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# Properties of Valtitudes and Vaxes of a Convex Quadrilateral

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**Abstract**. We introduce the vaxes relative to a v-parallelogram and determine several properties of valtitudes and of vaxes. In particular, we study the quadrilateral detected by the valtitudes and the one detected by the vaxes.

Given a convex quadrilateral $\mathbf{Q}$, we call maltitude of $\mathbf{Q}$ the perpendicular line through the midpoint of a side to the opposite side. Maltitudes have been investigated in several papers (see, for example, [2, 7, 8]). In particular in [7] it has been proved that they are concurrent in a point, called anticenter in [9], if and only if $\mathbf{Q}$ is cyclic. Valtitudes relative to a v-parallelogram of a convex quadrilateral $\mathbf{Q}$ were defined in [7]. This definition generalizes the one of maltitudes. Moreover the problem of concurrency of valtitudes relative to a v-parallelogram of a convex quadrilateral $\mathbf{Q}$ was investigated. In this paper we introduce the notion of vaxis relative to a v-parallelogram and we determine several properties of valtitudes and vaxes. In particular, we study the quadrilateral detected by the valtitudes and those detected by the vaxes.

## 1. v-parallelograms

Let $A_1A_2A_3A_4$ be a convex quadrilateral, that we denote by $\mathbf{Q}$. A v-parallelogram of $\mathbf{Q}$ is any parallelogram with vertices on the sides of $\mathbf{Q}$ and sides parallel to the diagonals of $\mathbf{Q}$.

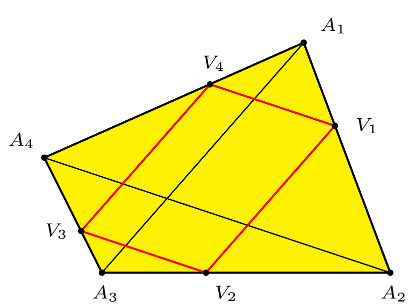


Figure 1.

To obtain a v-parallelogram of $\mathbf{Q}$ we can use the following construction. Fix an arbitrary point $V_1$ on the segment $A_1A_2$. Draw from $V_1$ the parallel to the diagonal $A_1A_3$ and let $V_2$ be the intersection point of this line with the side $A_2A_3$. Draw

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from $V_2$ the parallel to the diagonal $A_2A_4$ and let $V_3$ be the intersection point of this line with the side $A_3A_4$. Finally, draw from $V_3$ the parallel to the diagonal $A_1A_3$ and let $V_4$ be the intersection point of this line with the segment $A_4A_1$. The quadrilateral $V_1V_2V_3V_4$ is a v-parallelogram [7] and, by moving $V_1$ on the segment $A_1A_2$, we obtain all possible v-parallelograms of $\mathbf{Q}$ (see Figure 1).

In the following we will denote by $\mathbf{V}$ a v-parallelogram of $\mathbf{Q}$, with $V_i$ ($i = 1, 2, 3, 4$), vertex of $\mathbf{V}$ on the side $A_iA_{i+1}$ (with indices taken modulo 4) and with $G'$ the common point to the diagonals of $\mathbf{V}$. Observe that $\mathbf{V}$ is orthodiagonal. The v-parallelogram $M_1M_2M_3M_4$, with $M_i$ midpoint of the side $A_iA_{i+1}$, is the Varignon parallelogram of $\mathbf{Q}$. In this particular case $G'$ is the centroid $G$ of $\mathbf{Q}$. We recall that if $M_5$ and $M_6$ are the midpoints of the diagonals $A_1A_3$ and $A_2A_4$ of $\mathbf{Q}$ respectively, the segment $M_5M_6$, that we call the *third bimedian* of $\mathbf{Q}$, passes through $G$ that bisects this segment ([1, 5]).

**Theorem 1.** *The locus described by the common point of the diagonals of a v-parallelogram* $\mathbf{V}$ *of* $\mathbf{Q}$ *by varying* $\mathbf{V}$ *is the third bimedian of* $\mathbf{Q}$.

