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Construction of Circles Through Intercepts of Parallels to Cevians

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Abstract. From the traces of the cevians of a point in the plane of a given triangle, construct parallels to the cevians to intersect the sidelines at six points. We determine the points for which these six intersections are concyclic.

Given a point P in the plane of triangle ABC, with cevian triangle XYZ, construct parallels through X, Y, Z to the cevians to intersect the sidelines at the following points.

Point	Intersection of	with the parallel to	through	Coordinates
B_a	CA	CZ	X	$(-u : 0 : u+v+w)$
C_a	AB	BY	X	$(-u : u+v+w : 0)$
C_b	AB	AX	Y	$(u+v+w : -v : 0)$
A_b	BC	CZ	Y	$(0 : -v : u+v+w)$
A_c	BC	BY	Z	$(0 : u+v+w : -w)$
B_c	CA	AX	Z	$(u+v+w : 0 : -w)$

A simple application of Carnot's theorem shows that these six points lie on a conic $\mathcal{C}(P)$ (see Figure 1). In this note we inquire the possibility for this conic to be a circle, and give a complete answer. We work with homogeneous barycentric coordinates with reference to triangle ABC. Suppose the given point P has coordinates $(u : v : w)$. The coordinates of the six points are given in the rightmost column of the table above. It is easy to verify that these points are all on the conic

$$\begin{aligned} &u(u+v)(u+w)yz + v(v+w)(v+u)zx + w(w+u)(w+v)xy \\ &+ (u+v+w)(x+y+z)(vwx + wuy + uvz) = 0. \end{aligned} \tag{1}$$

Proposition 1. *The conic $\mathcal{C}(P)$ through the six points is a circle if and only if*

$$\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v} = a^2 : b^2 : c^2. \tag{2}$$

Proof. Note that the lines B_aC_a, C_bA_b, A_cB_c are parallel to the sidelines of ABC. These three lines bound a triangle homothetic to ABC at the point

$$\left(\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v}\right).$$

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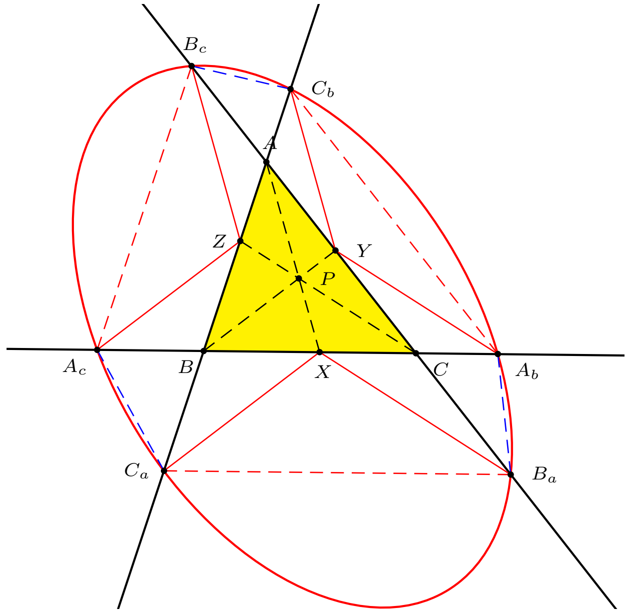

Figure 1.

It is known (see, for example, [2, §2]) that the hexagon $B_aC_aA_cB_cC_bA_b$ is a Tucker hexagon, *i.e.*, B_cC_b, C_aA_c, A_bB_a are antiparallels and the conic through the six points is a circle, if and only if this homothetic center is the symmedian point $K = (a^2 : b^2 : c^2)$. Hence the result follows. □

Corollary 2. *If $\mathcal{C}(P)$ is a circle, then it is a Tucker circle with center on the Brocard axis (joining the circumcenter and the symmedian point).*

Proposition 3. *If ABC is a scalene triangle, there are three distinct real points P for which the conic $\mathcal{C}(P)$ is a circle.*

Proof. Writing

$$\frac{u}{v+w} = \frac{a^2}{t}, \qquad \frac{v}{w+u} = \frac{b^2}{t}, \qquad \frac{w}{u+v} = \frac{c^2}{t}, \tag{3}$$

we have

$$\begin{array}{rcrcrcl} -tu & + & a^2v & + & a^2w & = & 0, \\ b^2u & - & tv & + & b^2w & = & 0, \\ c^2u & + & c^2v & - & tw & = & 0. \end{array}$$

