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Rational Steiner Porism

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Abstract. We establish formulas relating the radii of neighboring circles in a Steiner chain, a chain of mutually tangent circles each tangent to two given ones, one in the interior of the other. From such we parametrize, for $n = 3,\ 4,\ 6$, all configurations of Steiner n-cycles of rational radii.

1. Introduction

Given a circle $O(R)$ and a circle $I(r)$ in its interior, we write the distance d between their centers in the form

$$d^2 = (R-r)^2 - 4qRr. \tag{1}$$

It is well known [3, pp.98–100] that there is a closed chain of n mutually tangent circles each tangent internally to (O) and externally to (I) if and only if $q = \tan^2 \frac{\pi}{n}$. In this case we have a Steiner n-cycle, and such an n-cycle can be constructed beginning with any circle tangent to both (O) and (I). In this note we study the possibilities that the two given circles have *rational* radii and distance between their centers. Such is called a rational Steiner pair. Since $\cos\frac{2\pi}{n} = \frac{1-q}{1+q}$, we must have $\cos\frac{2\pi}{n}$ rational. By a classic theorem in algebraic number theory, $\cos\frac{2}{n}\pi$ is rational only for $n = 3, 4, 6$ (see, for example, [2, p.41, Corollary 3.12]). It follows that rational Steiner pairs exist only for $q = 3,\ 1,\ \frac{1}{3}$, corresponding to $n = 3, 4, 6$. We shall give a parametrization of such pairs and proceed to show how to construct Steiner n-cycles consisting of circles of rational radii. Here are some examples of symmetric rational Steiner n-cycles for these values of n.

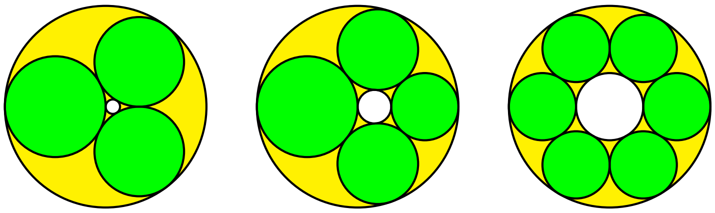

Figure 1. Symmetric Steiner n-cycles for $n = 3,\ 4,\ 6$

q	n	(R, r, d)	Radii $(\rho_1, \dots, \rho_n)$
3	3	$(14, 1, 1)$	$\left(7,\ \frac{56}{9},\ \frac{56}{9}\right)$
1	4	$(6, 1, 1)$	$\left(3,\ \frac{12}{5},\ 2,\ \frac{12}{5}\right)$
$\frac{1}{3}$	6	$(6, 1, 0)$	$(1, 1, 1, 1, 1, 1)$

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2. Construction of Steiner chains

In this section we consider two circles $O(R)$ and $I(r)$ with centers at a distance d apart, without imposing any relation on R, r, d, nor rationality assumption. By a Steiner circle we mean one which is tangent to both (O) and (I). We shall assume $d \neq 0$ so that the circles (O) and (I) are not concentric. Clearly there are unique Steiner circles of radii $\rho_0 := \frac{R-r-d}{2}$ and $\rho_1 := \frac{R-r+d}{2}$. For each $\rho \in (\rho_0,\ \rho_1)$, there are exactly two Steiner circles of radius ρ symmetric in the center line OI. The center of each is at distances $R-\rho$ from O and $r+\rho$ from I.

Proposition 1. *If $A(\rho)$ is a Steiner circle tangent to (O) at P and (I) at Q, then the line PQ contains T_+, the internal center of similitude of (O) and (I).*

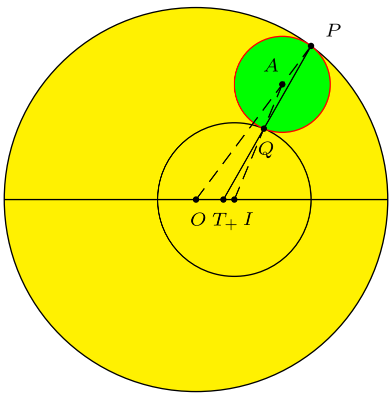


Figure 2.

Proof. Note that A divides OP internally in the ratio $OA : AP = R-\rho : \rho$, so that

$$A = \frac{\rho \cdot O + (R-\rho)P}{R}.$$

Similarly, the same point A divides IQ externally in the $IA : AQ = r+\rho : -\rho$, so that

$$A = \frac{-\rho \cdot I + (r+\rho)Q}{r}.$$

Eliminating A from these two equations, and rearranging, we obtain

$$\frac{-r(R-\rho)P + R(r+\rho)Q}{(R+r)\rho} = \frac{R \cdot I + r \cdot O}{R+r}.$$

This equation shows that a point on the line PQ is the same as a point on the line OI, which is the intersection of the lines PQ and OI. Note that the point on the line OI is independent of P. It is the internal center of similitude T_+ of the two circles, dividing O and I internally in the ratio $OT_+ : T_+I = R : r$. $\square$

Remark. The point T_+ can be constructed as the intersection of the line OI with the line joining the endpoints of a pair of oppositely parallel radii of the circles.

Two Steiner circles are neighbors if they are tangent to each other externally.

Proposition 2. *If two neighboring Steiner circles are tangent to each other at T, then T lies on a circle with center T_+.*

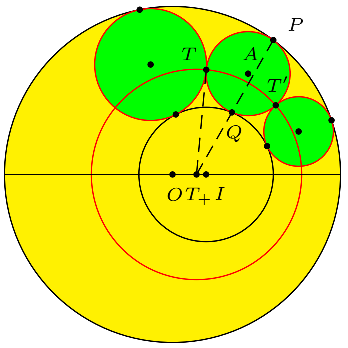


Figure 3

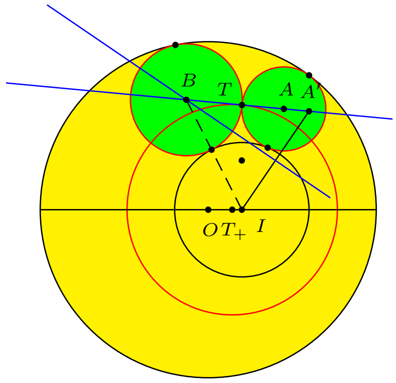


Figure 4

Proof. Applying the law of cosines to triangles POT_+ and AOI, we have

$$\frac{R^2 + \left(\frac{Rd}{R+r}\right)^2 - T_+P^2}{2R \cdot \frac{Rd}{R+r}} = \frac{(R-\rho)^2 + d^2 - (r+\rho)^2}{2(R-\rho)d}.$$