*Proof.* Let $\mathbf{V}$ be any v-parallelogram of $\mathbf{Q}$ and let $N_1N_2N_3N_4$ be the Varignon parallelogram of $\mathbf{V}$, with midpoint $N_i$ of $V_iV_{i+1}$ (see Figure 2).

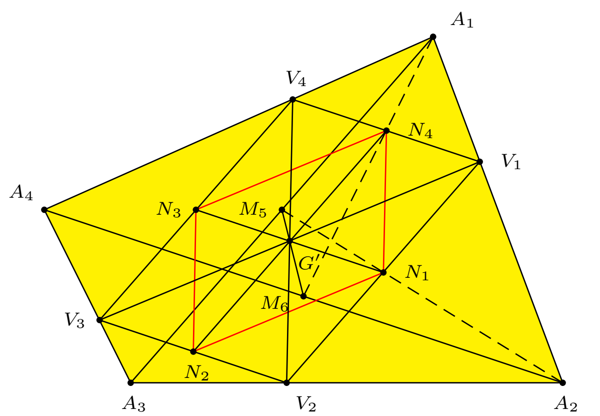


Figure 2.

The triangles $A_1A_2A_3$ and $V_1A_2V_2$ are correspondent in a homothetic transformation with center $A_2$. It follows that

$$\frac{A_1V_1}{A_1A_2} = \frac{A_3V_2}{A_2A_3}. \tag{1}$$

Moreover, $M_5$ and $N_1$ are collinear with $A_2$. Analogously, $M_6$ and $N_4$ are collinear with $A_1$.

Let $G'_1$ and $G'_2$ be the common points of the line $M_5M_6$ with $N_1N_3$ and $N_2N_4$, respectively. Because the triangles $M_5G'_1N_1$ and $M_5M_6A_2$ are similar, as are $V_2A_2N_1$ and $A_3A_2M_5$, we have

$$\frac{M_5G'_1}{M_5M_6} = \frac{M_5N_1}{M_5A_2} = \frac{A_3V_2}{A_2A_3}. \tag{2}$$

Analogously, because the triangles $M_6G'_2N_4$ and $M_5M_6A_1$ are similar, as are $A_1V_1N_4$ and $A_1A_2M_6$, we have

$$\frac{M_5G'_2}{M_5M_6} = \frac{A_1N_4}{A_1M_6} = \frac{A_1V_1}{A_1A_2}. \tag{3}$$

From (1), (2) and (3), it follows that $\frac{M_5G'_1}{M_5M_6} = \frac{M_5G'_2}{M_5M_6}$. Hence, $G'_1 = G'_2 = G'$, and $G'$ lies on the bimedian $M_5M_6$.

Conversely, fix a point $P$ on the bimedian $M_5M_6$. Let $N_1$ be the common point to the line $A_2M_5$ with the parallel line to $A_2A_4$ passing through $P$. Let $V_1$ be the common point to the line $A_1A_2$ with the parallel line to $A_1A_3$ passing through $N_1$. $V_1$ detectS a v-parallelogram $\mathbf{V}$ that has $P$ as common point of its diagonals. □

## 2. Valtitudes

Let $\mathbf{V}$ be a v-parallelogram of $\mathbf{Q}$ and $H_i$ be the foot of the perpendicular to $A_{i+2}A_{i+3}$ from $V_i$. The quadrilateral $H_1H_2H_3H_4$ is called the *orthic quadrilateral* of $\mathbf{Q}$ [6], and we will denote it by $\mathbf{H}$. The lines $V_iH_i$ are called the *valtitudes* of $\mathbf{Q}$ with respect to $\mathbf{V}$ (see Figure 3).

