

# Generalization and Extension of the Wallace Theorem

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**Abstract.** In the Wallace theorem we replace the projection directions (altitudes of the reference triangle) by all permutations of a general direction triple, and regard simultaneously the projections of a point  $P$  to each sideline. Introducing a pair of *Wallace points* and a pair of *Wallace triangles*, we present their properties and some connections to the Steiner ellipses.

## 1. Introduction

Most people interested in triangle geometry know the Wallace-Simson Theorem (see [2], [3] or [4]):

In the euclidean plane be  $ABC$  a triangle and  $P$  a point not on the sidelines. Then the feet of the perpendiculars from  $P$  to the sidelines are collinear (*Wallace-Simson line*), if and only if  $P$  is a point on the circumcircle of  $ABC$ .

This theorem is one of the gems of triangle geometry. For more than two centuries mathematicians are fascinated about its simplicity and beauty, and they reflected on generalizations or extensions up to the present time.

O. Giering [1] showed that not only the collinearity of the three pedals, but also the collinearities of other intersections of the projection lines (in direction of the altitudes) with the sidelines of the triangle are interesting in this respect.

In a paper of M. de Guzmán [2] it is shown that one can take instead altitude directions a general triple  $(\alpha, \beta, \gamma)$  of projection directions which are assigned to the oriented side triple  $(a, b, c)$ . One gets instead the circumcircle a circumconic for which it is easy to construct three points (apart from  $A, B, C$ ) and the center.

In this paper we aim at continuing some ideas of the above publications. We consider the permutations of a triple of projection directions simultaneously, and the concepts *Wallace points* and *Wallace triangles* yield new interesting insights.

## 2. Notations

First of all, we recall some concepts and connections of the euclidean triangle geometry. Detailed information can be found, for instance, in the books of R. A. Johnson [4] and P. Yiu [7], or in papers of S. Sigur [5].

Let  $\Delta = ABC$  be a triangle with the vertices  $A, B, C$ , the sides  $a, b, c$ , and the centroid  $G$ . For the representation of geometric elements we use homogeneous barycentric coordinates.

Suppose  $P = (u : v : w)$  is a general point. Reflecting the traces  $P_a, P_b, P_c$  of  $P$  in the midpoints  $G_a, G_b, G_c$  of the sides, respectively, then the points of reflection  $P_a^\bullet, P_b^\bullet, P_c^\bullet$  are the traces of the (*isotomic*) *conjugate*  $P^\bullet = (\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$  of  $P$ .

The line  $[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}]$  is the *trilinear polar* (*tripolar*)  $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$  of  $P$ , the line  $[u : v : w]$  is the *dual* (the tripolar of the conjugate) of  $P$  and  $\mathcal{C}_P : \frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0$  is a circumconic of  $\Delta$  with *perspector*  $P$  (*P-circumconic*). A perspector of a circumconic  $\mathcal{C}$  is the perspective center of  $\Delta$  and the triangle formed by the tangents of  $\mathcal{C}$  at  $A, B, C$ . The center  $M_P$  of  $\mathcal{C}_P$  has coordinates

$$(u(v+w-u) : v(w+u-v) : w(u+v-w)). \quad (1)$$

The point by point conjugation of  $\mathcal{C}_P$  yields the dual line of  $P$ . The duals of all points of  $\mathcal{C}_P$  form a family of lines whose envelope is the inconic associated to the circumconic  $\mathcal{C}_P$ .

The points of the infinite line  $l_\infty$  satisfy the equation  $x + y + z = 0$ .

The *medial* operation  $m$  and the *dilated* (*antimedial*) operation  $d$  carry a point  $P$  to the images  $mP = (v+w : w+u : u+v)$  and  $dP = (v+w-u : w+u-v : u+v-w)$ , respectively, which both lie on the line  $GP$ :

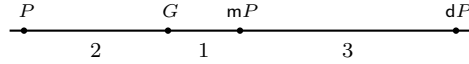


Figure 1. Medial and dilated operation

The point  $(u : v : w)$  forms together with the points  $(v : w : u)$  and  $(w : u : v)$  a *Brocardian triple* [6]; every two of these points are the right-right Brocardian and the left-left Brocardian, respectively, of the third point.

The *Steiner circumellipse*  $\mathcal{C}_G$  of  $\Delta$  has the equation

$$yz + zx + xy = 0, \quad (2)$$

and the *Steiner inellipse* is described by

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0. \quad (3)$$

The *Kiepert hyperbola* is the (rectangular) circumconic of  $\Delta$  through  $G$  and the orthocenter  $H$ .

### 3. Direction Stars, Projection Triples and their Normalized Representation

Let us call a *direction star* a set  $\{\alpha, \beta, \gamma\}$  of three pairwise different directions  $\alpha, \beta, \gamma$  not parallel to the sides of  $\Delta$ . It is described by three points

$$\alpha = (\alpha_1 : \alpha_2 : \alpha_3), \quad \beta = (\beta_1 : \beta_2 : \beta_3), \quad \gamma = (\gamma_1 : \gamma_2 : \gamma_3)$$

on the infinite line. Their barycentrics (different from zero) form a singular matrix

$$D = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

of rank 2. Since the coordinates of each point are defined except for a non-zero factor, we can adjust by suitable factors so that all cofactors of  $D$  are equal to unity. We call such representation of a direction star its *normalized representation*. In this case not only the row sums of  $D$  vanish, but also the column sums, and

$$\beta_2 - \gamma_3 = \gamma_1 - \alpha_2 = \alpha_3 - \beta_1 =: \lambda_1, \quad (4)$$