Hence,

$$\begin{vmatrix} -t & a^2 & a^2 \\ b^2 & -t & b^2 \\ c^2 & c^2 & -t \end{vmatrix} = 0,$$

or

$$F(t) := -t^3 + (a^2b^2 + b^2c^2 + c^2a^2)t + 2a^2b^2c^2 = 0. \tag{4}$$

Note that $F(0) > 0$ and $F(+\infty) = -\infty$. Furthermore, assuming $a > b > c$, we easily note that

$$F(-a^2) > 0, \qquad F(-b^2) < 0, \qquad F(-c^2) > 0.$$

Therefore, F has one positive and two negative roots. □

Theorem 4. *For a scalene triangle ABC with $\rho = \frac{2}{\sqrt{3}}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$ and $\theta_0 := \frac{1}{3}\arccos\frac{8a^2b^2c^2}{\rho^3}$, the three points for which the corresponding conics $\mathcal{P}$ are circle are*

$$P_k = \left(\frac{a^2}{a^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)} : \frac{b^2}{b^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)} : \frac{c^2}{c^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)}\right)$$

for $k = 0, \pm 1$.

Proof. From (3) the coordinates of P are

$$u : v : w = \frac{a^2}{a^2 + t} : \frac{b^2}{b^2 + t} : \frac{c^2}{c^2 + t},$$

with t a real root of the cubic equation (4). Writing $t = \rho\cos\theta$ we transform (4) into

$$\frac{1}{4}\rho^3\left(4\cos^3\theta - \frac{4(a^2b^2 + b^2c^2 + c^2a^2)}{\rho^2}\cos\theta\right) = 2a^2b^2c^2.$$

If $\rho = \frac{2}{\sqrt{3}}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$, this can be further reduced to

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{8a^2b^2c^2}{\rho^3}.$$

The three real roots of (4) $t_k = \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)$ for $k = 0, \pm 1$.

□

Remarks. (1) If the triangle is equilateral, the roots of the cubic equation (4) are $t = -a^2, -a^2, \frac{a^2}{2}$.

(2) If the triangle is isosceles at A (but not equilateral), we have two solutions P_1, P_2 on the line AG. The third one degenerates into the infinite point of BC. The two finite points can be constructed as follows. Let the tangent at B to the circumcircle intersects AC at U, and T be the projection of U on AG, $AW = \frac{3}{2}\cdot AT$. The circle centered at W and orthogonal to the circle $G(A)$ intersects AG at P_1 and P_2.

Henceforth, we shall assume triangle ABC scalene.

Proposition 5. *The conic $\mathcal{C}(P)$ is a circle if and only if P is an intersection, apart from the centroid G, of*

(i) *the rectangular hyperbola through G and the incenter I and their anticevian triangles,*

(ii) *the circum-hyperbola through G and the symmedian point K.*

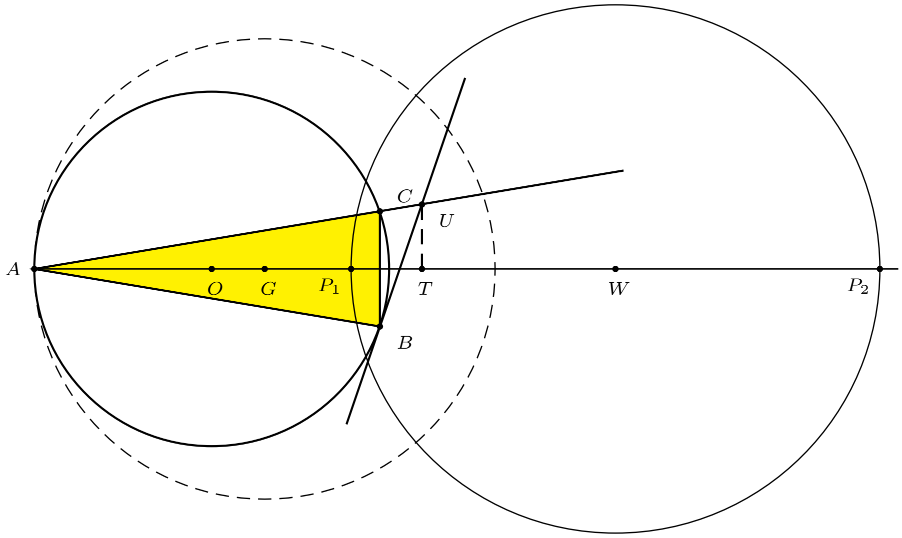

Figure 2

Proof. From (2), we have

$$f := c^2v(u+v) - b^2w(w+u) = 0, \tag{5}$$

$$g := a^2w(v+w) - c^2u(u+v) = 0, \tag{6}$$

$$h := b^2u(w+u) - a^2v(v+w) = 0. \tag{7}$$

From these,

$$0 = f + g + h = (b^2 - c^2)u^2 + (c^2 - a^2)v^2 + (a^2 - b^2)w^2.$$

This is the conic through the centroid $G = (1 : 1 : 1)$, the incenter $I = (a : b : c)$, and the vertices of their anticevian triangles.