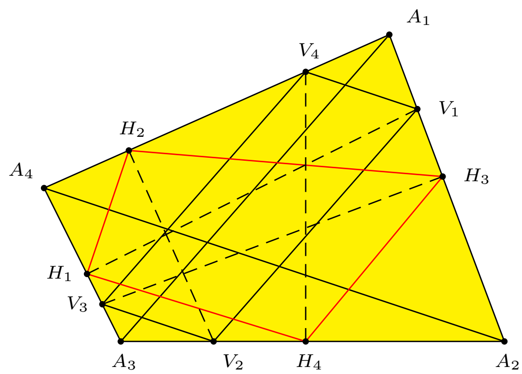


Figure 3.

In the following the valtitude $V_iH_i$ will be denoted by $h_i$. Observe that $\mathbf{H}$ can be a convex, concave, or crossed quadrilateral. If $\mathbf{V}$ is the Varignon parallelogram, the quadrilateral $\mathbf{H}$ is called the *principal orthic quadrilateral* of $\mathbf{Q}$ and the lines $M_iH_i$ are the maltitudes of $\mathbf{Q}$.

Given a v-parallelogram $\mathbf{V}$, *if the valtitudes of* $\mathbf{Q}$ *with respect to* $\mathbf{V}$ *are concurrent, then* $\mathbf{Q}$ *is cyclic or orthodiagonal* [7]. Moreover, *if* $\mathbf{Q}$ *is cyclic or orthodiagonal, there is only one v-parallelogram* $\mathbf{V}^*$ *with respect to which the valtitudes are concurrent*. Precisely,

(a) If $\mathbf{Q}$ is cyclic, $\mathbf{V}^*$ is the Varignon parallelogram of $\mathbf{Q}$ and then the valtitudes that are concurrent are the maltitudes of $\mathbf{Q}$; moreover the concurrency point of the maltitudes is the anticenter $H$ of $\mathbf{Q}$; $H$ is the symmetric of the circumcenter O with respect to the centroid $G$ of $\mathbf{Q}$ and the line containing the three points $H$, $O$ and $G$ is the Euler line of $\mathbf{Q}$ (see Figure 4).

The line through the midpoint $M_5$ of the diagonal $A_1A_3$ of $\mathbf{Q}$ perpendicular to the diagonal $A_2A_4$ and the line through the midpoint $M_6$ of $A_2A_4$ perpendicular

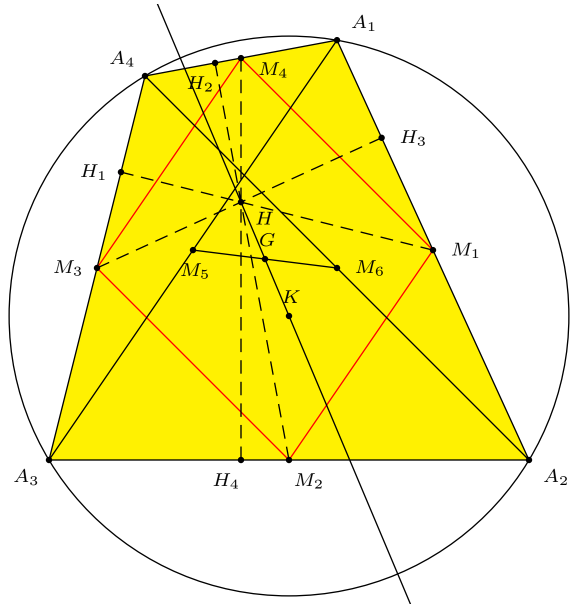


Figure 4.

to $A_1A_3$ are concurrent in $H$ [6]. Observe that $G$ is the midpoint of the segments $OH$ and $M_5M_6$, then the quadrilateral $OM_5HM_6$ is a parallelogram with $G$ as the common point to the diagonals.

(b) If $\mathbf{Q}$ is orthodiagonal, $\mathbf{V}^*$ is the v-parallelogram detected from the perpendiculars to the sides of $\mathbf{Q}$ through the common point $K$ of the diagonals of $\mathbf{Q}$, that is then the concurrency point of the valtitudes (see Figure 5).

