Forum Geometricorum
Volume 12 (2012) 13–25.

FORUM GEOM
ISSN 1534-1178

Characterizations of Orthodiagonal Quadrilaterals

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Abstract. We prove ten necessary and sufficient conditions for a convex quadrilateral to have perpendicular diagonals. One of these is a quite new eight point circle theorem and three of them are metric conditions concerning the nonoverlapping triangles formed by the diagonals.

1. A well known characterization

An *orthodiagonal quadrilateral* is a convex quadrilateral with perpendicular diagonals. The most well known and in problem solving useful characterization of orthodiagonal quadrilaterals is the following theorem. Five other different proofs of it was given in [19, pp.158–159], [11], [15], [2, p.136] and [4, p.91], using respectively the law of cosines, vectors, an indirect proof, a geometric locus and complex numbers. We will give a sixth proof using the Pythagorean theorem.

Theorem 1. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if*

$$AB^2 + CD^2 = BC^2 + DA^2.$$

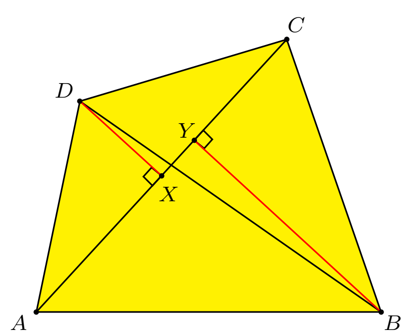

Figure 1. Normals to diagonal AC

Proof. Let X and Y be the feet of the normals from D and B respectively to diagonal AC in a convex quadrilateral $ABCD$, see Figure 1. By the Pythagorean theorem we have $BY^2+AY^2 = AB^2$, $BY^2+CY^2 = BC^2$, $DX^2+CX^2 = CD^2$

Publication Date: February 22, 2012. Communicating Editor: Paul Yiu.

and $AX^2 + DX^2 = DA^2$. Thus

$$\begin{aligned} & AB^2 + CD^2 - BC^2 - DA^2 \\ =\ & AY^2 - AX^2 + CX^2 - CY^2 \\ =\ & (AY + AX)(AY - AX) + (CX + CY)(CX - CY) \\ =\ & (AY + AX)XY + (CX + CY)XY \\ =\ & (AX + CX + AY + CY)XY \\ =\ & 2AC \cdot XY. \end{aligned}$$

Hence we have

$$AC \perp BD \qquad \Leftrightarrow \qquad XY = 0 \qquad \Leftrightarrow \qquad AB^2 + CD^2 = BC^2 + DA^2$$

since $AC > 0$. $\square$

Another short proof is the following. The area of a convex quadrilateral with sides a, b, c and d is given by the two formulas

$$K = \tfrac{1}{2}pq \sin\theta = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}$$

where θ is the angle between the diagonals p and q.[1] Hence we directly get

$$\theta = \frac{\pi}{2} \qquad \Leftrightarrow \qquad a^2 + c^2 = b^2 + d^2$$

completing this seventh proof.[2]

A different interpretation of the condition in Theorem 1 is the following. If four squares of the same sides as those of a convex quadrilateral are erected on the sides of that quadrilateral, then it is orthodiagonal if and only if the sum of the areas of two opposite squares is equal to the sum of the areas of the other two squares.

2. Two eight point circles

Another necessary and sufficient condition is that a convex quadrilateral is orthodiagonal if and only if the midpoints of the sides are the vertices of a rectangle ($EFGH$ in Figure 2). The direct theorem was proved by Louis Brand in the proof of the theorem about the *eight point circle* in [5], but was surely discovered much earlier since this is a special case of the Varignon parallelogram theorem.[3] The converse is an easy angle chase, as noted by "shobber" in post no 8 at [1]. In fact, the converse to the theorem about the eight point circle is also true, so we have the following condition as well. *A convex quadrilateral has perpendicular diagonals if and only if the midpoints of the sides and the feet of the maltitudes are*

[1]The first of these formulas yields a quite trivial characterization of orthodiagonal quadrilaterals: the diagonals are perpendicular if and only if the area of the quadrilateral is one half the product of the diagonals.

[2]This proof may be short, but the derivations of the two area formulas are a bit longer; see [17, pp.212–214] or [7] and [8].