$$\gamma_3 - \alpha_1 = \alpha_2 - \beta_3 = \beta_1 - \gamma_2 =: \lambda_2, \quad (5)$$

$$\alpha_1 - \beta_2 = \beta_3 - \gamma_1 = \gamma_2 - \alpha_3 =: \lambda_3 \quad (6)$$

and

$$\beta_3 - \gamma_2 = \gamma_1 - \alpha_3 = \alpha_2 - \beta_1 =: \mu_1, \quad (7)$$

$$\gamma_2 - \alpha_1 = \alpha_3 - \beta_2 = \beta_1 - \gamma_3 =: \mu_2, \quad (8)$$

$$\alpha_1 - \beta_3 = \beta_2 - \gamma_1 = \gamma_3 - \alpha_2 =: \mu_3. \quad (9)$$

Here is an example of a normalized representation of a direction star:

$$D = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & -4 \\ -2 & -5 & 7 \end{pmatrix}.$$

We will see below that two other matrices with the same elements as in  $D$  (but in other arrangements) are also involved. The rows of  $D_{\rightarrow}$  ( $D_{\leftarrow}$ ) consist of the elements of the main (skew) diagonal and their parallels:

$$D_{\rightarrow} := \begin{pmatrix} \alpha_1 & \beta_2 & \gamma_3 \\ \beta_1 & \gamma_2 & \alpha_3 \\ \gamma_1 & \alpha_2 & \beta_3 \end{pmatrix}, \quad D_{\leftarrow} := \begin{pmatrix} \alpha_1 & \gamma_2 & \beta_3 \\ \beta_1 & \alpha_2 & \gamma_3 \\ \gamma_1 & \beta_2 & \alpha_3 \end{pmatrix}.$$

From a direction star we form  $3! = 6$  ordered direction triples (permutations of the directions), which we can interpret as projection directions on the sidelines  $a$ ,  $b$ ,  $c$  (in this order). We denote these *projection triples* by

$$\begin{aligned} \alpha_{\rightarrow} &:= (\alpha, \beta, \gamma), & \alpha_{\leftarrow} &:= (\alpha, \gamma, \beta); \\ \beta_{\rightarrow} &:= (\beta, \gamma, \alpha), & \beta_{\leftarrow} &:= (\beta, \alpha, \gamma); \\ \gamma_{\rightarrow} &:= (\gamma, \alpha, \beta), & \gamma_{\leftarrow} &:= (\gamma, \beta, \alpha). \end{aligned}$$

The arrows indicate whether the permutation is even or odd. Interpreting as a map, for instance  $\alpha_{\leftarrow}(P)$  is a triple  $(P_{\alpha a}, P_{\gamma b}, P_{\beta c})$  of feet in which the first index indicates the projection direction, and the second one refers to the side on which  $P$  is projected.

The square matrices  $D$ ,  $D_{\rightarrow}$  and  $D_{\leftarrow}$  all have rank 2. Their kernels represent geometrically some points in the plane of  $\Delta$ . The kernel of  $D$  is obviously  $G =$

(1, 1, 1). For  $\ker D_{\rightarrow} =: (p_{\rightarrow} : q_{\rightarrow} : r_{\rightarrow})$  and  $\ker D_{\leftarrow} =: (p_{\leftarrow} : q_{\leftarrow} : r_{\leftarrow})$  we find

$$p_{\rightarrow} = \alpha_2\alpha_3 - \beta_3\gamma_2 = \beta_2\beta_3 - \gamma_3\alpha_2 = \gamma_2\gamma_3 - \alpha_3\beta_2, \quad (10)$$

$$q_{\rightarrow} = \alpha_3\alpha_1 - \beta_1\gamma_3 = \beta_3\beta_1 - \gamma_1\alpha_3 = \gamma_3\gamma_1 - \alpha_1\beta_3, \quad (11)$$

$$r_{\rightarrow} = \alpha_1\alpha_2 - \beta_2\gamma_1 = \beta_1\beta_2 - \gamma_2\alpha_1 = \gamma_1\gamma_2 - \alpha_2\beta_1, \quad (12)$$

and

$$p_{\leftarrow} = \alpha_2\alpha_3 - \beta_2\gamma_3 = \beta_2\beta_3 - \gamma_2\alpha_3 = \gamma_2\gamma_3 - \alpha_2\beta_3, \quad (13)$$

$$q_{\leftarrow} = \alpha_3\alpha_1 - \beta_3\gamma_1 = \beta_3\beta_1 - \gamma_3\alpha_1 = \gamma_3\gamma_1 - \alpha_3\beta_1, \quad (14)$$

$$r_{\leftarrow} = \alpha_1\alpha_2 - \beta_1\gamma_2 = \beta_1\beta_2 - \gamma_1\alpha_2 = \gamma_1\gamma_2 - \alpha_1\beta_2. \quad (15)$$

These satisfy

$$p_{\rightarrow} - p_{\leftarrow} = q_{\rightarrow} - q_{\leftarrow} = r_{\rightarrow} - r_{\leftarrow} = 1, \quad (16)$$

$$p_{\rightarrow}q_{\rightarrow} + q_{\rightarrow}r_{\rightarrow} + r_{\rightarrow}p_{\rightarrow} - p_{\rightarrow} - q_{\rightarrow} - r_{\rightarrow} = 0, \quad (17)$$

$$p_{\leftarrow}q_{\leftarrow} + q_{\leftarrow}r_{\leftarrow} + r_{\leftarrow}p_{\leftarrow} + p_{\leftarrow} + q_{\leftarrow} + r_{\leftarrow} = 0. \quad (18)$$