Also, from (5)–(7),

$$0 = a^2f + b^2g + c^2h = a^2(b^2 - c^2)vw + b^2(c^2 - a^2)wu + c^2(a^2 - b^2)uv = 0.$$

This shows that the point P also lies on the circumconic through G and the symmedian point $K = (a^2 : b^2 : c^2)$.

If P is the centroid, the conic through the six points has equation

$$4(yz + zx + xy) + 3(x + y + z)^2 = 0.$$

This is homothetic to the Steiner circum-ellipse and is not a circle since the triangle is scalene. Therefore, if $\mathcal{C}(P)$ is a circle, P is an intersection of the two conics above, apart from the centroid G. □

Remark. The positive root corresponds to the intersection which lies on the arc GK of the circum-hyperbola through these two points.

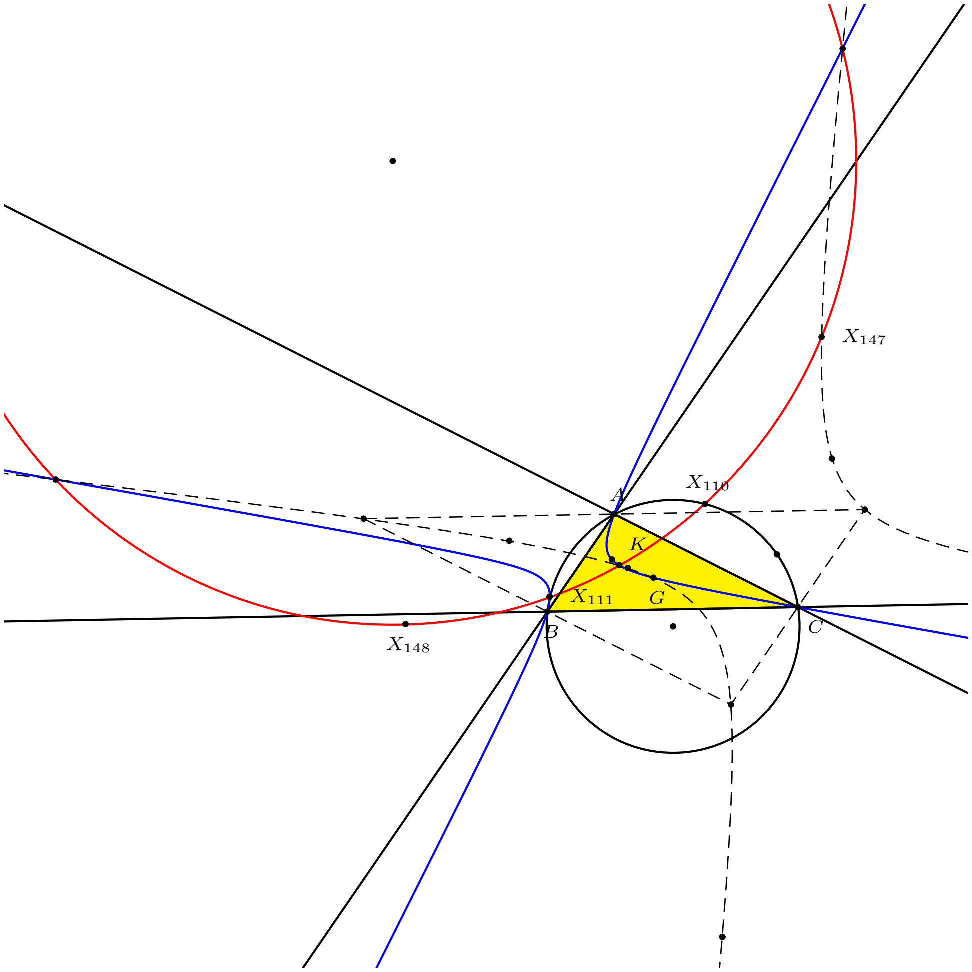

Figure 3.