From this,

$$T_+P^2 = \frac{R^2((R+r)^2 - d^2)}{(R+r)^2} \cdot \frac{r+\rho}{R-\rho}.$$

Similarly,

$$T_+Q^2 = \frac{r^2((R+r)^2 - d^2)}{(R+r)^2} \cdot \frac{R-\rho}{r+\rho}.$$

It follows that

$$T_+P \cdot T_+Q = Rr \cdot \frac{(R+r)^2 - d^2}{(R+r)^2}.$$

If we put $t^2 = Rr \cdot \frac{(R+r)^2 - d^2}{(R+r)^2}$ (independent on ρ), then the circle $T_+(t)$ intersects the Steiner circle $A(\rho)$ at a point T such that TT_+ is tangent to the Steiner circle (see Figure 3). □

This leads to an easy construction of the neighbor of $A(\rho)$ tangent at T (see Figure 4):

(1) Extend TA to A' such that $TA' = r$.

(2) Construct the perpendicular bisector of IA' to intersect the line AT at B.

Then the circle $B(T)$ is the Steiner circle tangent to $A(\rho)$ at T.

3. Radii of neighboring Steiner circles

Henceforth we write

$$(R, r, d;\ \rho_1,\ \ldots,\ \rho_n)$$

for a rational Steiner pair $(R, r, d)_q$ and an n-cycles of Steiner circles with rational radii $\rho_1, \ldots, \rho_n$. To relate the radii of neighboring Steiner circles, we make use of the following results.

Lemma 3. (a) $\rho_0\rho_1 = qRr$,
(b) $(R - \rho_0)(R - \rho_1) = (q + 1)Rr$.

Proposition 4 (Bottema [1])**.** *Given a triangle with sidelengths a_1, a_2, a_3, the distances d_1, d_2, d_3 from the opposite vertices of these sides to a point in the plane of the triangle satisfy the relation*

$$\begin{vmatrix} 2d_1^2 & -a_3^2 + d_1^2 + d_2^2 & -a_2^2 + d_3^2 + d_1^2 \\ -a_3^2 + d_1^2 + d_2^2 & 2d_2^2 & -a_1^2 + d_2^2 + d_3^2 \\ -a_2^2 + d_3^2 + d_1^2 & -a_1^2 + d_2^2 + d_3^2 & 2d_3^2 \end{vmatrix} = 0.$$

Proposition 5. *Let $A(\rho)$ be a Steiner circle between (O) and (I). The radii of its two neighbors are the roots of the quadratic polynomial $a\sigma^2 + b\sigma + c$, where*

$$\begin{aligned} a =&\ ((q+1)Rr - (R-r)\rho)^2 + 4Rr\rho^2, \\ b =&\ 2(q+1)Rr\rho((q-1)Rr - (R-r)\rho), \\ c =&\ (q+1)^2R^2r^2\rho^2. \end{aligned} \tag{2}$$

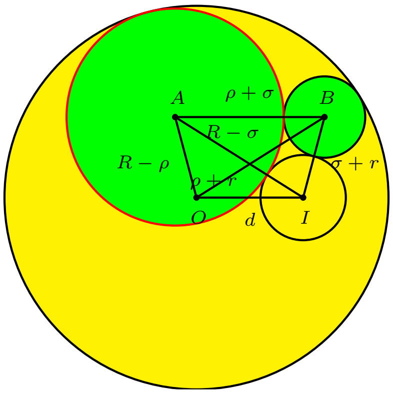


Figure 5

Proof. Let $B(\sigma)$ be a neighbor of $A(\rho)$. Apply Proposition 4 to triangle IAB with sides $\rho + \sigma$, $r + \sigma$, $r + \rho$, and the point O whose distances from I, A, B are respectively d, $R - \rho$, $R - \sigma$.

$$\begin{vmatrix} 2d^2 & R^2 - r^2 + d^2 - 2(R+r)\rho & R^2 - r^2 + d^2 - 2(R+r)\sigma \\ R^2 - r^2 + d^2 - 2(R+r)\rho & 2(R-\rho)^2 & 2(R(R-\rho) - (R+\rho)\sigma) \\ R^2 - r^2 + d^2 - 2(R+r)\sigma & 2(R(R-\rho) - (R+\rho)\sigma) & 2(R-\sigma)^2 \end{vmatrix} = 0.$$

It is clear that the determinant is a quadratic polynomial in σ. With the relation (1), we eliminate d and obtain

$$\begin{vmatrix} (R-r)^2-4qRr & R^2-Rr-2qRr-(R+r)\rho & R^2-Rr-2qRr-(R+r)\sigma \\ R^2-Rr-2qRr-(R+r)\rho & (R-\rho)^2 & R(R-\rho)-(R+\rho)\sigma \\ R^2-Rr-2qRr-(R+r)\sigma & R(R-\rho)-(R+\rho)\sigma & (R-\sigma)^2 \end{vmatrix} = 0.$$

This determinant, apart from a factor -4, is $a\sigma^2+b\sigma+c$, with coefficients given by (2) above. □

Lemma 6. $b^2-4ac = 16(q+1)^2R^3r^3\rho^2(\rho_1-\rho)(\rho-\rho_0)$.

Proof. $b^2-4ac = 4(q+1)^2R^2r^2\rho^2\cdot D$, where

$$\begin{aligned} D =\ & ((q-1)Rr-(R-r)\rho)^2-(((q+1)Rr-(R-r)\rho)^2+4Rr\rho^2) \\ =\ & 4Rr(-qRr+(R-r)\rho-\rho^2) \\ =\ & 4Rr(-\rho_0\rho_1+(\rho_0+\rho_1)\rho-\rho^2) \\ =\ & 4Rr(\rho_1-\rho)(\rho-\rho_0). \end{aligned}$$

□

Proposition 7. *The radius ρ of a Steiner circle and those of its two neighbors are rational if and only if*

$$\rho = \mathcal{R}(\tau) := \frac{\tau^2\rho_0+Rr\rho_1}{\tau^2+Rr} \tag{3}$$

for some rational number τ.