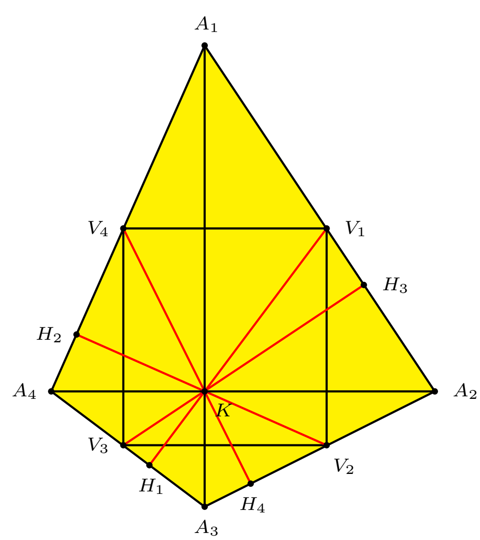


Figure 5.

## 3. Vaxes

Let $\mathbf{Q}$ be a convex quadrilateral and $\mathbf{V}$ a v-parallelogram of $\mathbf{Q}$.

We call the *vaxis* relative to the side $A_iA_{i+1}$ the perpendicular to $A_iA_{i+1}$ through $V_i$ and denote it by $k_i$.

**Theorem 2.** *If $\mathbf{V}$ is a v-parallelogram of $\mathbf{Q}$ and $G'$ is the common point of the diagonals of $\mathbf{V}$, in the symmetry with center $G'$ the valtitudes relative to $\mathbf{V}$ correspond with the vaxes relative to $\mathbf{V}$.*

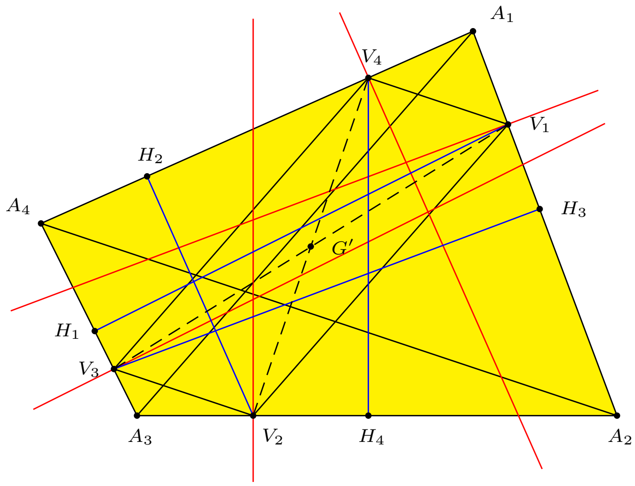


Figure 6.

*Proof.* In fact, $V_i$ and $V_{i+2}$ are symmetric with respect to $G'$ (see Figure 6). Then the vaxis $k_i$ and the line parallel to it passing through $V_{i+2}$, *i.e.*, the valtitude $h_{i+2}$, are correspondent in the symmetry with center $G'$. □

From Theorem 2 it follows that given a v-parallelogram $\mathbf{V}$, *the vaxes of $\mathbf{Q}$ relative to $\mathbf{V}$ are concurrent if and only if the valtitudes of $\mathbf{Q}$ relative to $\mathbf{V}$ are concurrent*.

Then, from the concurrency properties of valtitudes, it follows that *if the vaxes are concurrent, then $\mathbf{Q}$ is cyclic or orthodiagonal*. Moreover, *if $\mathbf{Q}$ is cyclic or orthodiagonal, there is only one v-parallelogram $\mathbf{V}^*$ such that the valtitudes relative to it are concurrent*. Precisely,

(a) If $\mathbf{Q}$ is cyclic, $\mathbf{V}^*$ is the Varignon parallelogram of $\mathbf{Q}$, and the vaxes that are concurrent are the axes of $\mathbf{Q}$ and the concurrency point is the circumcenter $O$ of $\mathbf{Q}$.