[3]The midpoints of the sides in any quadrilateral form a parallelogram named after the French mathematician Pierre Varignon (1654-1722). The diagonals in this parallelogram are the bimedians of the quadrilateral and they intersect at the centroid of the quadrilateral.

eight concyclic points,[4] see Figure 2. The center of the circle is the centroid of the quadrilateral (the intersection of EG and FH in Figure 2). This was formulated slightly different and proved as Corollary 2 in [10].[5]

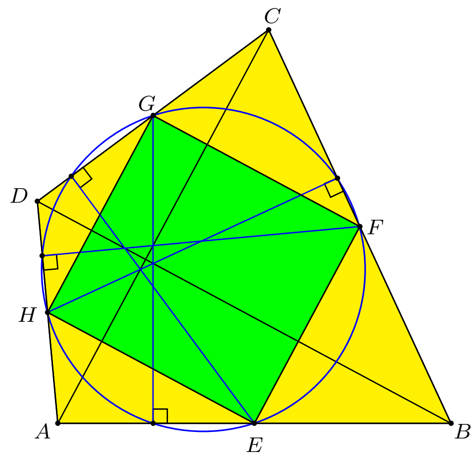

Figure 2. Brand's eight point circle and rectangle $EFGH$

There is also a second eight point circle characterization. Before we state and prove this theorem we will prove two other necessary and sufficient condition for the diagonals of a convex quadrilateral to be perpendicular, which are related to the second eight point circle.

Theorem 2. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if*

$$\angle PAB + \angle PBA + \angle PCD + \angle PDC = \pi$$

where P is the point where the diagonals intersect.

Proof. By the sum of angles in triangles ABP and CDP (see Figure 3) we have

$$\angle PAB + \angle PBA + \angle PCD + \angle PDC = 2\pi - 2\theta,$$

where θ is the angle between the diagonals. Hence $\theta = \frac{\pi}{2}$ if and only if the equation in the theorem is satisfied. □

Problem 6.17 in [14, p.139] is about proving that if the diagonals of a convex quadrilateral are perpendicular, then the projections of the point where the diagonals intersect onto the sides are the vertices of a cyclic quadrilateral.[6] The solution given by Prasolov in [14, p.149] used Theorem 2 and is, although not stated as such, also a proof of the converse. Our proof is basically the same.

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[4]A maltitude is a line segment in a quadrilateral from the midpoint of a side perpendicular to the opposite side.

[5]The quadrilateral formed by the feet of the maltitudes is called the principal orthic quadrilateral in [10].

[6]In [14] this is called an inscribed quadrilateral, but that is another name for a cyclic quadrilateral.

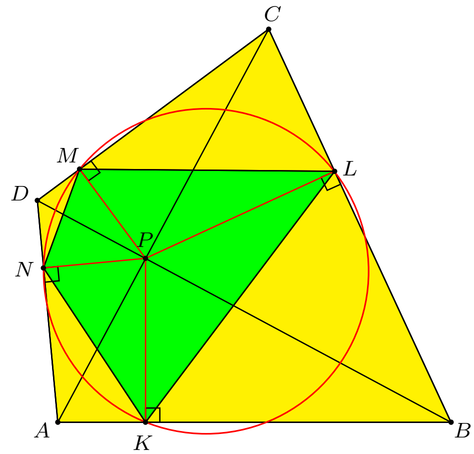

Figure 3. $ABCD$ is orthodiagonal iff $KLMN$ is cyclic

Theorem 3. *A convex quadrilateral is orthodiagonal if and only if the projections of the diagonal intersection onto the sides are the vertices of a cyclic quadrilateral.*

Proof. If the diagonals intersect in P, and the projection points on AB, BC, CD and DA are K, L, M and N respectively, then $AKPN$, $BLPK$, $CMPL$ and $DNPM$ are cyclic quadrilaterals since they all have two opposite right angles (see Figure 3). Then $\angle PAN = \angle PKN$, $\angle PBL = \angle PKL$, $\angle PCL = \angle PML$ and $\angle PDN = \angle PMN$. Quadrilateral $ABCD$ is by Theorem 2 orthodiagonal if and only if

$$\begin{aligned} & \angle PAN + \angle PBL + \angle PCL + \angle PDN = \pi \\ \Leftrightarrow \quad & \angle PKN + \angle PKL + \angle PML + \angle PMN = \pi \\ \Leftrightarrow \quad & \angle LKN + \angle LMN = \pi \end{aligned}$$

where the third equality is a well known necessary and sufficient condition for $KLMN$ to be a cyclic quadrilateral. □

Now we are ready to prove the second eight point circle theorem.