Proof. The roots of the quadratic polynomial $a\sigma^2+b\sigma+c$ are rational if and only if b^2-4ac is the square of a rational number. With a, b, c given in (2), this discriminant is given by Lemma 6. Writing $Rr(\rho_1-\rho)(\rho-\rho_0) = \tau^2(\rho-\rho_0)^2$ for a rational τ leads to the rational expression (3) above. □

Theorem 8. *For a Steiner circle with rational radius $\rho = \mathcal{R}(\tau)$, the two neighbors have radii $\rho_+ = \mathcal{R}(\tau_+)$ and $\rho_- = \mathcal{R}(\tau_-)$ where*

$$\tau_+ = \frac{Rr(\tau-\rho_1)}{Rr+\tau\rho_0} \quad \text{and} \quad \tau_- = \frac{Rr(\tau+\rho_1)}{Rr-\tau\rho_0}.$$

Proof. With ρ given by (3), we have
(i) $\rho_1-\rho = \frac{\tau^2(\rho_1-\rho_0)}{\tau^2+Rr}$ and $\rho-\rho_0 = \frac{Rr(\rho_1-\rho_0)}{\tau^2+Rr}$,
(ii) from (2)

$$\begin{aligned} b =\ & 2(q+1)Rr\rho((q-1)Rr-(\rho_0+\rho_1)\rho), \\ =\ & 2(q+1)Rr\rho\left(-R^2+R(\rho_0+\rho_1)+\rho_0\rho_1-(\rho_0+\rho_1)\cdot\frac{\tau^2\rho_0+Rr\rho_1}{\tau^2+Rr}\right) \\ =\ & -2(q+1)Rr\rho\cdot\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr}, \end{aligned}$$

(iii) by Lemma 6,

$$b^2 - 4ac = \frac{16(q+1)^2R^4r^4\rho^2(\rho_1-\rho_0)^2\tau^2}{(\tau^2+Rr)^2}.$$

The roots of the quadratic polynomial $a\sigma^2 + b\sigma + c$ are

$$\begin{aligned}
\frac{2c}{-b-\varepsilon\sqrt{b^2-4ac}} &= \frac{2(q+1)^2R^2r^2\rho^2}{2(q+1)Rr\rho\cdot\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr} - \varepsilon\frac{4(q+1)R^2r^2\rho(\rho_1-\rho_0)\tau}{\tau^2+Rr}} \\
&= \frac{(q+1)Rr\rho}{\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr} - \varepsilon\frac{2Rr(\rho_1-\rho_0)\tau}{\tau^2+Rr}} \\
&= \frac{(q+1)Rr(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau},
\end{aligned} \tag{4}$$

where $\varepsilon = \pm 1$. On the other hand,

$$\begin{aligned}
\mathcal{R}\left(\frac{Rr(\tau-\varepsilon\rho_1)}{Rr+\varepsilon\tau\rho_0}\right) &= \frac{R^2r^2(\tau-\varepsilon\rho_1)^2\rho_0+Rr(Rr+\varepsilon\tau\rho_0)^2\rho_1}{R^2r^2(\tau-\varepsilon\rho_1)^2+Rr(Rr+\varepsilon\tau\rho_0)^2} \\
&= \frac{Rr(\tau-\varepsilon\rho_1)^2\rho_0+(Rr+\varepsilon\tau\rho_0)^2\rho_1}{Rr(\tau-\varepsilon\rho_1)^2+(Rr+\varepsilon\tau\rho_0)^2} \\
&= \frac{(Rr+\rho_0\rho_1)(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau} \\
&= \frac{(q+1)Rr(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau}.
\end{aligned}$$

These are, according to (4) above, the radii of the two neighbors. □

4. Parametrizations

A rational Steiner pair (R, r, d) is *standard* if $R = 1$.

Proposition 9. *The standard rational Steiner pairs are parametrized by*

$$R = 1, \quad r = \frac{t}{(q+t)(q+1+t)}, \quad d = \frac{q(q+1)-t^2}{(q+t)(q+1+t)}. \tag{5}$$

Proof. Since $(R, r, d) = (1, 0, 1)$ is a rational solution of

$$d^2 = (1-r)^2 - 4qr,$$

every rational solution is of the form $d = 1 - (2q+1+2t)r$ for some rational number t. Direct substitution leads to

$$(q+t)(q+1+t)r - t = 0.$$

From this, the expressions of r and d follow. □

Remark. $\rho_0 = \frac{t}{q+1+t}$ and $\rho_1 = \frac{q}{q+t}$.

Proposition 10. *In a standard rational Steiner pair* $(R, r, d)_q$*, a Steiner circle* $A(\rho)$ *and its neighbors have rational radii if and only if*

$$\rho = \mathcal{R}(\tau) = \frac{t(q + (q+t)^2\tau^2)}{(q+t)(t + (q+t)(q+1+t)\tau^2)}$$

for a rational number τ*. The radii of the two neighbors are* $\mathcal{R}(\tau_\pm)$*, where*

$$\tau_+ = \frac{-q + (q+t)\tau}{(q+t)(1 + (q+t)\tau)} \quad \text{and} \quad \tau_- = \frac{q + (q+t)\tau}{(q+t)(1 - (q+t)\tau)}.$$

Proof. The neighbors of $A(\rho)$ have rational radii if and only if $a\sigma^2 + b\sigma + c$ (with a, b, c given in (2)) has rational roots. Therefore, the two neighbors have rational radii if and only if $Rr(\rho_1 - \rho)(\rho - \rho_0)$ is the square of a rational number by Proposition 7, Theorem 8, and Proposition 9. $\square$

Proposition 11. *The iterations of* τ_+ (respectively τ_-) *have periods* 3*,* 4*,* 6 *according as* $q = 3$*,* 1*, or* $\frac{1}{3}$*.*

Proof. The iterations of τ_+ are as follows.

$$\begin{tikzcd}
\frac{-3+(3+t)\tau}{(3+t)(1+(3+t)\tau)} \arrow[d] & \tau \arrow[l] \\
\frac{3+(3+t)\tau}{(3+t)(1-(3+t)\tau)} \arrow[ur] &
\end{tikzcd}$$

Figure 6. 3-cycle for $q = 3$

$$\begin{tikzcd}
& \frac{-1+(1+t)\tau}{(1+t)(1+(1+t)\tau)} \arrow[dl] & \\
\frac{-1}{(1+t)^2\tau} \arrow[dr] & & \tau \arrow[ul] \\
& \frac{1+(1+t)\tau}{(1+t)(1-(1+t)\tau)} \arrow[ur] &
\end{tikzcd}$$