(b) If $\mathbf{Q}$ is orthodiagonal, $\mathbf{V}^*$ is the v-parallelogram detected by the perpendiculars to the sides of $\mathbf{Q}$ through the common point $K$ of the diagonals of $\mathbf{Q}$ and the concurrency point of the vaxes is the point $K'$ symmetric of $K$ with respect to $G'$ (see Figure 7).

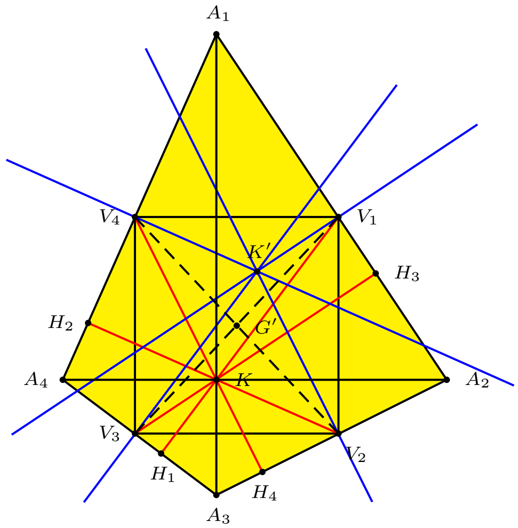


Figure 7.

## 4. The quadrilateral of valtitudes and the quadrilateral of vaxes

Let $\mathbf{Q}$ be a convex quadrilateral and $\mathbf{V}$ a v-parallelogram of $\mathbf{Q}$.

Let $B_i$ be the common point to the valtitudes $h_i$ and $h_{i+1}$. We call $B_1B_2B_3B_4$ the *quadrilateral of the valtitudes* and denote it by $\mathbf{Q}_h$.

Let $C_i$ be the common point of the vaxes $k_i$ and $k_{i+1}$. We call $C_1C_2C_3C_4$ the *quadrilateral of the vaxes* and denote it by $\mathbf{Q}_k$ (see Figure 8).

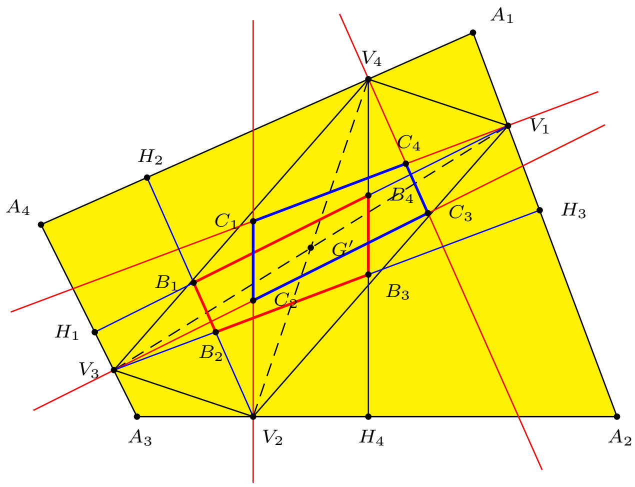


Figure 8.

If $\mathbf{V}$ is the Varignon parallelogram, the lines $h_i$ are the maltitudes and $\mathbf{Q}_h$ is called the *quadrilateral of the maltitudes* of $\mathbf{Q}$ [4]. The lines $k_i$ are the axes of $\mathbf{Q}$, $C_i$ is the circumcenter of the triangle $A_iA_{i+1}A_{i+2}$ and $\mathbf{Q}_k$ is called the *quadrilateral of the circumcenters* of $\mathbf{Q}$ [4]. Observe that when $\mathbf{V}$ is the Varignon parallelogram, if $\mathbf{Q}$ is cyclic, then $\mathbf{Q}_h$ and $\mathbf{Q}_k$ are reduced to a point.