Let us denote by  $\ell_{Qq}$  the line with direction  $q$  through a point  $Q$ . Then the direction stars localized at the vertices  $A, B, C$  are described by the following lines:

$$\begin{aligned} \ell_{A\alpha} &= [0 : \alpha_3 : -\alpha_2], & \ell_{B\alpha} &= [-\alpha_3 : 0 : \alpha_1], & \ell_{C\alpha} &= [\alpha_2 : -\alpha_1 : 0]; \\ \ell_{A\beta} &= [0 : \beta_3 : -\beta_2], & \ell_{B\beta} &= [-\beta_3 : 0 : \beta_1], & \ell_{C\beta} &= [\beta_2 : -\beta_1 : 0]; \\ \ell_{A\gamma} &= [0 : \gamma_3 : -\gamma_2], & \ell_{B\gamma} &= [-\gamma_3 : 0 : \gamma_1], & \ell_{C\gamma} &= [\gamma_2 : -\gamma_1 : 0]. \end{aligned}$$

Next we want to assign each projection triple to a specific line. We begin with the construction of such a line  $\ell_{\alpha_{\rightarrow}}$  for the projection triple  $\alpha_{\rightarrow}$ . Let

$$P_1 := \ell_{B\gamma} \cap \ell_{C\beta} = (\beta_1\gamma_1 : \beta_2\gamma_1 : \beta_1\gamma_3), \quad (19)$$

$$P_2 := \ell_{C\alpha} \cap \ell_{A\gamma} = (\gamma_2\alpha_1 : \gamma_2\alpha_2 : \gamma_3\alpha_2), \quad (20)$$

$$P_3 := \ell_{A\beta} \cap \ell_{B\alpha} = (\alpha_1\beta_3 : \alpha_3\beta_2 : \alpha_3\beta_3). \quad (21)$$

Their conjugates are

$$\begin{aligned} P_1^{\bullet} &= (\beta_2\gamma_3 : \beta_1\gamma_3 : \beta_2\gamma_1), \\ P_2^{\bullet} &= (\gamma_3\alpha_2 : \gamma_3\alpha_1 : \gamma_2\alpha_1), \\ P_3^{\bullet} &= (\alpha_3\beta_2 : \alpha_1\beta_3 : \alpha_1\beta_2). \end{aligned} \quad (22)$$

In view of (4), (5), (6) it is clear that  $\det(P_1^{\bullet}, P_2^{\bullet}, P_3^{\bullet}) = 0$ . Hence, these points are collinear and lie on the line

$$\ell_{\alpha_{\rightarrow}} := [\alpha_1 : \beta_2 : \gamma_3], \quad (23)$$

which intersects the infinite line in  $(\lambda_1 : \lambda_2 : \lambda_3)$ . By cyclic interchange of  $\alpha, \beta, \gamma$  we find

$$\ell_{\beta_{\rightarrow}} := [\beta_1 : \gamma_2 : \alpha_3], \quad \ell_{\gamma_{\rightarrow}} := [\gamma_1 : \alpha_2 : \beta_3], \quad (24)$$

and the intersections  $(\lambda_3 : \lambda_1 : \lambda_2)$  and  $(\lambda_2 : \lambda_3 : \lambda_1)$  with the infinite line, respectively. The barycentrics of these three lines form the rows of the matrix  $D_{\rightarrow}$ . In a similar fashion we find the lines

$$l_{\alpha\leftarrow} = [\alpha_1 : \gamma_2 : \beta_3], \quad l_{\beta\leftarrow} = [\beta_1 : \alpha_2 : \gamma_3], \quad l_{\gamma\leftarrow} = [\gamma_1 : \beta_2 : \alpha_3] \quad (25)$$

whose coordinates form the rows of  $D_{\leftarrow}$ . From these we have the theorem below.

**Theorem 1.** *The lines  $l_{\alpha\rightarrow}$ ,  $l_{\beta\rightarrow}$ ,  $l_{\gamma\rightarrow}$  are concurrent at the point*

$$W_{\rightarrow}^{\bullet} = (p_{\rightarrow} : q_{\rightarrow} : r_{\rightarrow}).$$

*Likewise, the lines  $l_{\alpha\leftarrow}$ ,  $l_{\beta\leftarrow}$ ,  $l_{\gamma\leftarrow}$  are concurrent at*

$$W_{\leftarrow}^{\bullet} = (p_{\leftarrow} : q_{\leftarrow} : r_{\leftarrow}).$$

Recall that the conjugates of the points of a line lie on a circumconic of  $\Delta$ . Hence the conjugates of the six lines in (23) - (25) are the circumconics

$$C_{\alpha\rightarrow} : \frac{\alpha_1}{x} + \frac{\beta_2}{y} + \frac{\gamma_3}{z} = 0, \quad C_{\beta\rightarrow} : \frac{\beta_1}{x} + \frac{\gamma_2}{y} + \frac{\alpha_3}{z} = 0, \quad C_{\gamma\rightarrow} : \frac{\gamma_1}{x} + \frac{\alpha_2}{y} + \frac{\beta_3}{z} = 0; \quad (26)$$

$$C_{\alpha\leftarrow} : \frac{\alpha_1}{x} + \frac{\gamma_2}{y} + \frac{\beta_3}{z} = 0, \quad C_{\beta\leftarrow} : \frac{\beta_1}{x} + \frac{\alpha_2}{y} + \frac{\gamma_3}{z} = 0, \quad C_{\gamma\leftarrow} : \frac{\gamma_1}{x} + \frac{\beta_2}{y} + \frac{\alpha_3}{z} = 0. \quad (27)$$

