu} < \frac{d}{\nu} < \frac{1}{3},$$

where ν is the length of the longest median, μ is the length of the longest side and s is the semi-perimeter of the triangle.

Proof. As the centroid lies on the Euler line, the distance from the incenter to the Euler line is at most the distance from the incenter to the centroid. By Lemma 3, this distance is one third the length of the longest median. But, by Lemma 4, the length of each median is less than $\mu < s$, and the result follows. □

4. In the limit

As the expressions $\frac{d}{\mathcal{E}}$, $\frac{d}{R}$ and $\frac{d}{s}$ are dimensionless we may choose our scale as suits us best. Consider the triangle with vertices $(0,0)$, $(1,0)$ and (ε,δ), where ε and δ are greater than but approximately equal to 0. The following information may be easily checked.

The coordinates of the orthocenter are

$$H\left(\varepsilon, \frac{\varepsilon-\varepsilon^2}{\delta}\right).$$

The coordinates of the circumcenter are

$$O\left(\frac{1}{2}, \frac{\delta^2+\varepsilon^2-\varepsilon}{2\delta}\right).$$

The Euler line has equation

$$l_{OH}: \quad \left(-\delta^2 + 3(1-\varepsilon)\varepsilon\right)x + (1-2\varepsilon)\delta y + \varepsilon\left(\delta^2+\varepsilon^2-1\right) = 0.$$

If we let $p = 2s = \sqrt{\delta^2+\varepsilon^2} + \sqrt{\delta^2+(1-\varepsilon)^2} + 1$, then the coordinates of the incenter are

$$I\left(\frac{\sqrt{\delta^2+\epsilon^2}+\epsilon}{p}, \frac{\delta}{p}\right).$$

We may now write down the value of d, being the perpendicular distance from I to l_{OH}.

$$d = \frac{\left|\left(-\delta^2+3(1-\varepsilon)\varepsilon\right)\left(\sqrt{\delta^2+\varepsilon^2}+\varepsilon\right)+(1-2\varepsilon)\delta^2+p\varepsilon\left(\delta^2+\varepsilon^2-1\right)\right|}{p\sqrt{\left(-\delta^2+3(1-\varepsilon)\varepsilon\right)^2+(1-2\varepsilon)^2\delta^2}}.$$

Suppose we let $\delta = \varepsilon^2$, then the expression for the ratio $\frac{d}{s}$ is

$$\frac{2\varepsilon\left|(-\varepsilon^3-3\varepsilon+3)(\sqrt{\varepsilon^4+\varepsilon^2}+\varepsilon)+\varepsilon(\varepsilon^2-2\varepsilon^3)+p(\varepsilon^4-\varepsilon^2-1)\right|}{p^2\varepsilon\sqrt{(-\varepsilon^3-3(\varepsilon-1))^2+(\varepsilon-2\varepsilon^2)^2}}$$

We cancel the common factor of ε and take the limit as $\varepsilon \to 0$. Noting that $p \to 2$ we see that the numerator approaches 4 while the denominator approaches 12, and we have proved the following.

Theorem 6. *If d is the distance from the incenter to the Euler line, s the semi-perimeter, μ the length of the longest side and ν the length of the longest median, then the least upper bound of $\frac{d}{s}$, and hence $\frac{d}{\mu}$ and $\frac{d}{\nu}$, over all non-degenerate triangles is $\frac{1}{3}$.*

Remark. In those cases where the distance ratio is close to the maximum, the line IG is nearly perpendicular to the Euler line. Thus the angle IGH will be close to $\frac{\pi}{2}$. In these cases the Euler line is extremely large compared to the triangle.

Similar calculations can be carried out for the ratios $\frac{d}{\mathcal{E}}$ and $\frac{d}{R}$. In those cases we take the point (ε, δ) to be a point on the circle through $(0,0)$ and $(1,0)$ with radius $\frac{\sqrt{10}}{6}$, or $\frac{\sqrt{3}}{3}$ respectively (remember that the values of $\sqrt{5}$ and $\sqrt{3}$ met earlier were relative to the length of the Euler line, not the length of a side).

5. Demonstrating the limits

We now have enough information to assist us in constructing diagrams that will demonstrate these limits using a suitable computer geometry package.

Taking the case of triangles with the ratio $\frac{d}{R}$ approaching $\frac{1}{2}$. Let AB be a line segment and define its length to be 1. Let G' be the point on AB one third of the way from A to B. Construct the line $G'T$ such that $\angle BG'T = \frac{\pi}{3}$ and let O be the point where this line meets the perpendicular bisector of AB. Draw the arc AB centered at O and let C be a point on that arc. Constructing the Euler line and incenter of triangle ABC will demonstrate that the ratio $\frac{d}{R}$ approaches $\frac{1}{2}$ as C approaches A. This construction is explained if you note that G' is the limiting position of the centroid, G, as C approaches A (see Figure 3).

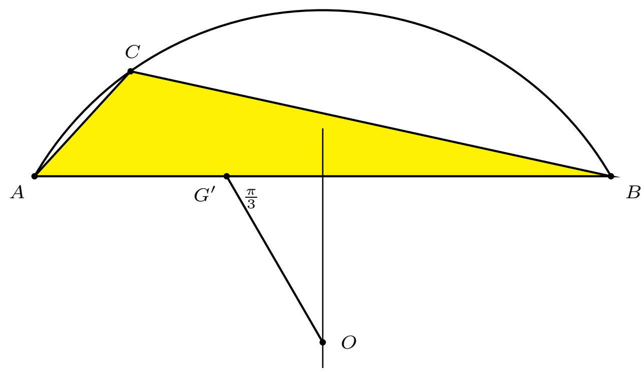

Figure 3.

A similar construction, except with $\angle BG'T = \frac{\pi}{4}$ will give a demonstration that $\frac{d}{\mathcal{E}}$ approaches $\frac{1}{3}$ as C approaches A.

Something different is required to demonstrate that $\frac{d}{s}$ approaches $\frac{1}{3}$. Given AB above, choose a point C' between A and B and let the length $AC' = \varepsilon$, with $0 < \varepsilon < 1$. Construct the perpendicular at C' and find the point C on the perpendicular with $CC' = \varepsilon^2$. Constructing the Euler line and incenter of this triangle will demonstrate that the ratio $\frac{d}{s}$ approaches $\frac{1}{3}$ as C approaches A.

References

[1] A. P. Guinand, Euler lines, tritangent centres, and their triangles, *Amer. Math. Monthly*, 91 (1984) 290–300.

[2] J. Stern, Euler's triangle determination problem, *Forum Geom.*, 7 (2007) 1–9.

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