The theorem below follows from Theorem 2.

**Theorem 3.** *If* $\mathbf{V}$ *is a v-parallelogram of* $\mathbf{Q}$ *and* $G'$ *is the common point of the diagonals of* $\mathbf{V}$*, the quadrilateral of the vaxes and the quadrilateral of the valtitudes are symmetric with respect to* $G'$*.*

*Proof.* In fact, the valtitude $h_{i+2}$ is the correspondent of the vaxis $k_i$ in the symmetry with center $G'$, and the point $B_{i+2}$ is the correspondent of the point $C_i$. □

**Corollary 4** ([4, p.474])**.** *If* $\mathbf{V}$ *is the Varignon parallelogram of* $\mathbf{Q}$*, the quadrilateral of the circumcenters and the quadrilateral of the maltitudes are symmetric with respect to the centroid* $G$ *of* $\mathbf{Q}$*.*

Let $K$ and $K'$ be the common points of the diagonals of $\mathbf{Q}$ and of $\mathbf{Q}_k$ respectively.

**Lemma 5.** *If* $\mathbf{Q}$ *is orthodiagonal, the triangles* $A_iA_{i+1}K$ *and* $C_iC_{i+3}K'$, $(i = 1, 2, 3, 4)$ *are similar.*

*Proof.* Since $\mathbf{Q}$ is orthodiagonal, the vertices $B_i$ of $\mathbf{Q}_h$ lie on the diagonals of $\mathbf{Q}$ [6]. The diagonals of $\mathbf{Q}_h$ and those of $\mathbf{Q}$ lie on the same lines (see Figure 9). It follows that $\mathbf{Q}_h$ is orthodiagonal. Then, by Theorem 3, $\mathbf{Q}_k$ is orthodiagonal as well, and the diagonals of $\mathbf{Q}_k$ are parallel to those of $\mathbf{Q}$. Then, the lines $C_1C_3$ and $C_2C_4$ are perpendicular to the lines $A_1A_3$ and $A_2A_4$ respectively. Moreover, the line $C_1C_4$ is perpendicular to $A_1A_2$. Therefore, the triangles $A_1A_2K$ and $C_1C_4K'$ are similar, because they have equal angles. Analogously, the similarlity of each of the pairs $A_2A_3K$, $C_2C_1K'$; $A_3A_4K$, $C_3C_2K'$; and $A_4A_1K$, $C_4C_3K'$ can be established. □

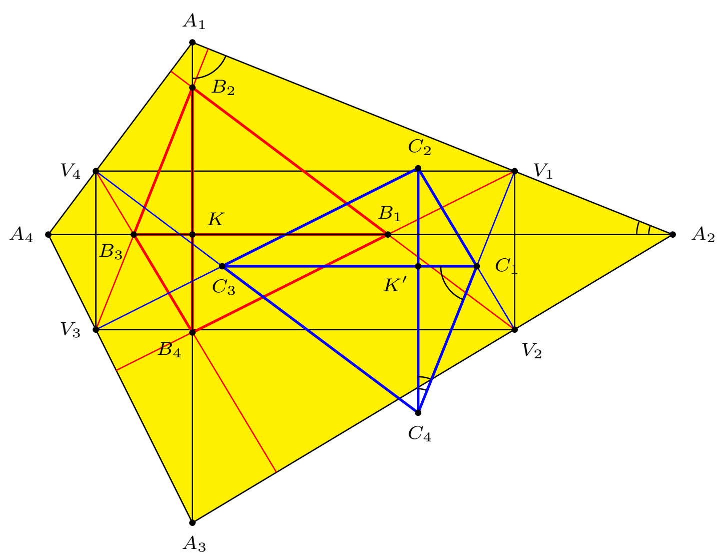


Figure 9.

Let us make some preliminary remarks.

