Forum Geometricorum
Volume 12 (2012) 39–46.

FORUM GEOM
ISSN 1534-1178

Synthetic Proofs of Two Theorems Related to the Feuerbach Point

Nguyen Minh Ha and Nguyen Pham Dat

Abstract. We give synthetic proofs of two theorems on the Feuerbach point of a triangle, one of Paul Yiu, and another of Lev Emelyanov and Tatiana Emelyanova theorem.

1. Introduction

If S is a point belonging to the circumcircle of triangle ABC, then the images of S through the reflections with axes BC, CA and AB respectively lie on the same line that passes through the orthocenter of ABC. This line is called the Steiner line of S with respect to triangle ABC.

If a line $\mathcal{L}$ passes through the orthocenter of ABC, then the images of $\mathcal{L}$ through the reflections with axes BC, CA and AB are concurrent at one point on the circumcircle of ABC. This point is named the anti-Steiner point of $\mathcal{L}$ with respect to ABC. Of course, $\mathcal{L}$ is Steiner line of S with respect to ABC if and only if S is the anti-Steiner point of $\mathcal{L}$ with respect to ABC. In 2005, using homogenous barycentric coordinates, Paul Yiu [5] established an interesting theorem related to the Feuerbach point of a triangle; see also [3, Theorem 5].

Theorem 1. *The Feuerbach point of triangle ABC is the anti-Steiner point of the Euler line of the intouch triangle of ABC with respect to the same triangle.*[1]

In 2009, J. Vonk [4] introduced a geometrically synthetic proof of Theorem 1. In 2001, by calculation, Lev Emelyanov and Tatiana Emelyanova [1] established a theorem that is also very interesting and also related to the Feuerbach point of a triangle.

Theorem 2. *The circle through the feet of the internal bisectors of triangle ABC passes through the Feuerbach point of the triangle.*

In this article, we present a synthetic proof of Theorem 1, which is different from Vonk's proof, and one for Theorem 2. We use (O), $I(r)$, (XYZ) to denote respectively the circle with center O, the circle with center I and radius r, and the circumcircle of triangle XYZ. As in [2, p.12], the directed angle from the line

---

Publication Date: March 22, 2012. Communicating Editor: J. Chris Fisher.

The authors thank Professor Chris Fisher for his valuable comments and suggestions.

[1]The anti-Steiner point of the Euler line is called the Euler reflection point in [3].

a to the line b denoted by (a, b). It measures the angle through which a must be rotated in the positive direction in order to become parallel to, or to coincide with, b. Therefore,
(i) $-90^\circ \leq (\mathsf{a},\ \mathsf{b}) \leq 90^\circ$,
(ii) $(\mathsf{a},\ \mathsf{b}) = (\mathsf{a},\ \mathsf{c}) + (\mathsf{c},\ \mathsf{b})$,
(iii) If a' and b' are the images of a and b respectively under a reflection, then $(\mathsf{a},\ \mathsf{b}) = (\mathsf{b}',\ \mathsf{a}')$,
(iv) Four noncollinear points A, B, C, D are concyclic if and only if $(AC, AD) = (BC, BD)$.

2. Preliminary results

Lemma 3. *Let ABC be a triangle inscribed in a circle (O), and $\mathcal{L}$ an arbitrary line. Let the parallels of $\mathcal{L}$ through A, B, C intersect the circle at D, E, F respectively. The lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are the perpendiculars to BC, CA, AB through D, E, F respectively.*
(a) *The lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent at a point S on the circle (O),*
(b) *The Steiner line of S with respect to ABC is parallel to $\mathcal{L}$.*

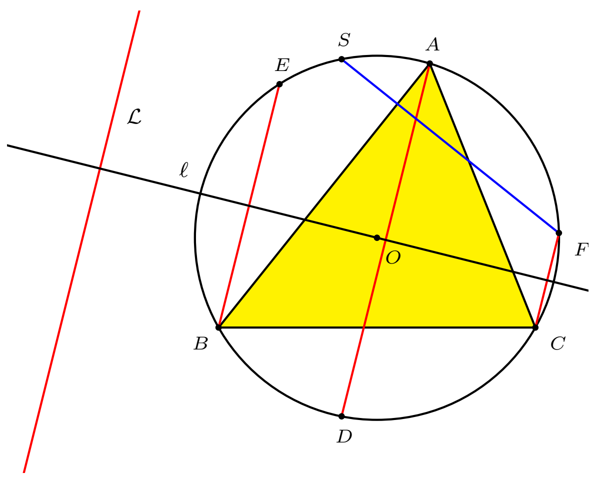

Figure 1.

