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The Golden Section with a Collapsible Compass Only

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Abstract. With the use of a collapsible compass, we divide a given segment in the golden ratio by drawing ten circles.

Kurt Hofstetter [4] has given an elegant euclidean construction in *five* steps for the division of a segment in the golden ratio. In Figure 1, AB is a given segment. The circles $c_1 := A(B)$ and $c_2 := B(A)$ intersect at C and C'. The circle $c_3 := C(A)$ intersects c_1 at D. Join C and C' to intersect c_3 at the midpoint M of the arc AB. Then the circle $c_4 := D(M)$ intersects the segment AB at a point G which divides it in the golden ratio. The validity of this construction depends on the fact that DM is a side of a square inscribed in the circle $C(D)$.

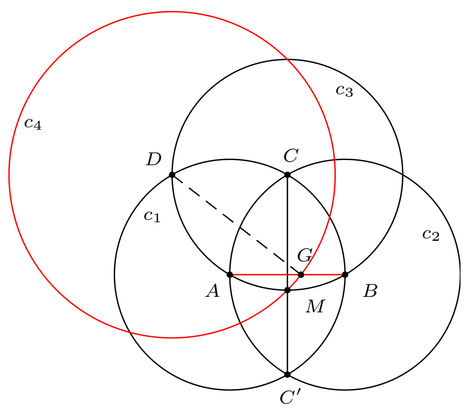

Figure 1. Hofstetter's division of a segment in the golden ratio

We shall modify this construction to one using only a collapsible compass, in *ten* steps (see Construction 5 below). Euclid, in his *Elements* I.2, shows how to construct, in *seven* steps, using a collapsible compass with the help of a straightedge, a circle with given center A and radius equal to a given segment BC (see [2, p.244]). His interest was not on the parsimoniousness of the construction, but rather on the justification of how his Postulate 3 (to describe a circle with any given center and distance) can be put into effect by the use of a straightedge (Postulates 1 and 2) and a collapsible compass (*Elements* I.1). Since we restrict to the use of a collapsible compass only, we show that this can be done without the use of a straightedge, more simply, in *five steps* (see Figure 2). There were two prior publications in this

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Forum on compass-only constructions. Note that Hofstetter [3] did not divide a given segment in the golden ratio. On the other hand, the one given by Bataille [1] requires a rusty compass.

The proof of Construction 1 below, though simple, makes use of *Elements* I.8.

Construction 1. *Given three points A, B, C, construct*
(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* P *and* Q,
(3,4) $c_3 := P(C)$ *and* $c_4 := Q(C)$ *to intersect at* D,
(5) $c_5 := A(D)$.
The circle c_5 *has radius congruent to* BC.

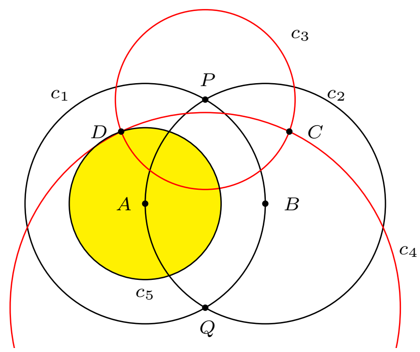

Figure 2. Construction of $A(BC)$ with a collapsible compass only

Lemma 2 helps simplify the proof of Construction 5.

Lemma 2. *Given a unit segment AB, let the circles $A(B)$ and $B(A)$ intersect at C. If the circle $C(A)$ intersects $A(B)$ at D and $B(A)$ at E, and the circles $D(B)$ and $E(A)$ intersect at H, then $CH = \sqrt{2}$, the side of a square inscribed in the circle $A(B)$.*

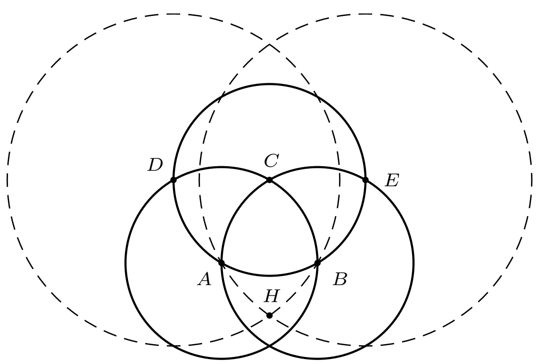

Figure 3.

We present two simple applications of Lemma 2.

Construction 3. *Given two points O and A, to construct a square $ABCD$ inscribed in the circle $O(A)$, construct*
(1,2) $c_1 := O(A)$ *and* $c_2 := A(O)$ *intersecting at* E *and* F,
(3) $c_3 := E(O)$ *intersecting* c_2 *at* F',
(4) $c_4 := F'(O)$,
(5) $c_5 := F(E)$ *intersecting at* c_1 *at* C *and* c_4 *at* H,
(6) $c_6 := A(H)$ *intersecting* c_1 *at* B *and* D.
$ABCD$ *is a square inscribed in the circle* $c_1 = O(A)$ (see Figure 4).