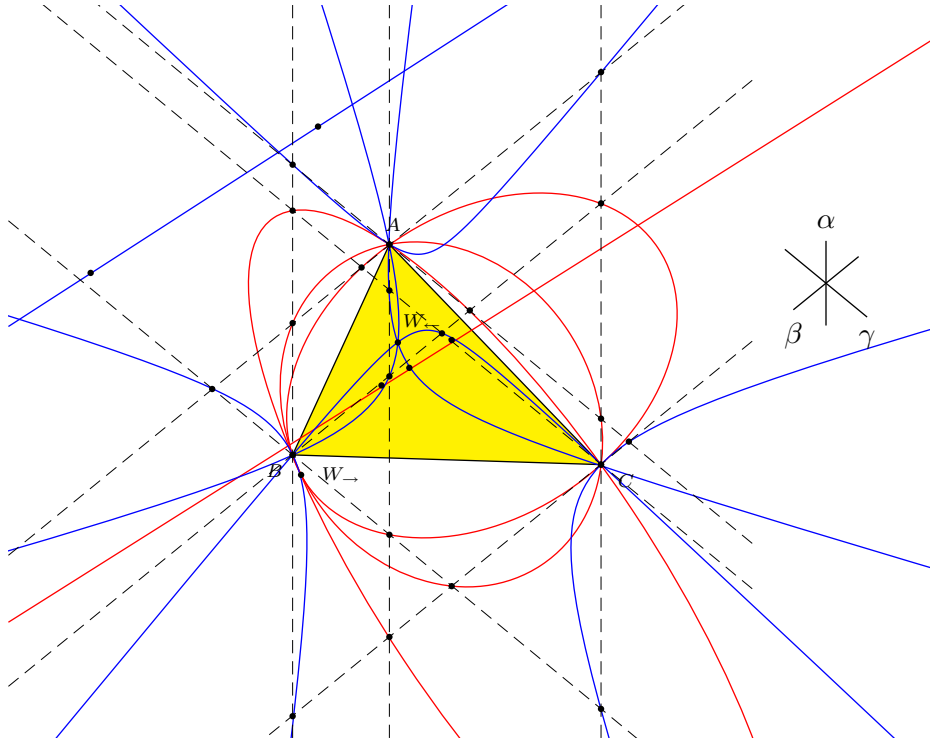


Figure 2.

Theorem 2 below follows easily from Theorem 1.

**Theorem 2.** *The circumconics  $\mathcal{C}_{\alpha\rightarrow}$ ,  $\mathcal{C}_{\beta\rightarrow}$ ,  $\mathcal{C}_{\gamma\rightarrow}$  (red in Figure 2) have the common point*

$$W_{\rightarrow} = \left( \frac{1}{p_{\rightarrow}} : \frac{1}{q_{\rightarrow}} : \frac{1}{r_{\rightarrow}} \right),$$

*the circumconics  $\mathcal{C}_{\alpha\leftarrow}$ ,  $\mathcal{C}_{\beta\leftarrow}$ ,  $\mathcal{C}_{\gamma\leftarrow}$  (blue in Figure 2) have the common point*

$$W_{\leftarrow} = \left( \frac{1}{p_{\leftarrow}} : \frac{1}{q_{\leftarrow}} : \frac{1}{r_{\leftarrow}} \right).$$

*Hence, their perspectors are collinear on the tripolars of  $W_{\rightarrow}$  and of  $W_{\leftarrow}$ , respectively. These lines are parallel and they intersect the infinite line at the point*

$$W_{\infty} = (q_{\rightarrow} - r_{\rightarrow} : r_{\rightarrow} - p_{\rightarrow} : p_{\rightarrow} - q_{\rightarrow})$$

*and define a direction  $\delta$ .*

In the special case of altitudes is  $W_{\rightarrow}$  the Tarry point and  $W_{\leftarrow}$  the orthocenter of  $\Delta$ . The circumconic  $\mathcal{C}_{\alpha\rightarrow}$  is the circumcircle. In [1],  $\mathcal{C}_{\beta\rightarrow}$  and  $\mathcal{C}_{\gamma\rightarrow}$  are called the right- and left-conics respectively.

#### 4. Wallace Points

In [2] it is shown that in the case of three directions  $\alpha, \beta, \gamma$  the points  $P_1, P_2, P_3$  constructed for the projection triple  $\alpha_{\rightarrow}$  lie on a circumconic with the property that for a point  $P$  on this circumconic the feet of the projections of  $P$  to  $a, b, c$  in direction  $\alpha, \beta, \gamma$ , respectively, are collinear. Now we want to look at this generalization of the theorem of Wallace *simultaneously* for all 6 projection triples belonging to the direction star  $\{\alpha, \beta, \gamma\}$ .

