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# On Six Circumcenters and Their Concyclicity

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**Abstract**. Given triangle $ABC$, let $P$ be a point with circumcevian triangle $A'B'C'$. We determine the positions of $P$ such that the circumcenters of the six circles $PBC'$, $PB'C$, $PCA'$, $PC'A$, $PAB'$, $PA'B$ are concyclic. There are two such real points $P$ which lie on the Euler line of $ABC$ provided the triangle is acute-angled. We provide two simple constructions of such points.

In the plane of a given triangle $ABC$ with circumcenter $O$, consider a point $P$ with its circumcevian triangle $A'B'C'$. In Theorem 1 below we show that the centers of the six circles $PBC'$, $PB'C$, $PCA'$, $PC'A$, $PAB'$, $PA'B$ form three segments sharing a common midpoint $M$ with $OP$. It follows that these six circumcenters lie on a conic $\mathcal{C}(P)$. We proceed to identify the point $P$ for which this conic is a circle. It turns out (Theorem 1 below) that there are two such real points lying on the Euler line when the given triangle is acute-angled, and these points can be easily constructed with ruler and compass.

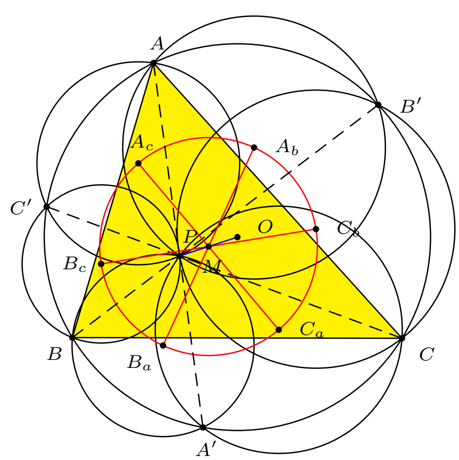


Figure 1. Six centers on a conic

Denote by $B_c$, $C_b$, $C_a$, $A_c$, $A_b$, $B_a$ the centers of the circles $PBC'$, $PCB'$, $PCA'$, $PAC'$, $PAB'$, $PBA'$ respectively, and by $r_{bc}$, $r_{cb}$, $r_{ca}$, $r_{ac}$, $r_{ab}$, $r_{ba}$ their radii. Let $R$ be the circumradius of triangle $ABC$.

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**Theorem 1.** *The segments $B_cC_b$, $C_aA_c$, $A_bB_a$, and $OP$ share a common midpoint.*