Figure 7. 4-cycle for $q = 1$

$$\begin{tikzcd}
& \frac{-3+(1+3t)\tau}{(1+3t)(1+(1+3t)\tau)} \arrow[dl] & \frac{3(-1+(1+3t)\tau)}{(1+3t)(3+(1+3t)\tau)} \arrow[l] & \\
\frac{-3}{(1+3t)^2\tau} \arrow[dr] & & & \tau \arrow[ul] \\
& \frac{3+(1+3t)\tau}{(1+3t)(1-(1+3t)\tau)} \arrow[r] & \frac{3(1+(1+3t)\tau)}{(1+3t)(3-(1+3t)\tau)} \arrow[ur] &
\end{tikzcd}$$

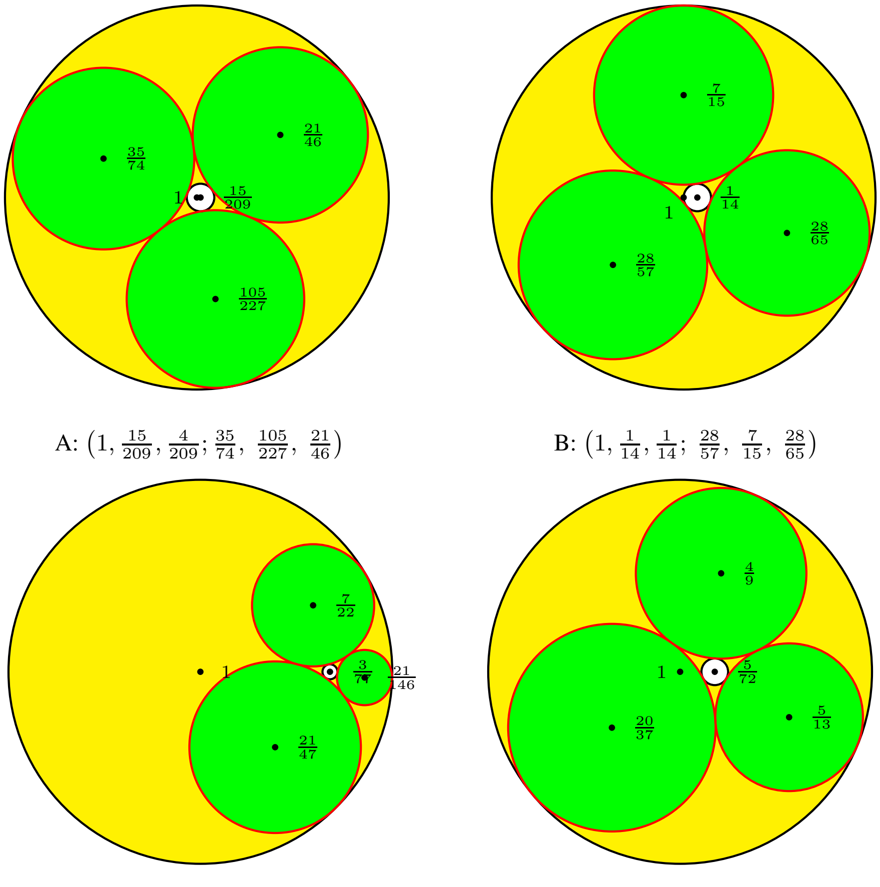
Figure 8. 6-cycle for $q = \frac{1}{3}$

The iterations of τ_- simply reverse the orientations of these cycles. $\square$

Example 1. Rational Steiner 3-cycles with simple rational radii:

t	(R, r, d)	τ	Steiner cycle
2	$\left(1, \frac{1}{15}, \frac{4}{15}\right)$	1	$\left(\frac{7}{12}, \frac{7}{17}, \frac{7}{20}\right)$
		$\frac{1}{3}$	$\left(\frac{13}{23}, \frac{13}{38}, \frac{13}{30}\right)$
		$\frac{4}{5}$	$\left(\frac{19}{32}, \frac{19}{48}, \frac{19}{53}\right)$
		$\frac{9}{5}$	$\left(\frac{21}{38}, \frac{21}{47}, \frac{21}{62}\right)$
		$\frac{11}{5}$	$\left(\frac{31}{57}, \frac{31}{68}, \frac{31}{92}\right)$
3	$\left(1, \frac{1}{14}, \frac{1}{14}\right)$	$\frac{3}{2}$	$\left(\frac{28}{57}, \frac{7}{15}, \frac{28}{65}\right)$
$\frac{3}{2}$	$\left(1, \frac{2}{33}, \frac{13}{33}\right)$	$\frac{4}{3}$	$\left(\frac{8}{13}, \frac{8}{21}, \frac{2}{7}\right)$
		$\frac{1}{9}$	$\left(\frac{24}{61}, \frac{24}{85}, \frac{3}{5}\right)$

t	(R, r, d)	τ	Steiner cycle
$\frac{2}{3}$	$\left(1, \frac{3}{77}, \frac{52}{77}\right)$	$\frac{5}{11}$	$\left(\frac{3}{5}, \frac{3}{20}, \frac{1}{4}\right)$
		$\frac{1}{3}$	$\left(\frac{7}{22}, \frac{21}{47}, \frac{21}{146}\right)$
$\frac{3}{4}$	$\left(1, \frac{4}{95}, \frac{61}{95}\right)$	$\frac{4}{13}$	$\left(\frac{16}{45}, \frac{4}{9}, \frac{16}{101}\right)$
$\frac{3}{5}$	$\left(1, \frac{5}{138}, \frac{97}{138}\right)$	$\frac{11}{6}$	$\left(\frac{20}{33}, \frac{5}{22}, \frac{4}{29}\right)$
$\frac{9}{5}$	$\left(1, \frac{15}{232}, \frac{73}{232}\right)$	$\frac{1}{2}$	$\left(\frac{60}{97}, \frac{20}{59}, \frac{3}{8}\right)$
$\frac{12}{5}$	$\left(1, \frac{5}{72}, \frac{13}{72}\right)$	$\frac{1}{3}$	$\left(\frac{20}{37}, \frac{5}{13}, \frac{4}{9}\right)$
$\frac{9}{7}$	$\left(1, \frac{21}{370}, \frac{169}{370}\right)$	$\frac{7}{5}$	$\left(\frac{12}{19}, \frac{28}{79}, \frac{21}{82}\right)$
$\frac{10}{3}$	$\left(1, \frac{15}{209}, \frac{4}{209}\right)$	$\frac{1}{19}$	$\left(\frac{35}{74}, \frac{105}{227}, \frac{21}{46}\right)$