**Theorem 3.** *The respective three feet of the three projection triples  $\alpha_{\rightarrow}(W_{\rightarrow})$ ,  $\beta_{\rightarrow}(W_{\rightarrow})$  and  $\gamma_{\rightarrow}(W_{\rightarrow})$  localized at  $W_{\rightarrow}$  are collinear on the Wallace lines  $w_{\alpha\rightarrow}$ ,  $w_{\beta\rightarrow}$ ,  $w_{\gamma\rightarrow}$ , respectively; there is analogy for the feet of  $\alpha_{\leftarrow}(W_{\leftarrow})$ ,  $\beta_{\leftarrow}(W_{\leftarrow})$ ,  $\gamma_{\leftarrow}(W_{\leftarrow})$ . We shall call the points  $W_{\rightarrow}$  and  $W_{\leftarrow}$  the Wallace-right- and Wallace-left-points respectively of the direction star  $\{\alpha, \beta, \gamma\}$ .*

*Proof.* Let  $g_{\alpha\rightarrow}$ ,  $g_{\beta\rightarrow}$ ,  $g_{\gamma\rightarrow}$  be the lines through  $W_{\rightarrow}$  in direction  $\alpha, \beta, \gamma$ , respectively. To simplify the equations we make use of the quantities

$$\begin{aligned} X_1 &:= \alpha_2 q_{\rightarrow} - \alpha_3 r_{\rightarrow} = \gamma_3 r_{\rightarrow} - \gamma_1 p_{\rightarrow} = \beta_1 p_{\rightarrow} - \beta_2 q_{\rightarrow} \\ X_2 &:= \beta_2 q_{\rightarrow} - \beta_3 r_{\rightarrow} = \alpha_3 r_{\rightarrow} - \alpha_1 p_{\rightarrow} = \gamma_1 p_{\rightarrow} - \gamma_2 q_{\rightarrow} \\ X_3 &:= \gamma_2 q_{\rightarrow} - \gamma_3 r_{\rightarrow} = \beta_3 r_{\rightarrow} - \beta_1 p_{\rightarrow} = \alpha_1 p_{\rightarrow} - \alpha_2 q_{\rightarrow}. \end{aligned}$$

These satisfy

$$X_1^2 - X_2 X_3 = X_2^2 - X_3 X_1 = X_3^2 - X_1 X_2, \quad (28)$$

and yield the equations of the lines

$$\begin{aligned} g_{\alpha\rightarrow} &= [p\rightarrow X_1 : q\rightarrow X_2 : r\rightarrow X_3] \\ g_{\beta\rightarrow} &= [p\rightarrow X_2 : q\rightarrow X_3 : r\rightarrow X_1] \\ g_{\gamma\rightarrow} &= [p\rightarrow X_3 : q\rightarrow X_1 : r\rightarrow X_2]. \end{aligned}$$

These projection lines intersect the sidelines in the points

$$\begin{aligned} Q_{\alpha\alpha} &= (0 : r\rightarrow X_3 : -q\rightarrow X_2), & Q_{\beta\alpha} &= (0 : r\rightarrow X_1 : -q\rightarrow X_3), & Q_{\gamma\alpha} &= (0 : r\rightarrow X_2 : -q\rightarrow X_1); \\ Q_{\alpha\beta} &= (-r\rightarrow X_3 : 0 : p\rightarrow X_1), & Q_{\beta\beta} &= (-r\rightarrow X_1 : 0 : p\rightarrow X_2), & Q_{\gamma\beta} &= (-r\rightarrow X_2 : 0 : p\rightarrow X_3); \\ Q_{\alpha\gamma} &= (q\rightarrow X_2 : -p\rightarrow X_1 : 0), & Q_{\beta\gamma} &= (q\rightarrow X_3 : -p\rightarrow X_2 : 0), & Q_{\gamma\gamma} &= (q\rightarrow X_1 : -p\rightarrow X_3 : 0). \end{aligned}$$

The feet  $Q_{\alpha\alpha}$ ,  $Q_{\beta\beta}$ ,  $Q_{\gamma\gamma}$  of the projection triple  $\alpha\rightarrow$  are collinear because their linear dependent coordinates. They yield a Wallace line

$$w_{\alpha\rightarrow} = Q_{\alpha\alpha}Q_{\beta\beta} = [p\rightarrow X_2 X_3 : q\rightarrow X_1 X_2 : r\rightarrow X_3 X_1].$$

Analogously it follows from the collinearity of  $Q_{\alpha\beta}$ ,  $Q_{\beta\gamma}$ ,  $Q_{\gamma\alpha}$  resp.  $Q_{\alpha\gamma}$ ,  $Q_{\beta\alpha}$ ,  $Q_{\gamma\beta}$

$$w_{\beta\rightarrow} = [p\rightarrow X_1 X_2 : q\rightarrow X_3 X_1 : r\rightarrow X_2 X_3], \quad w_{\gamma\rightarrow} = [p\rightarrow X_3 X_1 : q\rightarrow X_2 X_3 : r\rightarrow X_1 X_2].$$

The proof for the other Wallace point is analogous.  $\square$

## 5. Some circumconics generated by the Wallace points

The Wallace points generate some circumconics with notable properties:

- $W_{\rightarrow}^{\bullet}$ -circumconic  $\mathcal{C}_{W_{\rightarrow}^{\bullet}} : \frac{p\rightarrow}{x} + \frac{q\rightarrow}{y} + \frac{r\rightarrow}{z} = 0,$  (29)

- $W_{\leftarrow}^{\bullet}$ -circumconic  $\mathcal{C}_{W_{\leftarrow}^{\bullet}} : \frac{p\leftarrow}{x} + \frac{q\leftarrow}{y} + \frac{r\leftarrow}{z} = 0,$  (30)

- $W_{\rightarrow}$ -circumconic  $\mathcal{C}_{W_{\rightarrow}} : \frac{1}{p\rightarrow x} + \frac{1}{q\rightarrow y} + \frac{1}{r\rightarrow z} = 0,$  (31)

- $W_{\leftarrow}$ -circumconic  $\mathcal{C}_{W_{\leftarrow}} : \frac{1}{p\leftarrow x} + \frac{1}{q\leftarrow y} + \frac{1}{r\leftarrow z} = 0,$  (32)

- circumconic through  $W_{\rightarrow}$  and  $W_{\leftarrow}$ ,
- circumconics with the centers  $mW_{\rightarrow}$  resp.  $mW_{\leftarrow}$ ,
- circumconics of the medial triangle of  $ABC$  with the centers  $m^2W_{\rightarrow}$  and  $m^2W_{\leftarrow}$  respectively.