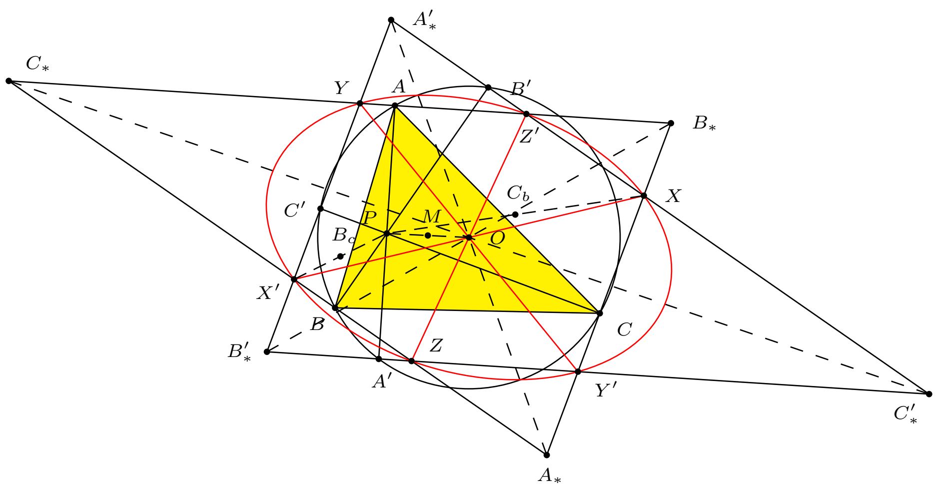


Figure 2. Antipedal triangle of $P$ and its reflection in $O$

*Proof.* Consider the lines perpendicular to $AP$, $BP$, $CP$ at $A$, $B$, $C$ respectively. These lines bound the antipedal triangle $A_*B_*C_*$ of $P$. If we draw the corresponding lines at $A'$, $B'$, $C'$ perpendicular to $A'P$, $B'P$, $C'P$, we obtain a triangle $A'_*B'_*C'_*$ oppositely homothetic to $A_*B_*C_*$. Since the parallel through the circumcenter $O$ (of triangle $ABC$) to $B_*C_*$ and $B'_*C'_*$ passes through the midpoint of $AA'$, $O$ is equidistant from the parallel lines $B_*C_*$ and $B'_*C'_*$. The same is true for the other two pairs of lines $C_*A_*$, $C'_*A'_*$, and $A_*B_*$, $A'_*B'_*$. Therefore, the two triangles $A_*B_*C_*$ and $A'_*B'_*C'_*$ are oppositely congruent at $O$ (see Figure 2). By symmetry, their sidelines intersect at six points which are pairwise symmetric in $O$. These are the points

$$
\begin{aligned}
X &:= A_*B_* \cap C'_*A'_*, &\qquad X' &:= A'_*B'_* \cap C_*A_*;\\
Y &:= B_*C_* \cap A'_*B'_*, &\qquad Y' &:= B'_*C'_* \cap A_*B_*;\\
Z &:= C_*A_* \cap B'_*C'_*, &\qquad Z' &:= C'_*A'_* \cap B_*C_*.
\end{aligned}
$$

The six circumcenters $B_c$, $C_b$, $C_a$, $A_c$, $A_b$, $B_a$ are the images of $X'$, $X$, $Y'$, $Y$, $Z'$, $Z$ under the homothety $\mathsf{h}\left(P, \frac{1}{2}\right)$. It follows that the segments $B_cC_b$, $C_aA_c$, $A_bB_a$ share a common midpoint, which is $\mathsf{h}\left(P, \frac{1}{2}\right)(O)$, the midpoint of $OP$. □

We determine the location of $P$ for which the conic through these six circumcenters is a circle. Clearly, this is case if and only if $B_cC_b = C_aA_c = A_bB_a$.

**Lemma 2.** *The radii of the circles $PBC'$, $PCB'$, $PCA'$, $PAC'$, $PAB'$, $PBA'$ are*

$$r_{bc} = \frac{R}{a} \cdot BP, \qquad r_{cb} = \frac{R}{a} \cdot CP;$$
$$r_{ca} = \frac{R}{b} \cdot CP, \qquad r_{ac} = \frac{R}{b} \cdot AP;$$
$$r_{ab} = \frac{R}{c} \cdot AP, \qquad r_{ba} = \frac{R}{c} \cdot BP.$$

*Proof.* It is enough to establish the expression for $r_{bc}$. The others follow similarly.

Note that $\angle BC'P = \angle BC'C = \angle BAC$. Applying the law of sines to triangles$PBC'$ and $BCC'$, we have

$$r_{bc} = \frac{BP}{2\sin BC'P} = \frac{BP}{2\sin BAC} = \frac{R}{BC} \cdot BP = \frac{R}{a} \cdot BP.$$

□

**Theorem 3.** *Let $A_1$, $B_1$, $C_1$ be the midpoints of $BC$, $CA$, $AB$ respectively. The six circumcenters lie on a circle if and only if*

$$A_1P : B_1P : C_1P = B_1C_1 : C_1A_1 : A_1B_1. \tag{1}$$

*Proof.* Let $M$ be the common midpoint of $OP$, $B_cC_b$, $C_aA_c$, $A_bB_a$. Clearly the conic through the six circumcenter is a circle if and only if $B_cC_b = C_aA_c = A_bB_a$. Applying Apollonius' theorem to the triangles $PB_cC_b$ and $PBC$, making use of Lemma 2, we have

$$2PM^2 + \frac{B_cC_b^2}{2} = r_{bc}^2 + r_{cb}^2 = \frac{R^2}{a^2}(BP^2 + CP^2) = \frac{R^2}{2}\left(\frac{A_1P^2}{B_1C_1^2} + 1\right). \tag{2}$$