Theorem 4. *In a convex quadrilateral $ABCD$ where the diagonals intersect at P, let K, L, M and N be the projections of P onto the sides, and let R, S, T and U be the points where the lines KP, LP, MP and NP intersect the opposite sides. Then the quadrilateral $ABCD$ is orthodiagonal if and only if the eight points K, L, M, N, R, S, T and U are concyclic.*

Proof. $(\Rightarrow)$ If $ABCD$ is orthodiagonal, then K, L, M and N are concyclic by Theorem 3. We start by proving that $KTMN$ has the same circumcircle as $KLMN$. To do this, we will prove that $\angle MNK + \angle MTK = \pi$, which is equivalent to proving that $\angle MTK = \angle ANK + \angle DNM$ since $\angle AND = \pi$ (see Figure 4). In cyclic quadrilaterals $ANPK$ and $DNPM$, we have $\angle ANK = \angle APK = \angle TPC$ and $\angle DNM = \angle MPD$. By the exterior angle theorem $\angle MTP = \angle TPC + \angle TCP$.

In addition $\angle MPD = \angle TCP$ since CPD is a right triangle with altitude MP. Hence

$$\angle MTK = \angle TPC + \angle TCP = \angle ANK + \angle MPD = \angle ANK + \angle DNM$$

which proves that T lies on the circumcircle of $KLMN$, since K, M and N uniquely determine a circle. In the same way it can be proved that R, S and U lies on this circle.

($\Leftarrow$) If K, L, M, N, R, S, T and U are concyclic, then $NMTK$ is a cyclic quadrilateral. By using some of the angle relations from the first part, we get

$$\begin{aligned} & \angle MTK = \pi - \angle MNK \\ \Rightarrow \quad & \angle MTP = \angle ANK + \angle DNM \\ \Rightarrow \quad & \angle TPC + \angle TCP = \angle APK + \angle MPD \\ \Rightarrow \quad & \angle TCP = \angle MPD. \end{aligned}$$

Thus triangles MPC and MDP are similar since angle MDP is common. Then

$$\angle CPD = \angle PMD = \tfrac{\pi}{2}$$

so $AC \perp BD$. $\square$

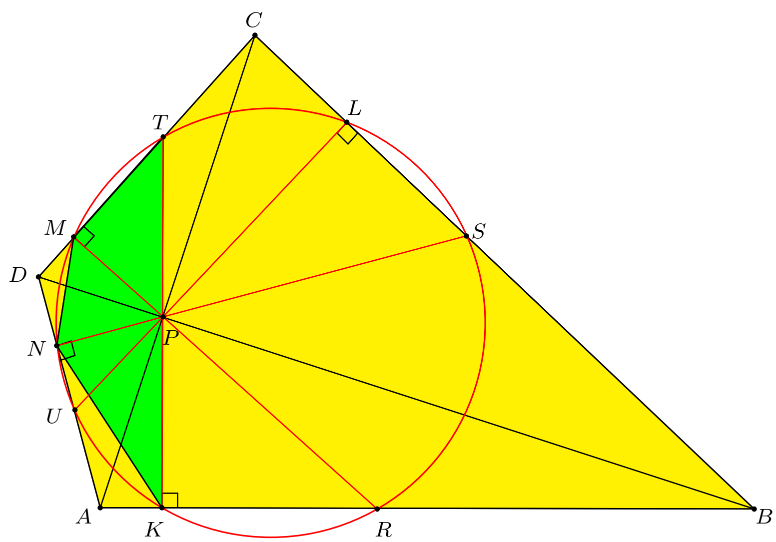

Figure 4. The second eight point circle

In the next theorem we prove that quadrilateral $RSTU$ in Figure 4 is a rectangle if and only if $ABCD$ is an orthodiagonal quadrilateral.