**Theorem 4.** (a) *The circumconics  $\mathcal{C}_{W_{\rightarrow}^{\bullet}}$  and  $\mathcal{C}_{W_{\leftarrow}^{\bullet}}$  intersect at the point  $S_{\delta} := W_{\infty}^{\bullet}$  on the Steiner circumellipse.*

(b) *The circumconic through  $W_{\rightarrow}$  and  $W_{\leftarrow}$  has perspector  $W_{\infty}$ . Hence it is the circumconic  $\mathcal{C}_{W_{\infty}}$*

$$\frac{q\rightarrow - r\rightarrow}{x} + \frac{r\rightarrow - p\rightarrow}{y} + \frac{p\rightarrow - q\rightarrow}{z} = 0 \quad (33)$$

*passing through  $G$ . Its center  $M_{\infty}$  lies on the Steiner inellipse. The Wallace points are antipodes.*

*Proof.* (a) The conjugates of the circumconics  $\mathcal{C}_{W_{\rightarrow}}$  and  $\mathcal{C}_{W_{\leftarrow}}$ , that are the lines  $[p_{\rightarrow} : q_{\rightarrow} : r_{\rightarrow}]$  and  $[p_{\leftarrow} : q_{\leftarrow} : r_{\leftarrow}]$ , respectively, intersect on the infinite line at the point  $W_{\infty}$ . Hence its conjugate lies on the Steiner circumellipse.

(b) The line through the conjugates of the Wallace points is

$$[q_{\rightarrow} - r_{\rightarrow} : r_{\rightarrow} - p_{\rightarrow} : p_{\rightarrow} - q_{\rightarrow}].$$

Its conjugate (a circumconic) has the perspector  $W_{\infty}$ . The point  $G = (1 : 1 : 1)$  obviously satisfies the circumconic equation (33). The center of the  $W_{\infty}$  - circumconic according to (1) is

$$M_{\infty} = ((q_{\rightarrow} - r_{\rightarrow})^2 : (r_{\rightarrow} - p_{\rightarrow})^2 : (p_{\rightarrow} - q_{\rightarrow})^2). \quad (34)$$

It satisfies equation (3) of the Steiner inellipse and is - how one finds out by a longer computation in accordance with (17) - collinear with the two Wallace points, hence they must be antipodes.  $\square$

In the special case of the altitude directions the point  $S_{\delta}$  is the Steiner point of  $ABC$  and  $\mathcal{C}_{W_{\infty}}$  is the Kiepert hyperbola.

An interesting property of (31) and (32) is presented in Theorem 7 below.

The following theorem involves circumconics that are in connection with the 6 centers of the circumconics (26), (27).

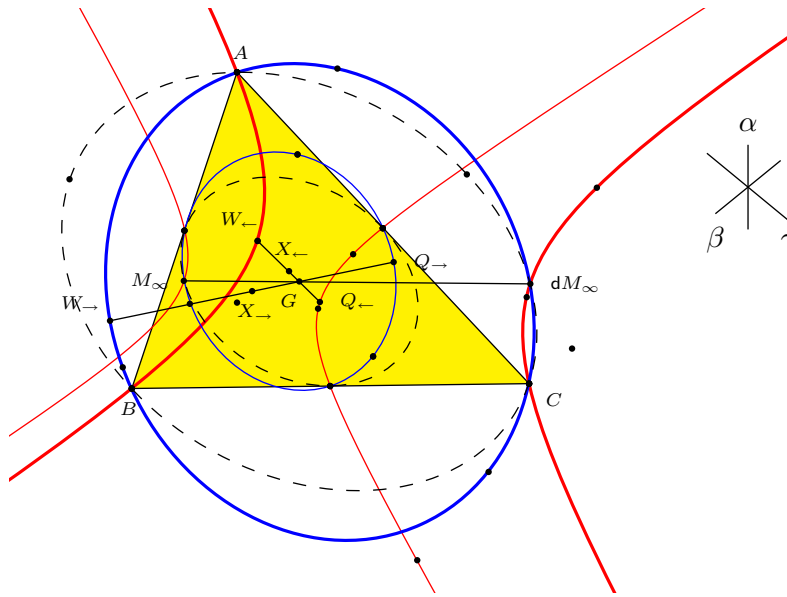


Figure 3.

**Theorem 5.** (a) Suppose the Wallace point  $W_{\rightarrow}$  (respectively  $W_{\leftarrow}$ ) is reflected in the centers of the three circumconics in (26) (respectively (27)). Then the three reflection points lie on a circumconic through  $W_{\leftarrow}$  (respectively  $W_{\rightarrow}$ ). Its center is  $Q_{\rightarrow} = mW_{\rightarrow}$  (respectively  $Q_{\leftarrow} = mW_{\leftarrow}$ ). These two circumconics (thick red and blue respectively in Figure 3) intersect the Steiner circumellipse at point  $dM_{\infty}$ .