Proposition 6. *The three real points P for which $\mathcal{C}(P)$ is a Tucker circle lie on a circle containing the following triangle centers:* (i) *the Euler reflection point*

$$X_{110} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right),$$

(ii) *the Parry point*

$$X_{111} = \left(\frac{a^2}{b^2+c^2-2a^2} : \frac{b^2}{c^2+a^2-2b^2} : \frac{c^2}{a^2+b^2-2c^2}\right),$$

(iii) *the Tarry point of the superior triangle X_{147},*
(iv) *the Steiner point of the superior triangle X_{148}.*

Proof. The combination

$$a^2(c^2-a^2)(a^2-b^2)f + b^2(a^2-b^2)(b^2-c^2)g + c^2(b^2-c^2)(c^2-a^2)h \quad (8)$$

of (5)–(7) (with x, y, z replacing u, v, w) yields the circle through the three points:

$$(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)x\right) = 0. \tag{9}$$

Since the line

$$\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)x = 0$$

contains the Euler reflection point and the Parry point, as is easily verified, so does the circle (9).

If we replace in (5)–(7) u, v, w by $y+z-x$, $z+x-y$, $x+y-z$ respectively, the combination (8) yields the circle

$$2(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2yz+b^2zx+c^2xy) - (x+y+z)\left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^4+c^4-a^2(b^2+c^2))x\right) = 0, \tag{10}$$

which is the inferior of the circle (9). Since the line

$$\sum_{\text{cyclic}} a^2(b^2-c^2)(b^4+c^4-a^2(b^2+c^2))x = 0$$

clearly contains the Tarry point

$$\left(\frac{1}{b^4+c^4-a^2(b^2+c^2)} : \frac{1}{c^4+a^4-b^2(c^2+a^2)} : \frac{1}{a^4+b^4-c^2(a^2+b^2)}\right),$$

and the Steiner point

$$\left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right),$$

so does the circle (10). It follows that the circle (9) contains these two points of the superior triangle. □

Remark. (1) The triangle center X_{147} also lies on the hyperbola through the hyperbola in Proposition 5(i).

(2) The Parry point X_{111} also lies on the circum-hyperbola through G and K (in Proposition 5(ii)). It is the isogonal conjugate of the infinite point of the line GK.

We conclude this note by briefly considering a conic companion to $\mathcal{C}(P)$.

With the same parallel lines through the traces of P on the sidelines, consider the intersections

Point	Intersection of	with the parallel to	through	Coordinates
B'_a	CA	BY	X	$(uv : 0 : w(u+v+w))$
C'_a	AB	CZ	X	$(wu : v(u+v+w) : 0)$
C'_b	AB	CZ	Y	$(u(u+v+w) : vw : 0)$
A'_b	BC	AX	Y	$(0 : uv : w(u+v+w))$
A'_c	BC	AX	Z	$(0 : v(u+v+w) : uw)$
B'_c	CA	BY	Z	$(u(u+v+w) : 0 : vw)$

These six points also lie on a conic $\mathcal{C}'(P)$, which has equation

$$(u+v)(v+w)(w+u)\sum_{\text{cyclic}} u(v+w)yz-(u+v+w)(x+y+z)\sum_{\text{cyclic}} v^2w^2x=0.$$

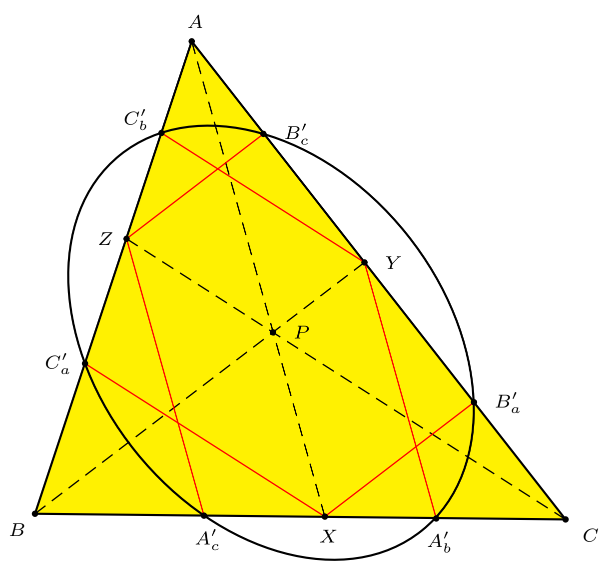

Figure 4.

In this case, the lines $B'_cC'_b$, $C'_aA'_c$, $A'_bB'_a$ are parallel to the sidelines, and bound a triangle homothetic to ABC at the point

$$(u(v+w):\ v(w+u):\ w(u+v)),$$

which is the inferior of the isotomic conjugate of P. The lines $B'_aC'_a$, $C'_bA'_b$, $A'_cB'_c$ are antiparallels if and only if the homothetic center is the symmedian point. Therefore, the conic $\mathcal{C}'(P)$ is a circle if and only if P is the isotomic conjugate of the superior of K, namely, the orthocenter H. The resulting circle is the Taylor circle.

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