A: $\left(1, \frac{15}{209}, \frac{4}{209}; \frac{35}{74}, \frac{105}{227}, \frac{21}{46}\right)$

B: $\left(1, \frac{1}{14}, \frac{1}{14}; \frac{28}{57}, \frac{7}{15}, \frac{28}{65}\right)$

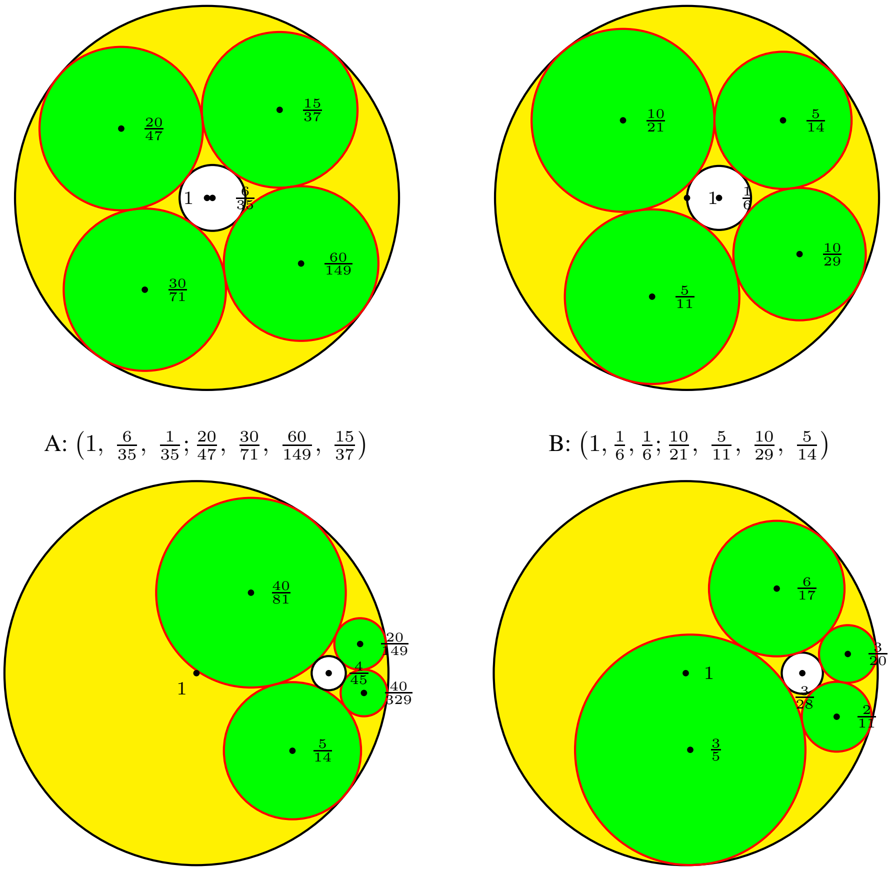
C: $\left(1, \frac{3}{77}, \frac{52}{77}; \frac{7}{22}, \frac{21}{47}, \frac{21}{146}\right)$

D: $\left(1, \frac{5}{72}, \frac{13}{72}; \frac{20}{37}, \frac{5}{13}, \frac{4}{9}\right)$

Figure 9. Rational Steiner 3-cycles

Example 2. Rational Steiner 4-cycles with simple rational radii:

t	(R, r, d)	τ	Steiner cycle
1	$\left(1, \frac{1}{6}, \frac{1}{6}\right)$	1	$\left(\frac{10}{21}, \frac{5}{11}, \frac{10}{29}, \frac{5}{14}\right)$
		2	$\left(\frac{34}{77}, \frac{17}{35}, \frac{34}{93}, \frac{17}{50}\right)$
		$\frac{1}{3}$	$\left(\frac{26}{53}, \frac{13}{35}, \frac{26}{77}, \frac{13}{30}\right)$
$\frac{1}{2}$	$\left(1, \frac{2}{15}, \frac{7}{15}\right)$	2	$\left(\frac{5}{11}, \frac{20}{37}, \frac{10}{43}, \frac{20}{93}\right)$
$\frac{1}{3}$	$\left(1, \frac{3}{28}, \frac{17}{28}\right)$	3	$\left(\frac{6}{17}, \frac{3}{5}, \frac{2}{11}, \frac{3}{20}\right)$
		$\frac{1}{4}$	$\left(\frac{15}{37}, \frac{30}{193}, \frac{15}{88}, \frac{10}{19}\right)$
$\frac{1}{4}$	$\left(1, \frac{4}{35}, \frac{31}{45}\right)$	$\frac{2}{5}$	$\left(\frac{5}{14}, \frac{40}{81}, \frac{20}{149}, \frac{40}{329}\right)$
$\frac{3}{5}$	$\left(1, \frac{15}{104}, \frac{41}{104}\right)$	$\frac{1}{2}$	$\left(\frac{30}{49}, \frac{15}{49}, \frac{10}{43}, \frac{3}{8}\right)$
$\frac{4}{5}$	$\left(1, \frac{10}{63}, \frac{17}{63}\right)$	$\frac{1}{3}$	$\left(\frac{10}{19}, \frac{20}{61}, \frac{5}{17}, \frac{4}{9}\right)$
$\frac{4}{3}$	$\left(1, \frac{6}{35}, \frac{1}{35}\right)$	$\frac{1}{7}$	$\left(\frac{20}{47}, \frac{30}{71}, \frac{60}{149}, \frac{15}{37}\right)$



A: $\left(1, \frac{6}{35}, \frac{1}{35}; \frac{20}{47}, \frac{30}{71}, \frac{60}{149}, \frac{15}{37}\right)$

B: $\left(1, \frac{1}{6}, \frac{1}{6}; \frac{10}{21}, \frac{5}{11}, \frac{10}{29}, \frac{5}{14}\right)$