Proof. Let S be the intersection of $\mathcal{L}_a$ and (O). Let ℓ be the line through O perpendicular to $\mathcal{L}$ (see Figure 1).

(a) Because A, B, and C are the images of D, E, and F through the reflections with axis $\mathcal{L}$ respectively,

$$(FE, FD) = (CA, CB). \tag{1}$$

Therefore, we have

$$\begin{aligned}(SE, AC) &= (SE, SD) + (SD, BC) + (BC, AC) \\ &= (FE, FD) + 90^\circ + (BC, AC) && (F \in (SDE), SD \perp BC) \\ &= (CA, CB) + 90^\circ + (BC, AC) \\ &= 90^\circ.\end{aligned}$$

Therefore, SE coincides $\mathcal{L}_b$, *i.e.*, S lies on $\mathcal{L}_b$. Similarly, S also lies on $\mathcal{L}_c$, and the three lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent at S on the circle (O).

(b) Let B_1, C_1 respectively be the images of S through the reflections with axes CA, AB. Let B_2, C_2 respectively be the intersection points of SB_1, SC_1 with AC, AB (see Figure 2). Obviously, B_2, C_2 are the midpoints of SB_1, SC_1 respectively. Thus,

$$B_2C_2 // B_1C_1. \tag{2}$$

Since SB_2, SC_2 are respectively perpendicular to AC, AB,

$$S \in (AB_2C_2). \tag{3}$$

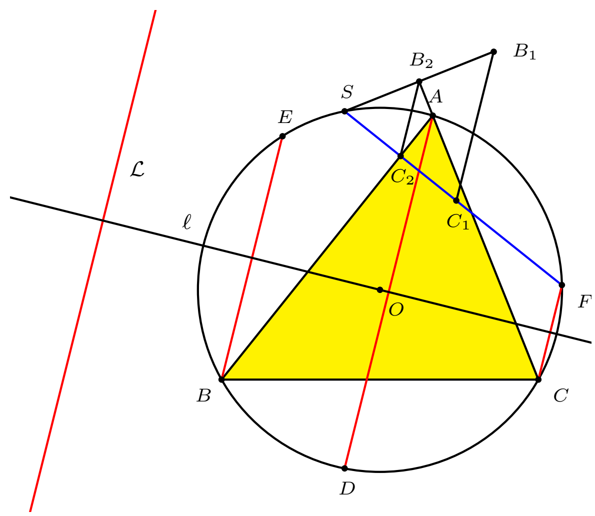

Figure 2.

Therefore, we have

$$\begin{aligned}(B_1C_1, \mathcal{L}) &= (B_1C_1, AD) && (\mathcal{L} // AD) \\ &= (B_2C_2, AD) && \text{(by (2))} \\ &= (B_2C_2, AC_2) + (AB, AD) && (B \in AC_2) \\ &= (B_2S, AS) + (AB, AD) && \text{(by (3))} \\ &= (ES, AS) + (AB, AD) && (E \in B_2S) \\ &= (ED, AD) + (DA, DE) && (D \in (SEA)) \\ &= 0^\circ.\end{aligned}$$

Therefore, $B_1C_1//\mathcal{L}$. This means that the Steiner line of S with respect to triangle ABC is parallel to $\mathcal{L}$. □

Before we go on to Lemma 4, we review a very interesting concept in plane geometry called the orthopole. Let triangle ABC and the line $\mathcal{L}$. A', B', C' are the feet of the perpendiculars from A, B, C to $\mathcal{L}$ respectively. The lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ pass through A', B', C' and are perpendicular to BC, CA, AB respectively. Then $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent at one point called the orthopole of the line $\mathcal{L}$ with respect to triangle ABC. The following result is one of the most important results related to the concept of the orthopole. This result is often attributed to Griffiths, whose proof can be found in [2, pp.246–247].

Lemma 4. *Let ABC be a triangle inscribed in the circle (O), and P be an arbitrary point other than O. The orthopole of the line OP with respect to triangle ABC belongs to the circumcircle of the pedal triangle of P with respect to ABC.*

Lemma 5. *Let ABC be a triangle inscribed in (O). A_1, B_1, C_1 are the images of A, B, C respectively through the symmetry with center O. A_2, B_2, C_2 are the images of O through the reflections with axes BC, CA, AB respectively. A_3, B_3, C_3 are the feet of the perpendiculars from A, B, C to the lines OA_2, OB_2, OC_2 respectively. Then,*
(a) *The circles (OA_1A_2), (OB_1B_2), (OC_1C_2) all pass through the anti-Steiner point of the Euler line of triangle ABC with respect to the same triangle.*
(b) *The circle $(A_3B_3C_3)$ also passes through the same anti-Steiner point.*

Proof. (a) Let H be the orthocenter of ABC. Take the points D, S belonging to (O) such that $AD//OH$ and $DS \perp BC$ (see Figure 3).