Similarly,

$$2PM^2 + \frac{C_aA_c^2}{2} = \frac{R^2}{2}\left(\frac{B_1P^2}{C_1A_1^2} + 1\right), \tag{3}$$

$$2PM^2 + \frac{A_bB_a^2}{2} = \frac{R^2}{2}\left(\frac{C_1P^2}{A_1B_1^2} + 1\right). \tag{4}$$

Comparison of (2), (3) and (4) yields (1) as a necessary and sufficient condition for $B_cC_b = C_aA_c = A_bB_a$; hence for the six circumcenters to lie on a circle. □

Now we identify the points $P$ satisfying the condition (1).

Let $A_2$, $B_2$, $C_2$ be the midpoints of $B_1C_1$, $C_1A_1$, $A_1B_1$ respectively. Consider the reflections $P_a$, $P_b$, $P_c$ of $P$ in $A_2$, $B_2$, $C_2$ respectively. Since $P_aB_1 = PC_1$ and $P_aC_1 = PB_1$, we have $\frac{P_aC_1}{C_1A_1} = \frac{P_aB_1}{A_1B_1}$ or $\frac{P_aB_1}{P_aC_1} = \frac{A_1B_1}{A_1C_1}$. This means that $P_a$ is on the $A_1$-Apollonian circle of triangle $A_1B_1C_1$. Equivalently, $P$ is a point on the circle $\mathscr{C}_a$ which is the reflection of the $A_1$-Apollonian circle of $A_1B_1C_1$ in the perpendicular bisector of $B_1C_1$. For the same reason, $P$ also lies on the two circles $\mathscr{C}_b$ and $\mathscr{C}_c$, which are the reflections of the $B_1$- and $C_1$-Apollonian circles in the perpendicular bisectors of $C_1A_1$ and $A_1B_1$ respectively.

The circle $\mathscr{C}_a$ passes through $A$, $H_a$ the trace of $H$ on $BC$, and the intersection of $B_1C_1$ with the internal bisector of angle $A$ of $ABC$. Hence the diameter $AO$ of $ABC$ is tangent to this circle. This leads to the following simple barycentric equations of $\mathscr{C}_a$, and the other two circles.

$$\mathscr{C}_a: \quad (S_B - S_C)(a^2yz + b^2zx + c^2xy) - (x+y+z)(c^2S_By - b^2S_Cz) = 0,$$
$$\mathscr{C}_b: \quad (S_C - S_A)(a^2yz + b^2zx + c^2xy) - (x+y+z)(a^2S_Cz - c^2S_Ax) = 0,$$
$$\mathscr{C}_c: \quad (S_A - S_B)(a^2yz + b^2zx + c^2xy) - (x+y+z)(b^2S_Ax - a^2S_By) = 0.$$