(b) *The centers of the three circumconics in (26) (respectively (27)) lie on a circumconic of the medial triangle through  $Q_{\leftarrow}$  (respectively  $Q_{\rightarrow}$ ) with center  $X_{\rightarrow} = m^2W_{\rightarrow}$  (respectively  $X_{\leftarrow} = m^2W_{\leftarrow}$ ). Both circumconics (red and green respectively in Figure 3) intersect on the Steiner inellipse at point  $M_{\infty}$ .*

## 6. Wallace Triangles

The Wallace lines  $w_{\alpha\rightarrow}, w_{\beta\rightarrow}, w_{\gamma\rightarrow}$  belonging to  $W_{\rightarrow}$  form a triangle  $\Delta_{\rightarrow}$  (Wallace-right-triangle) and the Wallace lines  $w_{\alpha\leftarrow}, w_{\beta\leftarrow}, w_{\gamma\leftarrow}$  belonging to  $W_{\leftarrow}$  form a triangle  $\Delta_{\leftarrow}$  (Wallace-left-triangle).

**Theorem 6.** *Each of the Wallace triangles and  $\Delta$  are triply perspective.*

(a) *The 3 centers of perspective of  $(\Delta, \Delta_{\rightarrow})$  are collinear on the tripolar of  $W_{\rightarrow}$ .*

(b) *The 3 centers of perspective of  $(\Delta, \Delta_{\leftarrow})$  are collinear on the tripolar of  $W_{\leftarrow}$ .*

*Proof.* With (28), the vertices of the Wallace-right-triangle  $\Delta_{\rightarrow}$  are

$$A_{\rightarrow} := \left( \frac{1}{p_{\rightarrow}X_1} : \frac{1}{q_{\rightarrow}X_3} : \frac{1}{r_{\rightarrow}X_2} \right), \quad (35)$$

$$B_{\rightarrow} := \left( \frac{1}{p_{\rightarrow}X_3} : \frac{1}{q_{\rightarrow}X_2} : \frac{1}{r_{\rightarrow}X_1} \right), \quad (36)$$

$$C_{\rightarrow} := \left( \frac{1}{p_{\rightarrow}X_2} : \frac{1}{q_{\rightarrow}X_1} : \frac{1}{r_{\rightarrow}X_3} \right). \quad (37)$$

The triple perspectivity of  $\Delta$  and  $\Delta_{\rightarrow}$  follows from the concurrency of the lines

$$AA_{\rightarrow}, \quad BB_{\rightarrow}, \quad CC_{\rightarrow} \quad \text{at} \quad \left( \frac{X_1}{p_{\rightarrow}} : \frac{X_2}{q_{\rightarrow}} : \frac{X_3}{r_{\rightarrow}} \right) =: P_{A_{\rightarrow}}$$

$$AB_{\rightarrow}, \quad BC_{\rightarrow}, \quad CA_{\rightarrow} \quad \text{at} \quad \left( \frac{X_3}{p_{\rightarrow}} : \frac{X_1}{q_{\rightarrow}} : \frac{X_2}{r_{\rightarrow}} \right) =: P_{B_{\rightarrow}}$$

$$AC_{\rightarrow}, \quad BA_{\rightarrow}, \quad CB_{\rightarrow} \quad \text{at} \quad \left( \frac{X_2}{p_{\rightarrow}} : \frac{X_3}{q_{\rightarrow}} : \frac{X_1}{r_{\rightarrow}} \right) =: P_{C_{\rightarrow}}.$$

These three centers of perspectivity are obviously collinear on the line  $[p_{\rightarrow} : q_{\rightarrow} : r_{\rightarrow}]$ , which is the tripolar of  $\left( \frac{1}{p_{\rightarrow}} : \frac{1}{q_{\rightarrow}} : \frac{1}{r_{\rightarrow}} \right) = W_{\rightarrow}$ .

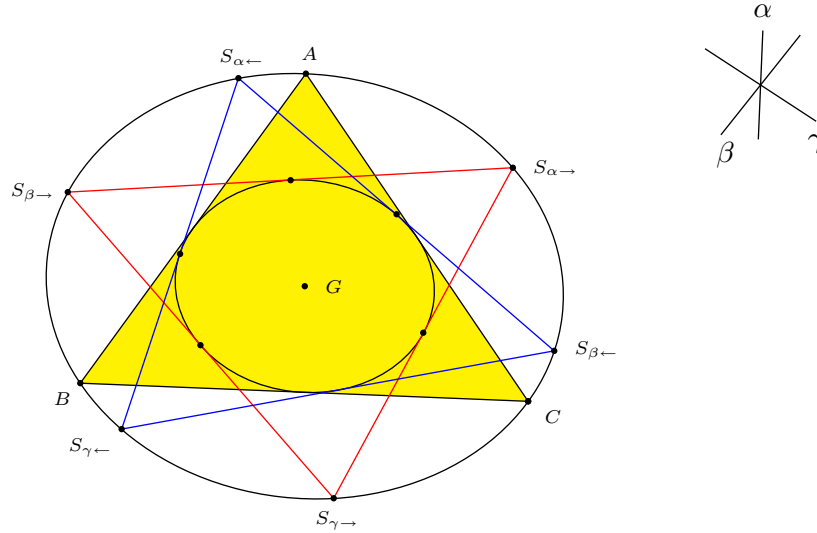
The proof for  $\Delta_{\leftarrow}$  is analogous.  $\square$