According to Lemma 3, the Steiner line of S with respect to ABC is parallel to AD. On the other hand, the Steiner line of S with respect to ABC passes through H. Hence, OH is the Steiner line of S with respect to ABC. In other words,

$$S \text{ is the anti-Steiner point of the Euler line of } ABC \text{ with respect to the same triangle.} \quad (4)$$

Let S_a be the intersection of SD and OH. By (4), S_a is the images of S through the reflection with axis BC. From this, note that A_2 is the image of O through the reflection with axis BC, we have:

$$OA_2SS_a \text{ is an isosceles trapezium with } OA_2//S_a. \quad (5)$$

Therefore, we have

$$\begin{aligned}(A_2O, A_2S) &= (S_aO, S_aS) && (\text{by } (5))\\ &= (DA, DS) && (DA//S_aO \text{ and } D \in S_aS)\\ &= (A_1A, A_1S) && (A_1 \in (DAS))\\ &= (A_1O, A_1S) && (O \in A_1A).\end{aligned}$$

It follows that $S \in (OA_1A_2)$. Similarly, $S \in (OB_1B_2)$ and $S \in (OC_1C_2)$. Therefore,

$$\text{the circles } (OA_1A_2), (OB_1B_2), (OC_1C_2) \text{ all pass through } S. \quad (6)$$

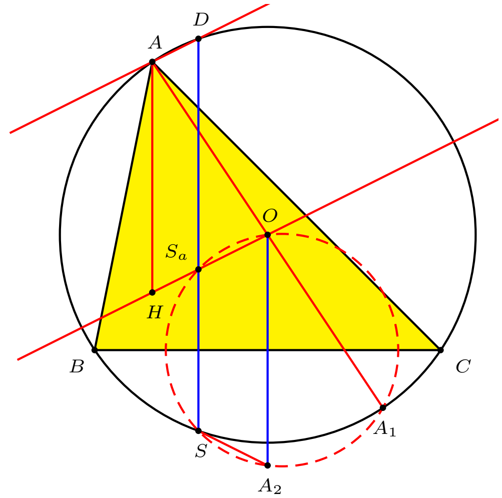

Figure 3.

From (4) and (6), we can deduce that (OA_1A_2), (OB_1B_2), (OC_1C_2) all pass through the anti-Steiner point of the Euler line of triangle ABC with respect to ABC.

(b) Take the points A_0, B_0, C_0 such that A, B, C are the midpoints of B_0C_0, C_0A_0, A_0B_0 respectively. Let M be the mid-point of BC (see Figure 4). Since $AB//CA_0$ and $AC//BA_0$, ABA_0C is a parallelogram. On the other hand, noting that $HB \perp AC$ and $CA_1 \perp AC$, $HC \perp AB$, and $BA_1 \perp AB$, we have $HB//CA_1$, $HC//BA_1$. This means that HBA_1C is a parallelogram. Thus, A_0, A_1 are the images of A, H respectively through the symmetry with center M. Therefore, the vectors $\mathbf{A_1A_0}$ and $\mathbf{AH}$ are equal.

On the other hand, since AHS_aD is a parallelogram, the vectors $\mathbf{DS_a}$ and $\mathbf{AH}$ are equal.

Hence, under the translation by the vector $\mathbf{AH}$, the points A_1, D are transformed into the points A_0, S_a respectively. This means that $A_0S_a//A_1D$.

From this, noting that $AD \perp A_1D$ and $AD//OH$, we deduce that

$$A_0S_a \perp OH. \tag{7}$$

On the other hand, because $SS_a \perp BC$ and $BC//B_0C_0$, we have

$$SS_a \perp B_0C_0. \tag{8}$$

From (7) and (8), we see that the orthopole of OH with respect to triangle $A_0B_0C_0$ lies on the line SS_a. Similarly, the orthopole of OH with respect to $A_0B_0C_0$ also lies on SS_b and SS_c, where S_b, S_c are defined in the same way with S_a. Thus,

$$S \text{ is the orthopole of } OH \text{ with respect to triangle } A_0B_0C_0. \tag{9}$$

It is also clear that H is the center of the circle $(A_0B_0C_0)$ and

$$A_3B_3C_3 \text{ is the pedal triangle of } O \text{ with respect to triangle } A_0B_0C_0. \tag{10}$$