Theorem 5. *If the normals to the sides of a convex quadrilateral $ABCD$ through the diagonal intersection intersect the opposite sides in R, S, T and U, then $ABCD$ is orthodiagonal if and only if $RSTU$ is a rectangle whose sides are parallel to the diagonals of $ABCD$.*

Proof. ($\Rightarrow$) If $ABCD$ is orthodiagonal, then $UTMN$ is a cyclic quadrilateral according to Theorem 4 (see Figure 5). Thus

$$\angle MTU = \angle DNM = \angle MPD = \angle TCP,$$

so $UT \parallel AC$. In the same way it can be proved that $RS \parallel AC$, $UR \parallel DB$ and $TS \parallel DB$. Hence $RSTU$ is a parallelogram with sides parallel to the perpendicular lines AC and BD, so it is a rectangle.

($\Leftarrow$) If $RSTU$ is a rectangle with sides parallel to the diagonals AC and BD of a convex quadrilateral, then

$$\angle DPC = \angle UTS = \tfrac{\pi}{2}.$$

Hence $AC \perp BD$. $\square$

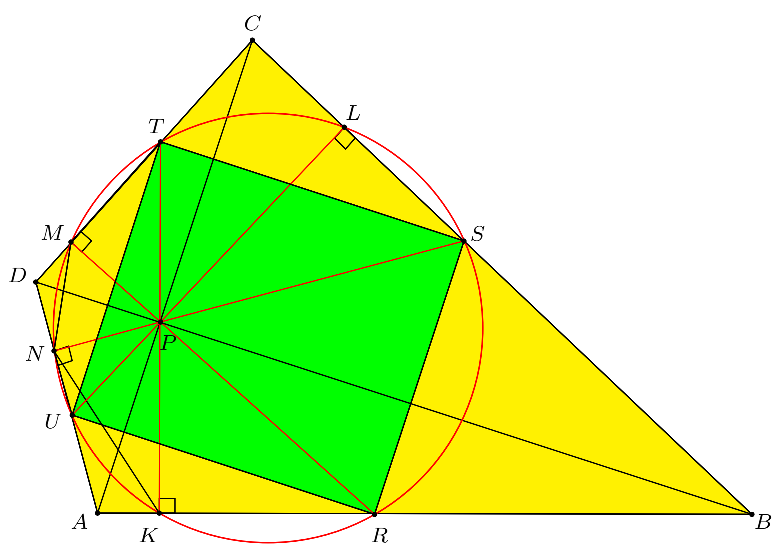

Figure 5. $ABCD$ is orthodiagonal iff $RSTU$ is a rectangle

Remark. Shortly after we had proved Theorems 4 and 5 we found out that the direct parts of these two theorems was proved in 1998 [20]. Thus, in [20] Zaslavsky proved that in an orthodiagonal quadrilateral, the eight points K, L, M, N, R, S, T and U are concyclic, and that $RSTU$ is a rectangle with sides parallel to the diagonals. We want to thank Vladimir Dubrovsky for the help with the translation of the theorems in [20].

Let's call the eight point circle due to Louis Brand the *first eight point circle* and the one in Theorem 4 the *second eight point circle*. Since $RSTU$ is a rectangle, the center of the second eight point circle is the point where the diagonals in $RSTU$ intersect.

Theorem 6. *The first and second eight point circle of an orthodiagonal quadrilateral coincide if and only if the quadrilateral is also cyclic.*

Proof. Since the second eight point circle is constructed from line segments through the diagonal intersection, the two eight point circles coincide if and only if the four maltitudes are concurrent at the diagonal intersection. The maltitudes of a convex quadrilateral are concurrent if and only if the quadrilateral is cyclic according to [12, p.19]. □