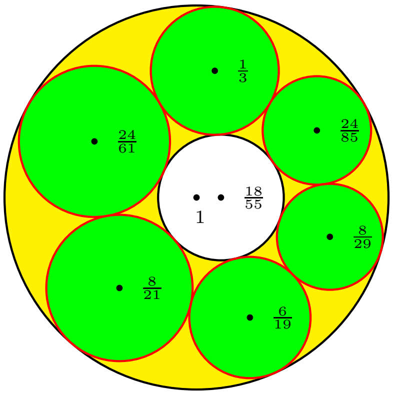
C: $\left(1, \frac{4}{45}, \frac{31}{45}; \frac{5}{14}, \frac{40}{81}, \frac{20}{149}, \frac{40}{329}\right)$

D: $\left(1, \frac{3}{28}, \frac{17}{28}; \frac{6}{17}, \frac{3}{5}, \frac{2}{11}, \frac{3}{20}\right)$

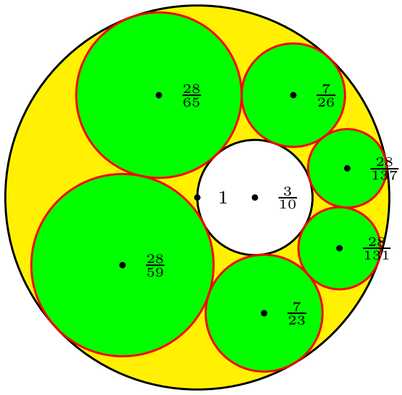
Figure 10. Rational Steiner 4-cycles

Example 3. Rational Steiner 6-cycles with simple rational radii:

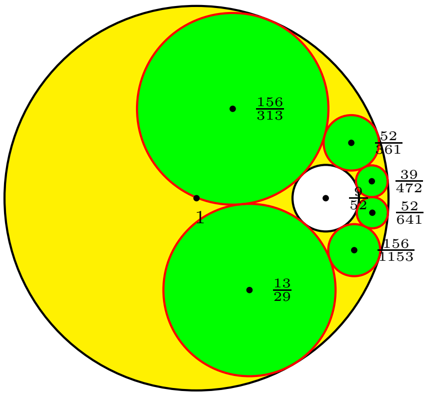
t	(R, r, d)	τ	Steiner cycle
$\frac{1}{2}$	$\left(1,\ \frac{18}{55},\ \frac{7}{55}\right)$	2	$\left(\frac{1}{3},\ \frac{24}{61},\ \frac{8}{21},\ \frac{6}{19},\ \frac{8}{29},\ \frac{24}{85}\right)$
$\frac{1}{3}$	$\left(1,\ \frac{3}{10},\ \frac{3}{10}\right)$	1	$\left(\frac{28}{65},\ \frac{28}{59},\ \frac{7}{23},\ \frac{28}{131},\ \frac{28}{137},\ \frac{7}{26}\right)$
$\frac{1}{5}$	$\left(1,\ \frac{45}{184},\ \frac{91}{184}\right)$	$\frac{5}{3}$	$\left(\frac{20}{57},\ \frac{60}{97},\ \frac{5}{17},\ \frac{12}{77},\ \frac{20}{153},\ \frac{15}{88}\right)$
$\frac{1}{9}$	$\left(1,\ \frac{9}{52},\ \frac{35}{52}\right)$	$\frac{3}{2}$	$\left(\frac{12}{31},\ \frac{4}{7},\ \frac{3}{19},\ \frac{4}{47},\ \frac{12}{151},\ \frac{1}{8}\right)$
		$\frac{5}{4}$	$\left(\frac{52}{361},\ \frac{156}{313},\ \frac{13}{29},\ \frac{156}{1153},\ \frac{52}{641},\ \frac{39}{472}\right)$
$\frac{2}{9}$	$\left(1,\ \frac{9}{35},\ \frac{16}{35}\right)$	3	$\left(\frac{21}{83},\ \frac{7}{13},\ \frac{21}{47},\ \frac{7}{33},\ \frac{21}{143},\ \frac{7}{45}\right)$
$\frac{4}{15}$	$\left(1,\ \frac{5}{18},\ \frac{7}{18}\right)$	5	$\left(\frac{5}{21},\ \frac{4}{9},\ \frac{20}{39},\ \frac{5}{18},\ \frac{20}{111},\ \frac{20}{117}\right)$



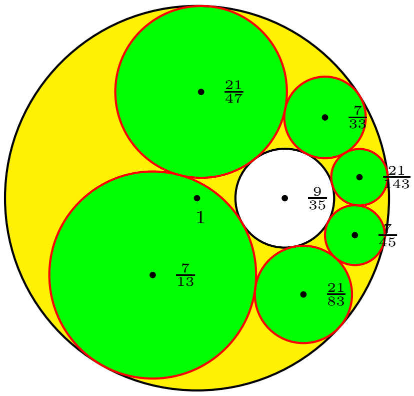
A: $\left(1, \frac{18}{55}, \frac{7}{55}; \frac{1}{3},\ \frac{24}{61},\ \frac{8}{21},\ \frac{6}{19},\ \frac{8}{29},\ \frac{24}{85}\right)$



B: $\left(1, \frac{3}{10}, \frac{3}{10}; \frac{28}{65},\ \frac{28}{59},\ \frac{7}{23},\ \frac{28}{131},\ \frac{28}{137},\ \frac{7}{26}\right)$



C: $\left(1, \frac{9}{52}, \frac{35}{52}; \frac{52}{361},\ \frac{156}{313},\ \frac{13}{29},\ \frac{156}{1153},\ \frac{52}{641},\ \frac{39}{472}\right)$



D: $\left(1, \frac{9}{35}, \frac{16}{35}; \frac{21}{83},\ \frac{7}{13},\ \frac{21}{47},\ \frac{7}{33},\ \frac{21}{143},\ \frac{7}{45}\right)$

Figure 11. Rational Steiner 6-cycles

5. Inversion

By inverting a Steiner n-cycle configuration $(R, r, d; \rho_1, \ldots, \rho_n)$ in the circle (O), we obtain a new n-cycle $(R, r', d'; \rho'_1, \ldots, \rho'_n)$ in which

$$r' = \frac{-R^2 r}{d^2 - r^2}, \qquad d' = \frac{R^2 d}{d^2 - r^2}, \qquad \rho'_i = \frac{R\rho_i}{2\rho_i - R}, \ i = 1, \ldots, n.$$

Regarding the circles in a Steiner configuration with (I) in the interior of (O) all positively oriented, we interpret circles with negative radii as those oppositely oriented to the circle (O).