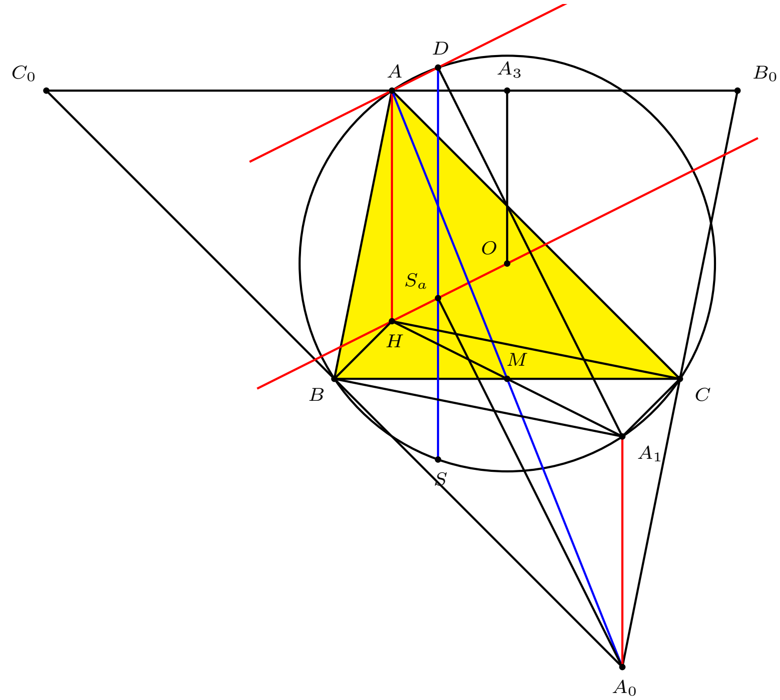

Figure 4.

From (9) and (10), and by Lemma 4, we have $S \in (A_3B_3C_3)$. □

Lemma 6. *If any of the three points in A, B, C, D are not collinear, then the nine-point circles of triangles BCD, CDA, DAB, ABC all pass through one point.*

Lemma 6 is familiar and its simple proof can be found in [2, p.242].

3. Main results

3.1. *A synthetic proof of Theorem 1.* Assume that the circle $I(r)$ inscribed in ABC touches BC, CA, AB at A_0, B_0, C_0 respectively. Let A_1, B_1, C_1 be the images of A_0, B_0, C_0 respectively through the symmetry with center I. Let A_2, B_2, C_2 be the images of I through the reflections with axes B_0C_0, C_0A_0, A_0B_0 respectively. Let A_3, B_3, C_3 be the mid-points of AI, BI, CI respectively (see Figure 5).

Under the inversion in $I(r)$, the points A_2, B_2, C_2 are transformed into the points A_3, B_3, C_3 respectively. As a result, the circles (IA_1A_2), (IB_1B_2), (IC_1C_2) are transformed into the lines A_1A_3, B_1B_3, C_1C_3 respectively. According to Lemma 5(a),
the circles (IA_1A_2), (IB_1B_2), (IC_1C_2) all pass through one point lying on the circle (I), the anti-Steiner point of the Euler line of triangle $A_0B_0C_0$ with respect to the same triangle. We call this point F. (11)
Hence, A_1A_3, B_1B_3, C_1C_3 are also concurrent at F. (12)

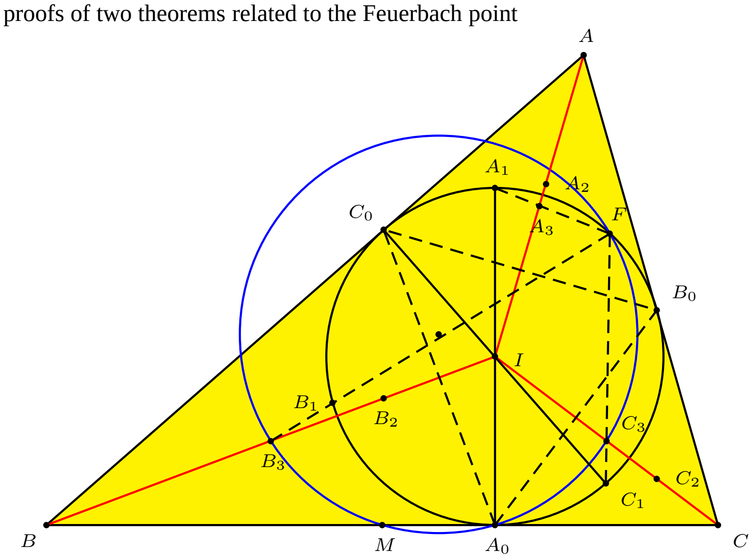

Figure 5.