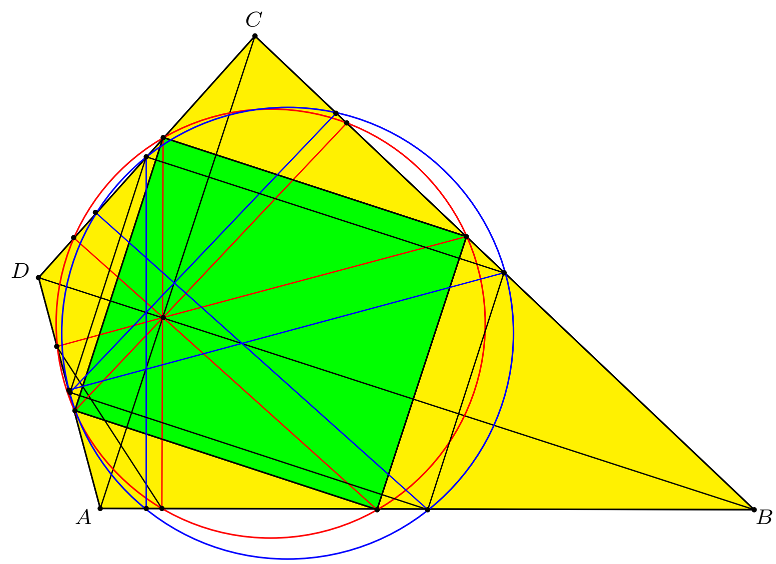

Figure 6. The two eight point circles

That the point where the maltitudes intersect (the anticenter) in a cyclic orthodiagonal quadrilateral coincide with the diagonal intersection was proved in another way in [2, p.137].

3. A duality between the bimedians and the diagonals

The next theorem gives an interesting sort of dual connection between the bimedians and the diagonals of a convex quadrilateral. The first part is a characterization of orthodiagonal quadrilaterals. Another proof of (i) using vectors was given in [6, p.293].

Theorem 7. *In a convex quadrilateral we have the following conditions:*
(i) *The bimedians are congruent if and only if the diagonals are perpendicular.*
(ii) *The bimedians are perpendicular if and only if the diagonals are congruent.*

Proof. (i) According to the proof of Theorem 7 in [9], the bimedians m and n in a convex quadrilateral satisfy

$$4(m^2 - n^2) = -2(a^2 - b^2 + c^2 - d^2)$$

where a, b, c and d are the sides of the quadrilateral. Hence

$$m = n \qquad \Leftrightarrow \qquad a^2 + c^2 = b^2 + d^2$$

which proves the condition according to Theorem 1.

(ii) Consider the Varignon parallelogram of a convex quadrilateral (see Figure 7). Its diagonals are the bimedians m and n of the quadrilateral. It is well known that the length of the sides in the Varignon parallelogram are one half the length of the diagonals p and q in the quadrilateral. Applying Theorem 1 to the Varignon parallelogram yields

$$m \perp n \qquad \Leftrightarrow \qquad 2\left(\frac{p}{2}\right)^2 = 2\left(\frac{q}{2}\right)^2 \qquad \Leftrightarrow \qquad p = q$$

since opposite sides in a parallelogram are congruent. □

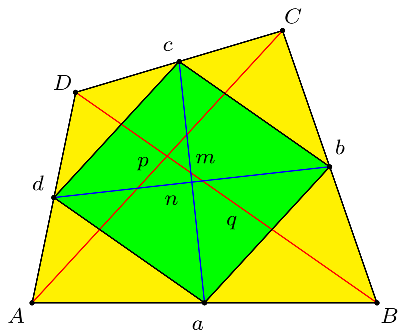

Figure 7. The Varignon parallelogram

4. Three metric conditions in the four subtriangles

Now we will use Theorem 1 to prove two more characterizations resembling it.

Theorem 8. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if*

$$m_1^2 + m_3^2 = m_2^2 + m_4^2$$

where m_1, m_2, m_3 and m_4 are the medians in the triangles ABP, BCP, CDP and DAP from the intersection P of the diagonals to the sides AB, BC, CD and DA respectively.