The tables below show the rational Steiner n-cycles obtained by inverting those in Figures 9-11 in the circle (O). Figure 12 illustrates those obtained from the 4-cycles in Figure 10.

	Steiner 3-cycle	Inversive images in (O)
A	$\left(1, \frac{15}{209}, \frac{4}{209}; \frac{35}{74}, \frac{105}{227}, \frac{21}{46}\right)$	$\left(1, 15, -4; -\frac{35}{4}, -\frac{105}{17}, -\frac{21}{4}\right)$
B	$\left(1, \frac{1}{14}, \frac{1}{14}; \frac{28}{57}, \frac{7}{15}, \frac{28}{65}\right)$	$\left(1, \infty, 7; -28, -7, -\frac{28}{9}\right)$
C	$\left(1, \frac{3}{77}, \frac{52}{77}; \frac{7}{22}, \frac{21}{47}, \frac{21}{146}\right)$	$\left(1, -\frac{3}{35}, \frac{52}{35}; -\frac{7}{8}, -\frac{21}{5}, -\frac{21}{104}\right)$
D	$\left(1, \frac{5}{72}, \frac{13}{72}; \frac{20}{37}, \frac{5}{13}, \frac{4}{9}\right)$	$\left(1, -\frac{5}{2}, \frac{13}{2}; \frac{20}{3}, -\frac{5}{3}, -4\right)$

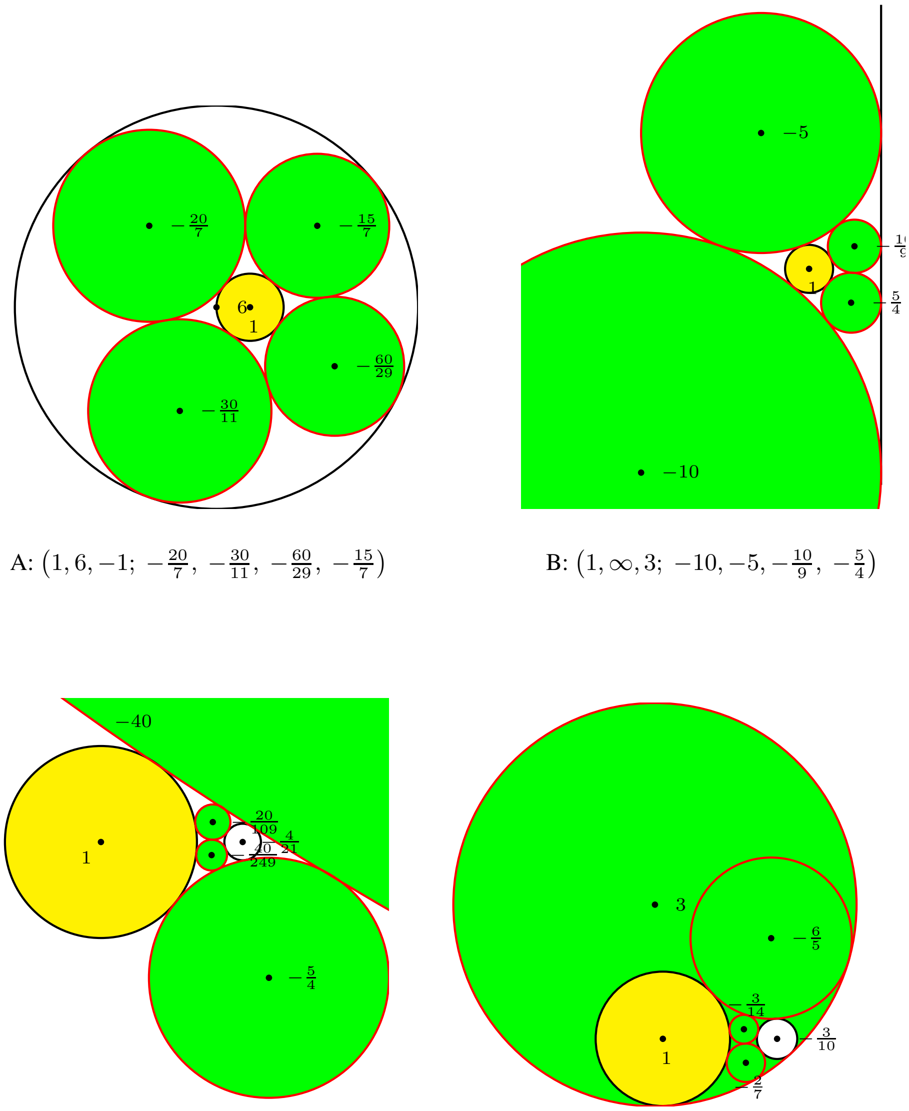
	Steiner 4-cycle	Inversive images in (O)
A	$\left(1, \frac{6}{35}, \frac{1}{35}; \frac{20}{47}, \frac{30}{71}, \frac{60}{149}, \frac{15}{37}\right)$	$\left(1, 6, -1; -\frac{20}{7}, -\frac{30}{11}, -\frac{60}{29}, -\frac{15}{7}\right)$
B	$\left(1, \frac{1}{6}, \frac{1}{6}; \frac{10}{21}, \frac{5}{11}, \frac{10}{29}, \frac{5}{14}\right)$	$\left(1, \infty, 3; -10, -5, -\frac{10}{9}, -\frac{5}{4}\right)$
C	$\left(1, \frac{4}{45}, \frac{31}{45}; \frac{5}{14}, \frac{40}{81}, \frac{20}{149}, \frac{40}{329}\right)$	$\left(1, -\frac{4}{21}, \frac{31}{21}; -\frac{5}{4}, -40, -\frac{20}{109}, -\frac{40}{249}\right)$
D	$\left(1, \frac{3}{28}, \frac{17}{28}; \frac{6}{17}, \frac{3}{5}, \frac{2}{11}, \frac{3}{20}\right)$	$\left(1, -\frac{3}{10}, \frac{17}{10}; -\frac{6}{5}, 3, -\frac{2}{7}, -\frac{3}{14}\right)$