For the two ratios $\frac{A_1K}{KA_3}$ and $\frac{A_3K}{KA_1}$ let $r$ be the one not greater than 1. Also, for the two ratios $\frac{A_2K}{KA_4}$ and $\frac{A_4K}{KA_2}$, let $r'$ be the one not greater than 1. The pair $\{r, r'\}$ is called the characteristic of $\mathbf{Q}$. In [3] it was proved that two quadrilaterals are affine if and only if they have the same characteristic.

**Theorem 6.** *If* $\mathbf{Q}$ *is orthodiagonal and* $\mathbf{V}$ *is a v-parallelogram of* $\mathbf{Q}$*, the quadrilateral of the vaxes and the quadrilateral of the valtitudes are affine to* $\mathbf{Q}$*.*

*Proof.* From Lemma 5, we have

$$\frac{A_1K}{A_2K} = \frac{C_1K'}{C_4K'}, \tag{4}$$

$$\frac{A_2K}{A_3K} = \frac{C_2K'}{C_1K'}, \tag{5}$$

$$\frac{A_3K}{A_4K} = \frac{C_3K'}{C_2K'}. \tag{6}$$

By multiplying (4) and (5), and also (5) and (6), we obtain:

$$\frac{A_1K}{A_3K} = \frac{C_2K'}{C_4K'},$$

$$\frac{A_2K}{A_4K} = \frac{C_3K'}{C_1K'}.$$

Thus the quadrilaterals $\mathbf{Q}$ and $\mathbf{Q}_k$ have the same characteristic, and therefore are affine. From theorem 3, also $\mathbf{Q}_h$ is affine to $\mathbf{Q}$. □

**Lemma 7.** *If* $\mathbf{Q}$ *is cyclic, the angles of* $\mathbf{Q}_k$ *are equal to those of* $\mathbf{Q}$*. Precisely,* $\angle C_iC_{i+1}C_{i+2} = \angle A_{i-1}A_iA_{i+1}$ (*i*=1,2,3,4).

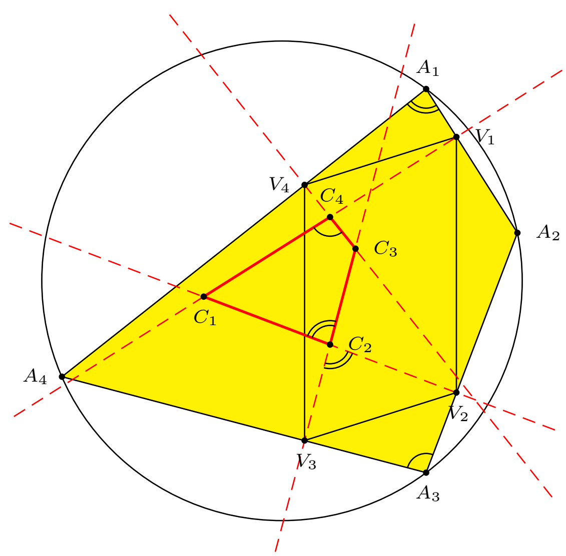


Figure 10.

*Proof.* Let us prove that $\angle C_1C_2C_3 = \angle A_4A_1A_2$ (see Figure 10). The other cases can be established analogously. Since $\mathbf{Q}$ is cyclic, $\angle A_4A_1A_2$ and $\angle A_2A_3A_4$ are supplementary angles. Moreover, the angles at $V_2$ and $V_4$ of the quadrilateral $V_3C_2V_2A_3$ are right angles. Therefore, $\angle C_1C_2C_3$ and $\angle A_2A_3A_4$ are supplementary angles. It follows that $\angle C_1C_2C_3 = \angle A_4A_1A_2$. □

**Theorem 8.** *If $\mathbf{Q}$ is cyclic, then the quadrilateral of the vaxes and the quadrilateral of the valtitudes are cyclic.*