Proof. Let P divide the diagonals in parts w, x and y, z (see Figure 8). By applying Apollonius' theorem in triangles ABP, CDP, BCP and DAP we get

$$\begin{aligned} & m_1^2 + m_3^2 = m_2^2 + m_4^2 \\ \Leftrightarrow \quad & 4m_1^2 + 4m_3^2 = 4m_2^2 + 4m_4^2 \\ \Leftrightarrow \quad & 2(w^2 + y^2) - a^2 + 2(x^2 + z^2) - c^2 = 2(y^2 + x^2) - b^2 + 2(z^2 + w^2) - d^2 \\ \Leftrightarrow \quad & a^2 + c^2 = b^2 + d^2 \end{aligned}$$

which by Theorem 1 completes the proof. □

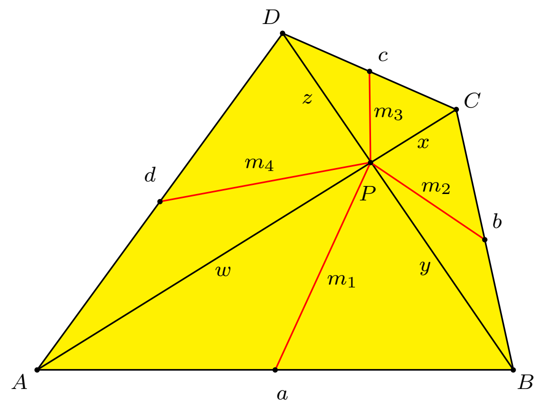

Figure 8. The subtriangle medians m_1, m_2, m_3 and m_4

Theorem 9. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if*

$$R_1^2 + R_3^2 = R_2^2 + R_4^2$$

where R_1, R_2, R_3 and R_4 are the circumradii in the triangles ABP, BCP, CDP and DAP respectively and P is the intersection of the diagonals.

Proof. According to the extended law of sines applied in the four subtriangles, $a = 2R_1 \sin\theta$, $b = 2R_2 \sin(\pi - \theta)$, $c = 2R_3 \sin\theta$ and $d = 2R_4 \sin(\pi - \theta)$, see Figure 9. We get

$$a^2 + c^2 - b^2 - d^2 = 4\sin^2\theta \left(R_1^2 + R_3^2 - R_2^2 - R_4^2\right)$$

where we used that $\sin(\pi - \theta) = \sin\theta$. Hence

$$a^2 + c^2 = b^2 + d^2 \qquad \Leftrightarrow \qquad R_1^2 + R_3^2 = R_2^2 + R_4^2$$

since $\sin\theta > 0$ for $0 < \theta < \pi$. □

When studying Figure 9 it is easy to realize the following result, which gives a connection between the previous two theorems.

Theorem 10. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if the circumcenters of the triangles ABP, BCP, CDP and DAP are the midpoints of the sides of the quadrilateral, where P is the intersection of its diagonals.*

Proof. The quadrilateral $ABCD$ is orthodiagonal if and only if one of the triangles ABP, BCP, CDP and DAP have a right angle at P; then all of them have it. Hence we only need to prove that the circumcenter of one triangle is the midpoint of a side if and only if the opposite angle is a right angle. But this is an immediate consequence of Thales' theorem and its converse, see [18]. □

The next theorem is our main result and concerns the altitudes in the four nonoverlapping subtriangles formed by the diagonals.

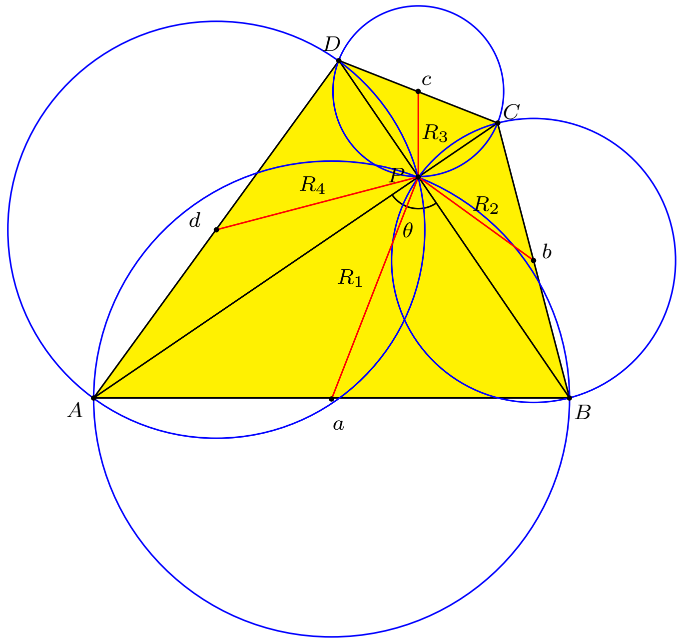

Figure 9. The circumradii R_1, R_2, R_3 and R_4

Theorem 11. *A convex quadrilateral $ABCD$ is orthodiagonal if and only if*

$$\frac{1}{h_1^2}+\frac{1}{h_3^2}=\frac{1}{h_2^2}+\frac{1}{h_4^2}$$

where h_1, h_2, h_3 and h_4 are the altitudes in the triangles ABP, BCP, CDP and DAP from the intersection P of the diagonals to the sides AB, BC, CD and DA respectively.