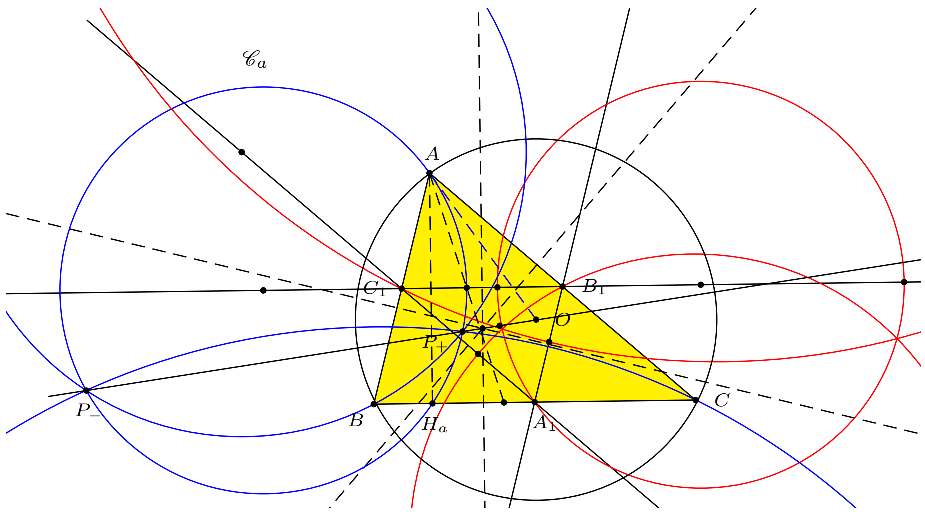


Figure 3. $P_\pm$ as intersections of reflections of Apollonian circles

**Proposition 4.** *The two points $P_\pm$ lie on the Euler line of triangle $ABC$.*

*Proof.* If $AP = \lambda$, $BP = \mu$, $CP = \nu$ are the tripolar coordinates of $P$ with reference to $ABC$ (see [1]), then from (1), we have

$$\frac{2(\mu^2+\nu^2)-a^2}{a^2} = \frac{2(\nu^2+\lambda^2)-b^2}{b^2} = \frac{2(\lambda^2+\mu^2)-c^2}{c^2},$$

or

$$\frac{\mu^2+\nu^2}{a^2} = \frac{\nu^2+\lambda^2}{b^2} = \frac{\lambda^2+\mu^2}{c^2} = k,$$

for some $k$. Hence,

$$\lambda^2 = \frac{k(b^2+c^2-a^2)}{2}, \qquad \mu^2 = \frac{k(c^2+a^2-b^2)}{2}, \qquad \nu^2 = \frac{k(a^2+b^2-c^2)}{2}, \tag{5}$$

and

$$(b^2-c^2)\lambda^2 + (c^2-a^2)\mu^2 + (a^2-b^2)\nu^2 = 0.$$

This is the equation of the Euler line in tripolar coordinates ([1, Proposition 3]). □

Representing the circle $\mathscr{C}_a$ by the matrix

$$M_a = \begin{pmatrix} 0 & -S_C(S_A+S_B) & S_B(S_C+S_A) \\ -S_C(S_A+S_B) & -2S_B(S_A+S_B) & -S_A(S_B-S_C) \\ S_B(S_C+S_A) & -S_A(S_B-S_C) & 2S_C(S_C+S_A) \end{pmatrix},$$

we compute the equation of the polar of $G$ in the circle. This gives

$$\begin{pmatrix} x & y & z \end{pmatrix} M_a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0,$$

or

$$S_A(S_B-S_C)x - S_B(3S_A+2S_B+S_C)y + S_C(3S_A+S_B+2S_C)z = 0.$$

Clearly, this polar contains the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$. This shows that $G$ and $H$ are conjugate in the circle $\mathscr{C}_a$; similarly also in the circles $\mathscr{C}_b$ and $\mathscr{C}_c$. Therefore, $G$ and $H$ divide $P_+$ and $P_-$ harmonically.

**Theorem 5.** *The two points satisfying* (1) *are*

$$P_\varepsilon := (\sqrt{S_A+S_B+S_C}\cdot S_{BC} + \varepsilon S\cdot\sqrt{S_{ABC}} : \cdots : \cdots), \quad \varepsilon = \pm 1$$

*in homogeneous barycentric coordinates.*

*Proof.* Let $P = (S_{BC}+t : S_{CA}+t : S_{AB}+t)$. We have

$$\begin{aligned} 0 = & \begin{pmatrix} S_{BC}+t & S_{CA}+t & S_{AB}+t \end{pmatrix} M_a \begin{pmatrix} S_{BC}+t \\ S_{CA}+t \\ S_{AB}+t \end{pmatrix} \\ = & \begin{pmatrix} S_{BC} & S_{CA} & S_{AB} \end{pmatrix} M_a \begin{pmatrix} S_{BC} \\ S_{CA} \\ S_{AB} \end{pmatrix} + \begin{pmatrix} t & t & t \end{pmatrix} M_a \begin{pmatrix} t \\ t \\ t \end{pmatrix} \\ = & 2S_{ABC}(S_B-S_C)(S_{BC}+S_{CA}+S_{AB}) - 2t^2(S_A+S_B+S_C)(S_B-S_C). \end{aligned}$$