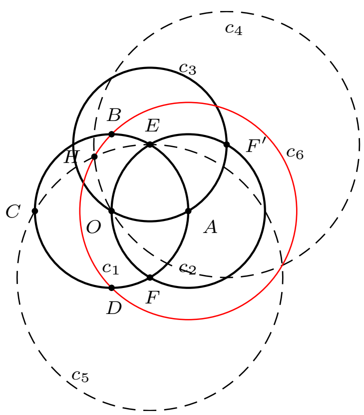

Figure 4

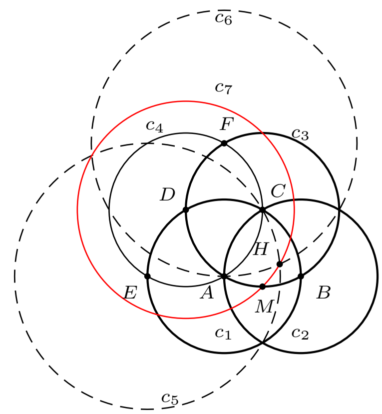

Figure 5

Construction 4. *To construct the midpoint M of the arc AB of the circle c_3 in Figure 1, we construct*
(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* C,
(3) $c_3 := C(A)$ *to intersect* c_1 *at* D,
(4) $c_4 := D(A)$ *to intersect* c_1 *at* E *and* c_3 *at* F,
(5,6) $c_5 := E(C)$ *and* $c_6 := F(A)$ *to intersect at* H,
(7) $c_7 := D(H)$ *to intersect* c_3 *at* M.
M *is the midpoint of the arc* AB (see Figure 5).

Finally, we present a division of a segment in the golden ratio in *ten* steps.

Construction 5. *Given two points A and B, construct*
(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* C.
(3) $c_3 := C(A)$ *to intersect* c_1 *at* D *and* c_2 *at* E.
(4) $c_4 := E(A)$ *intersects* c_3 *at* A'.
(5) $c_5 := D(B)$ *intersects* c_1 *at* D' *and* c_4 *at* H.
(6,7) $c_6 := A(H)$ *and* $c_7 := A'(H)$ *to intersect at* F.
(8) $c_8 := B(F)$ *intersects* c_6 (which is also $A(F)$) *at* F'.
(9,10) $c_9 := D(F)$ *and* $c_{10} := D'(F')$ *to intersect at* G.
The point G divides AB in the golden ratio (see Figure 6).

In Step (5), either of the intersections may be chosen as H. In Figure 6, H and C are on opposite sides of AB.

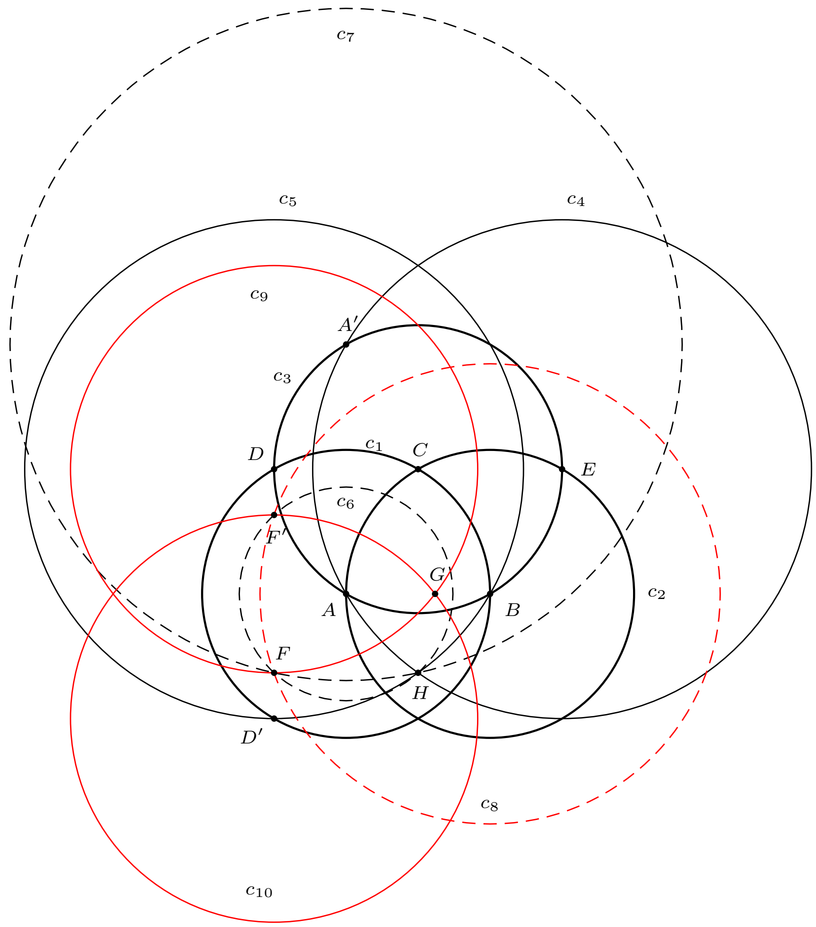

Figure 6. Golden section with collapsible compass only

Proof. Assume AB has unit length. We have
(i) $CH = \sqrt{2}$ by Lemma 2.
(ii) D and C are symmetric in the line AA'.
(iii) F and H are also symmetric in the line AA' by construction.
(iv) Therefore, $DF = \sqrt{2}$.
(v) D' and F' are the reflections of D and F in the line AB by construction. Therefore, $D(F)$ and $D'(F')$ intersect at a point G on the line AB. The fact that G divides AB in the golden ratio follows from Hofstetter's construction. □

References

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