Because A_1, B_1, C_1 be the images of A_0, B_0, C_0 respectively through the symmetry with center I, A_1B_1, A_1C_1 are parallel to A_0B_0, A_0C_0 respectively.

From this, noting that A_0B_0, A_0C_0 are perpendicular to IC, IB respectively, we deduce that

$$A_1B_1, A_1C_1 \text{ are perpendicular to } IC, IB. \tag{13}$$

Let M be the mid-point of BC. Noting that B_3, C_3 are the mid-points of BI, CI respectively, we have

$$IC//MB_3 \quad \text{and} \quad IB//MC_3. \tag{14}$$

Therefore, we have

$$\begin{aligned} (FB_3, FC_3) &= (FB_1, FC_1) && \text{(by (12))} \\ &= (A_1B_1, A_1C_1) && (A_1 \in (FB_1C_1)) \\ &= (IC, IB) && \text{(by (13))} \\ &= (MB_3, MC_3) && \text{(by (14))}. \end{aligned}$$

From this, $F \in (MB_3C_3)$, the nine-point circle of triangle IBC.

Similarly, F also belongs to the nine-point circles of triangles ICA, IAB.

Thus, from Lemma 6, F belongs to the nine-point circle of triangle ABC. This means that

$$F \text{ is the Feuerbach point of triangle } ABC. \tag{15}$$

From (11) and (15), F is not only the anti-Steiner point of the Euler line of $A_0B_0C_0$ with respect to $A_0B_0C_0$, but also the Feuerbach point of ABC.

Thus, we can conclude that the Feuerbach point of ABC is the anti-Steiner point of the Euler line of $A_0B_0C_0$.

3.2. *A synthetic proof of Theorem 2.* Suppose that the inscribed circle $I(r)$ of triangle ABC touches BC, CA, AB at A_0, B_0, C_0 respectively. Let A', B', C' be the intersections of AI, BI, CI with BC, CA, AB respectively; A'', B'', C'' be the feet of the perpendiculars from A_0, B_0, C_0 to AI, BI, CI respectively and F be the Feuerbach point of ABC (see Figure 6).

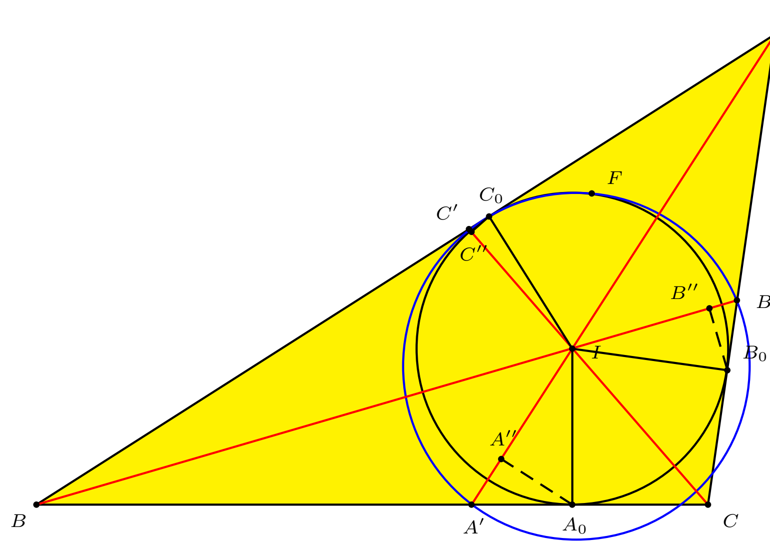

Figure 6.

From Lemma 5(b) and Theorem 1, $F \in (A''B''C'')$. (16)

On the other hand, under inversion in the incircle $I(r)$, F, A'', B'', C'' are transformed into F, A', B', C' respectively. (17)

From (16) and (17), we can conclude that In conclusion, the circumcircle of $A'B'C'$ passes through the Feuerbach point F of ABC.

References

[1] L. Emelyanov and T. Emelyanov, A note on the Feuerbach point, *Forum Geom.*, 1 (2001) 121–124.

[2] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.

[3] B. D. Suceavă and P. Yiu, The Feuerbach point and Euler lines, *Forum Geom*, 6 (2006) 191–197.

[4] J. Vonk, The Feuerbach point and reflections of the Euler line, *Forum Geom*, 9 (2009) 47–55.

[5] P. Yiu, Hyacinthos message 11652, October 18, 2005.

Nguyen Minh Ha: Hanoi University of Education, Hanoi, Vietnam.
E-mail address: minhha27255@yahoo.com

Nguyen Pham Dat: Hanoi University of Education, Hanoi, Vietnam.
E-mail address: datpn@chuyensphn.com