Proof. Let P divide the diagonals in parts w, x and y, z. From expressing twice the area of triangle ABP in two different ways we get (see Figure 10)

$$ah_1 = wy\sin\theta$$

where θ is the angle between the diagonals. Thus

$$\frac{1}{h_1^2}=\frac{a^2}{w^2y^2\sin^2\theta}=\frac{w^2+y^2-2wy\cos\theta}{w^2y^2\sin^2\theta}=\left(\frac{1}{y^2}+\frac{1}{w^2}\right)\frac{1}{\sin^2\theta}-\frac{2\cos\theta}{wy\sin^2\theta}$$

where we used the law of cosines in triangle ABP in the second equality. The same resoning in triangle CDP yields

$$\frac{1}{h_3^2}=\left(\frac{1}{x^2}+\frac{1}{z^2}\right)\frac{1}{\sin^2\theta}-\frac{2\cos\theta}{xz\sin^2\theta}.$$

In triangles BCP and DAP we have respectively

$$\frac{1}{h_2^2}=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\frac{1}{\sin^2\theta}+\frac{2\cos\theta}{yx\sin^2\theta}$$

and

$$\frac{1}{h_4^2} = \left(\frac{1}{w^2} + \frac{1}{z^2}\right)\frac{1}{\sin^2\theta} + \frac{2\cos\theta}{zw\sin^2\theta}$$

since $\cos(\pi - \theta) = -\cos\theta$. From the last four equations we get

$$\frac{1}{h_1^2} + \frac{1}{h_3^2} - \frac{1}{h_2^2} - \frac{1}{h_4^2} = -\frac{2\cos\theta}{\sin^2\theta}\left(\frac{1}{wy} + \frac{1}{yx} + \frac{1}{xz} + \frac{1}{zw}\right).$$

Hence

$$\frac{1}{h_1^2} + \frac{1}{h_3^2} = \frac{1}{h_2^2} + \frac{1}{h_4^2} \qquad \Leftrightarrow \qquad \cos\theta = 0 \qquad \Leftrightarrow \qquad \theta = \frac{\pi}{2}$$

since $(\sin\theta)^{-2} \neq 0$ and the expression in the parenthesis is positive. □

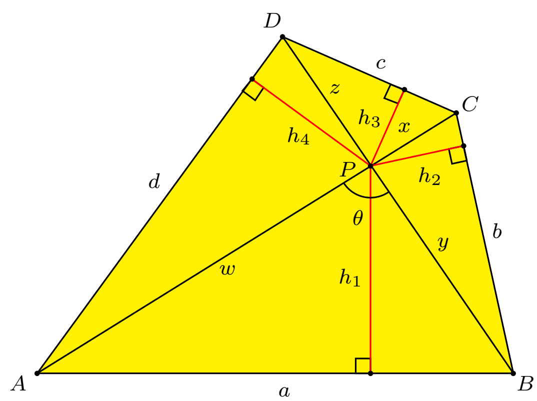

Figure 10. The subtriangle altitudes h_1, h_2, h_3 and h_4

5. Similar metric conditions in tangential and orthodiagonal quadrilaterals

A tangential quadrilateral is a quadrilateral with an incircle. A convex quadrilateral with the sides a, b, c and d is tangential if and only if

$$a + c = b + d$$

according to the well known Pitot theorem [3, pp.65–67]. In Theorem 1 we proved the well known condition that a convex quadrilateral with the sides a, b, c and d is orthodiagonal if and only if

$$a^2 + c^2 = b^2 + d^2.$$

Here all terms are squared compared to the Pitot theorem.