**Theorem 7.** *The vertices of  $\Delta_{\rightarrow}$  and  $\Delta_{\leftarrow}$  lie on the  $W_{\rightarrow}$  - circumconic and on the  $W_{\leftarrow}$  - circumconic, respectively.*

*Proof.* Easy verification.  $\square$

## 7. Direction Star and Steiner Circumellipse

Each of the 6 circumconics in (26) and (27) assigned to a direction star has a fourth common point  $(S_{\alpha\rightarrow}, \dots, S_{\gamma\leftarrow})$  with the Steiner circumellipse. These points

Figure 4. The triangles  $\Delta S_{\rightarrow}$  and  $\Delta S_{\leftarrow}$ 

form two triangles  $\Delta S_{\rightarrow}$  and  $\Delta S_{\leftarrow}$  (Figure 4). The point  $S_{\alpha\rightarrow}$  is the conjugate of the intersection of  $\ell_{\alpha\rightarrow}$  with the infinite line, thus according to (4) - (6) follows

$$S_{\alpha\rightarrow} = \left( \frac{1}{\beta_2 - \gamma_3} : \frac{1}{\gamma_3 - \alpha_1} : \frac{1}{\alpha_1 - \beta_2} \right) = \left( \frac{1}{\lambda_1} : \frac{1}{\lambda_2} : \frac{1}{\lambda_3} \right), \quad (38)$$

for the other vertices of the triangle  $\Delta S_{\rightarrow}$ , we find

$$S_{\beta\rightarrow} = \left( \frac{1}{\gamma_2 - \alpha_3} : \frac{1}{\alpha_3 - \beta_1} : \frac{1}{\beta_1 - \gamma_2} \right) = \left( \frac{1}{\lambda_3} : \frac{1}{\lambda_1} : \frac{1}{\lambda_2} \right), \quad (39)$$

$$S_{\gamma\rightarrow} = \left( \frac{1}{\alpha_2 - \beta_3} : \frac{1}{\beta_3 - \gamma_1} : \frac{1}{\gamma_1 - \alpha_2} \right) = \left( \frac{1}{\lambda_2} : \frac{1}{\lambda_3} : \frac{1}{\lambda_1} \right). \quad (40)$$

The coordinates of these points are connected by cyclic interchange. Hence they form a Brocardian triple [6]. The same is valid for the triangle  $\Delta S_{\leftarrow}$ .

**Theorem 8.** (a) *The triangles  $\Delta S_{\rightarrow}$  and  $\Delta S_{\leftarrow}$  have the centroid  $G$ .*

(b) *The 6 sidelines of these triangles are the duals of the respective opposite vertices and hence tangents at the Steiner inellipse. The points of contact are the midpoints of the sides of these triangles.*

(c) *The triangles  $\Delta S_{\rightarrow}$  and  $\Delta S_{\leftarrow}$  have the same area like  $ABC$ , because each Brocardian triple with vertices on the Steiner circumellipse has this property.*

**Theorem 9.** *The triangles  $\Delta$ ,  $\Delta S_{\rightarrow}$  and  $\Delta S_{\leftarrow}$  are pairwise triply perspective. The 9 centers of perspective lie on the infinite line, and the 9 axes of perspective pass through  $G$ .*

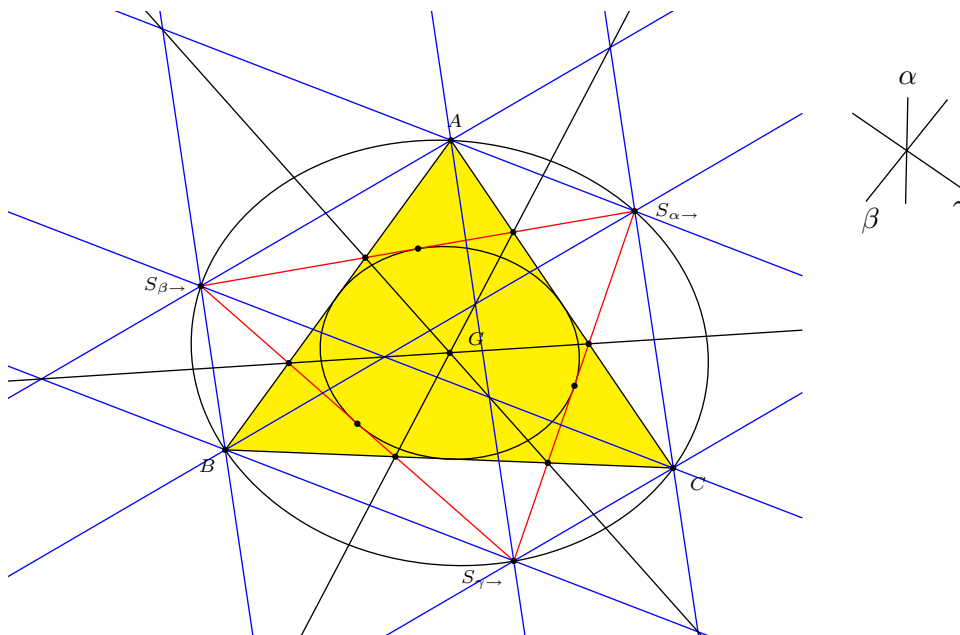


Figure 5. Triple perspectivity of  $\Delta$  and  $\Delta S_{\rightarrow}$ .

We omit the elementary but long computational proof. Figure 5 illustrates the triple perspectivity of  $\Delta$  and  $\Delta S_{\rightarrow}$ .

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