It follows that $t^2 = \frac{S^2\cdot S_{ABC}}{S_A+S_B+S_C}$. From these we obtain the coordinates of the two points $P_\pm$ given above. □

**Proposition 6.** *The midpoint of the segment $P_+P_-$ is the point*

$$Q = (S_{BC}(S_B+S_C-2S_A) : S_{CA}(S_C+S_A-2S_B) : S_{AB}(S_A+S_B-2S_C)).$$

*Proof.* The midpoint between the two points $(S_{BC}+t : S_{CA}+t : S_{AB}+t)$ and $(S_{BC}-t : S_{CA}-t : S_{AB}-t)$ has coordinates

$$\begin{aligned} & (S_{BC}+S_{CA}+S_{AB}-3t)(S_{BC}+t, S_{CA}+t, S_{AB}+t)) \\ + & (S_{BC}+S_{CA}+S_{AB}+3t)(S_{BC}-t, S_{CA}-t, S_{AB}-t)) \\ = & (S_{BC}S^2-3t^2, S_{CA}S^2-3t^2, S_{AB}S^2-3t^2). \end{aligned}$$

For $t^2 = \frac{S_{ABC}\cdot S^2}{S_A+S_B+S_C}$, this simplifies into the form given above. □

The point $Q$ is the intersection of the Euler line and the orthic axis. [1]

This leads to the simple construction of the two points $P_+$ and $P_-$ as the intersections of the circle with center $Q$, orthogonal to the orthocentroidal circle (see Figure 4).

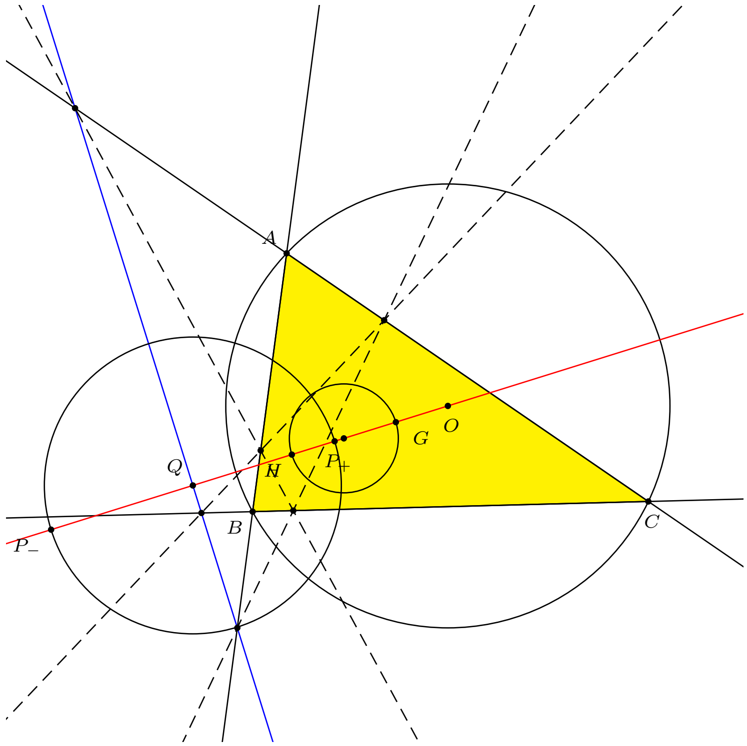


Figure 4. Construction of $P_+$ and $P_-$

We conclude this note with the remark that the problem of construction of points whose distances from the vertices of a given triangle are proportional to the lengths of the opposite sides was addressed in [4]. Also, according to [5], these two points can also be constructed as the common points of the triad of generalized Apollonian circles for the isotomic conjugate of the incenter, namely, the triangle center $X_{75} = \left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

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