From the extended law of sines (see the proof of Theorem 9) we have that

$$a + c - b - d = 2\sin\theta(R_1 + R_3 - R_2 - R_4)$$

where R_1, R_2, R_3 and R_4 are the circumradii in the triangles ABP, BCP, CDP and DAP respectively, P is the intersection of the diagonals and θ is the angle between them. Hence

$$a + c = b + d \quad \Leftrightarrow \quad R_1 + R_3 = R_2 + R_4$$

since $\sin\theta > 0$, so a convex quadrilateral is tangential if and only if

$$R_1 + R_3 = R_2 + R_4.$$

In Theorem 9 we proved that the quadrilateral is orthodiagonal if and only if

$$R_1^2 + R_3^2 = R_2^2 + R_4^2.$$

All terms in this condition are squared compared to the tangential condition.

In [16] and [13] it is proved that a convex quadrilateral is tangential if and only if

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}$$

where h_1, h_2, h_3 and h_4 are the same altitudes as in Figure 10. We have just proved in Theorem 11 that a convex quadrilateral is orthodiagonal if and only if

$$\frac{1}{h_1^2} + \frac{1}{h_3^2} = \frac{1}{h_2^2} + \frac{1}{h_4^2},$$

that is, all terms in the orthodiagonal condition are squared compared to the tangential condition. We find these similarities between these two types of quadrilaterals very interesting and remarkable.

References

[1] 4everwise and shobber (usernames), Quadrilateral, *Art of Problem Solving*, 2005, `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=48225`

[2] N. Altshiller-Court, *College Geometry*, Dover reprint, 2007.

[3] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.

[4] T. Andreescu and D. Andrica, *Complex Numbers from A to... Z*, Birkhäuser, 2006.

[5] L. Brand, The Eight-Point Circle and the Nine-Point Circle, *Amer. Math. Monthly*, 51 (1944) 84–85.

[6] A. Engel, *Problem-Solving Strategies*, Springer, New York, 1998.

[7] J. Harries, Area of a Quadrilateral, *The Mathematical Gazette*, 86 (2002) 310–311.

[8] V. F. Ivanoff, C. F. Pinzka and J. Lipman, Problem E1376: Bretschneider's Formula, *Amer. Math. Monthly*, 67 (1960) 291.

[9] M. Josefsson, The area of a bicentric quadrilateral, *Forum Geom.*, 11 (2011) 155–164.

[10] M. F. Mammana, B. Micale and M. Pennisi, The Droz-Farny circles of a convex quadrilateral, *Forum Geom.*, 11 (2011) 109–119.

[11] P. Maynard and G. Leversha, Pythagoras' Theorem for Quadrilaterals, *The Mathematical Gazette*, 88 (2004) 128–130.

[12] B. Micale and M. Pennisi, On the altitudes of quadrilaterals, *Int. J. Math. Educ. Sci. Technol.* 36 (2005) 15–24.

[13] N. Minculete, Characterizations of a tangential quadrilateral, *Forum Geom.* 9 (2009) 113–118.

[14] V. Prasolov (translated and edited by D. Leites), *Problems in Plane and Solid Geometry*, 2005, available at `http://students.imsa.edu/~liu/Math/planegeo.pdf`

[15] K. R. S. Sastry and L. Hoehn, Problem 227: Orthodiagonal quadrilaterals, *The College Mathematics Journal*, 15 (1984) 165–166.

[16] I. Vaynshtejn, N. Vasilyev and V. Senderov, Problem M1495, *Kvant* (in Russian) no 6, 1995 pp. 27–28, available at
http://kvant.mirror1.mccme.ru/1995/06/resheniya_zadachnika_kvanta_ma.htm

[17] J. A. Vince, *Geometry for Computer Graphics. Formulae, Examples and Proofs*, Springer, 2005.

[18] *Wikipedia*, Thales' theorem, http://en.wikipedia.org/wiki/Thales%27_theorem, accessed October 5, 2011.

[19] P. Yiu, *Notes on Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

[20] A. A. Zaslavsky, The Orthodiagonal Mapping of Quadrilaterals, *Kvant* (in Russian) no 4, 1998 pp. 43–44, available at
http://kvant.mirror1.mccme.ru/pdf/1998/04/kv0498zaslavsky.pdf

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