	Steiner 6-cycle
A	$\left(1, \frac{18}{55}, \frac{7}{55}; \frac{1}{3}, \frac{24}{61}, \frac{8}{21}, \frac{6}{19}, \frac{8}{29}, \frac{24}{85}\right)$
Inversive images	$\left(1, \frac{18}{5}, -\frac{7}{5}; -1, -\frac{24}{13}, -\frac{8}{5}, -\frac{6}{7}, -\frac{8}{13}, -\frac{24}{37}\right)$
B	$\left(1, \frac{3}{10}, \frac{3}{10}; \frac{28}{65}, \frac{28}{59}, \frac{7}{23}, \frac{28}{131}, \frac{28}{137}, \frac{7}{26}\right)$
Inversive images	$\left(1, \infty, \frac{5}{3}; -\frac{28}{9}, -\frac{28}{3}, -\frac{7}{9}, -\frac{28}{75}, -\frac{28}{81}, -\frac{7}{12}\right)$
C	$\left(1, \frac{9}{52}, \frac{35}{52}; \frac{52}{361}, \frac{156}{313}, \frac{13}{29}, \frac{156}{1153}, \frac{52}{641}, \frac{39}{472}\right)$
Inversive images	$\left(1, -\frac{9}{22}, \frac{35}{22}; -\frac{52}{257}, -156, -\frac{13}{3}, -\frac{156}{841}, -\frac{52}{537}, -\frac{39}{394}\right)$
D	$\left(1, \frac{9}{35}, \frac{16}{35}; \frac{21}{83}, \frac{7}{13}, \frac{21}{47}, \frac{7}{33}, \frac{21}{143}, \frac{7}{45}\right)$
Inversive images	$\left(1, -\frac{9}{5}, \frac{16}{5}; -\frac{21}{41}, 7, -\frac{21}{5}, -\frac{7}{19}, -\frac{21}{101}, -\frac{7}{31}\right)$

The rational Steiner pairs in Figures 9-11A all have $r > d$. In the inversive images, (I) contains (O) in its interior. The new configuration is equivalent to a standard one with $r < 1$. For example, the Steiner pair in Figure 12A is equivalent to $\left(1, \frac{1}{6}, \frac{1}{6}\right)$.

The Steiner pairs in Figures 9-11B all have $r = d$. The inversive image of (I) is a line at a distance $\frac{1}{2r}$ from O.

In Figures 9-11C and D, $r < d$. The images of (O) and (I) are disjoint circles. For the cycles in C, none of the Steiner circles contains O. Their images are all externally tangent to the images of (O) and (I). On the other hand, for the cycles in D, one of the Steiner circles contains O in its interior. Therefore, its inversive image contains those of (O) and (I) in its interior.



A: $\left(1, 6, -1;\ -\frac{20}{7}, -\frac{30}{11}, -\frac{60}{29}, -\frac{15}{7}\right)$

B: $\left(1, \infty, 3;\ -10, -5, -\frac{10}{9}, -\frac{5}{4}\right)$

C: $\left(1, -\frac{4}{21}, \frac{31}{21};\ -\frac{5}{4}, -40, -\frac{20}{109}, -\frac{40}{249}\right)$

D: $\left(1,\ -\frac{3}{10},\ \frac{17}{10};\ -\frac{6}{5},\ 3,\ -\frac{2}{7},\ -\frac{3}{14}\right)$

Figure 12. Rational Steiner 4-cycles by inversion

6. Relations among standard rational Steiner pairs

We conclude this note with a brief explanation of the relations among Steiner pairs with different parameters. Denote by $\mathcal{S}_q(t)$ the standard rational Steiner pair given in Proposition 9. By allowing t to take on negative values, we also include the disjoint pairs and those with (I) containing (O).

Proposition 12. *Let (O) and (I) be the circles in $\mathcal{S}_q(t)$.*
(a) *(I) is in the interior of (O) if and only if $t > 0$.*
(b) *(I) contains (O) in its interior if and only if $-(q+1) < t < -q$.*
(c) *(O) and (I) are disjoint if and only if $-q < t < 0$ or $t < -(q+1)$.*

Proof. (a) The circle (I) is contained in the interior of (O) if and only if $d+r < R$ and $d - r > -R$. This means $t(q+1+t) > 0$ and $q+t > 0$. Therefore, $t > 0$.

(b) The circle (I) contains (O) in its interior if and only if $d+r > R$ and $d-r < -R$. This means $t(q+1+t) < 0$ and $q+t < 0$. Therefore, $-(q+1) < t < -q$.

(c) follows from (a) and (b). □

Remarks. (1) If $-(q+1) < t < -q$, $\mathcal{S}_q(t)$ is homothetic, by the homothety at I with ratio $\frac{R}{r}$, to $\mathcal{S}_q(t')$, where $t' = -\frac{(q+1)(q+t)}{q+1+t} > 0$.

(2) For standard pairs $\mathcal{S}_q(t)$ with $t > 0$, we may restrict to $0 < t < \sqrt{q(q+1)}$. If $t^2 > q(q+1)$, $\mathcal{S}_q(t)$ is the reflection of $\mathcal{S}_q\left(\frac{q(q+1)}{t}\right)$ in O.

Proposition 13. *The inversive image of $\mathcal{S}_q(t)$ in the circle (O) is $\mathcal{S}_q(-t)$.*

Proof. Let $I'(r')$ be the inversive image of (I) in (O), with $d' = OI'$.

$$r' = \frac{1}{2}\left(\frac{R^2}{d+r} - \frac{R^2}{d-r}\right) = \frac{-t}{(q-t)(q+1-t)},$$
$$d' = \frac{1}{2}\left(\frac{R^2}{d+r} + \frac{R^2}{d-r}\right) = \frac{q(q+1)-t^2}{(q-t)(q+1-t)}.$$

From this it is clear that the inversive image of the pair $\mathcal{S}_q(t)$ is $\mathcal{S}_q(-t)$. □

Remarks. (1) If $t = 0$, (I) reduces to a point on the circle (O).

(2) If $t^2 = q(q+1)$, then $d = 0$. The circles (O) and (I) are concentric.

(3) If $t = -q$ or $-(q+1)$, the circle (I) degenerates into a line. This means that with $t = q$ or $q+1$, the circle (I) passes through O. It has radius $\frac{1}{2(2q+1)}$. Therefore, the line in $\mathcal{S}_q(-q)$ is at a distance $2q+1$ from O.

(4) If the circle (I) contains the center O, then the inversive image of (I) contains (O). This means that $-(q+1) < -t < -q$, and $q < t < q+1$. It follows that if $0 < t < q$, the inversive images of the Steiner pair of circles are disjoint.

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