*Proof.* Since $\mathbf{Q}$ is cyclic, $\angle A_4A_1A_2$ and $\angle A_2A_3A_4$ are supplementary angles. Therefore, from Lemma 7, $\angle C_1C_2C_3$ and $\angle C_1C_4C_2$ are supplementary angles. Then, $\mathbf{Q}_k$ is cyclic and, from Theorem 3, $\mathbf{Q}_h$ is cyclic as well. □

**Theorem 9.** *If $\mathbf{Q}$ is cyclic and orthodiagonal and $\mathbf{V}$ is a v-parallelogram of $\mathbf{Q}$, the quadrilateral of the vaxes and the quadrilateral of the valtitudes are similar to $\mathbf{Q}$.*

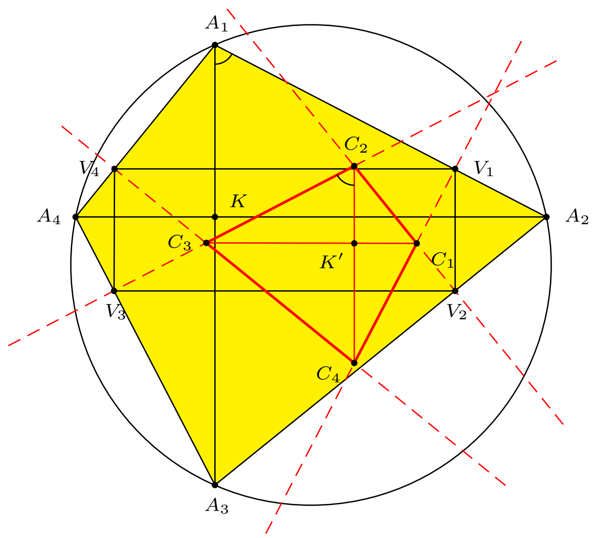


Figure 11.

*Proof.* From Lemma 7, $\mathbf{Q}$ and $\mathbf{Q}_k$ have equal angles. Let us prove now that the sides of $\mathbf{Q}$ are proportional to those of $\mathbf{Q}_k$. Consider the triangles $A_1A_2A_3$ and $C_2C_3C_4$ (see Figure 11). From Lemma 5 the triangles $A_1A_2K$ and $C_2C_3K'$ are similar, and $\angle KA_1A_2 = \angle K'C_2C_3$. Since, from Lemma 7, $\angle A_1A_2A_3 = \angle C_2C_3C_4$, the triangles $A_1A_2A_3$ and $C_2C_3C_4$ are similar.

Analogously, the similarity of each of the following pairs of triangles can be established: $A_2A_3A_4$, $C_3C_4C_1$; $A_3A_4A_1$, $C_4C_1C_2$; and $A_4A_1A_2$, $C_1C_2C_3$. It follows that

$$\frac{A_1A_2}{C_2C_3} = \frac{A_2A_3}{C_3C_4} = \frac{A_3A_4}{C_4C_1} = \frac{A_4A_1}{C_1C_2},$$

and the sides of $\mathbf{Q}$ are proportional to those of $\mathbf{Q}_k$.

Therefore, $\mathbf{Q}_k$ is similar to $\mathbf{Q}$, and from Theorem 3, $\mathbf{Q}_h$ is also similar to $\mathbf{Q}$. □

**Lemma 10.** *If* $\mathbf{V}$ *is a v-parallelogram of* $\mathbf{Q}$ *and* $M_i$ *is the midpoint of the side* $A_iA_{i+1}$ *of* $\mathbf{Q}$ $(i = 1, 2, 3, 4)$*, then*

$$\frac{A_1V_1}{A_1M_1} = \frac{A_1V_4}{A_1M_4} = \frac{A_3V_2}{A_3M_2} = \frac{A_3V_3}{A_3M_3}, \tag{7}$$

$$\frac{A_2V_1}{A_2M_1} = \frac{A_2V_2}{A_2M_2} = \frac{A_4V_3}{A_4M_3} = \frac{A_4V_4}{A_4M_4}. \tag{8}$$

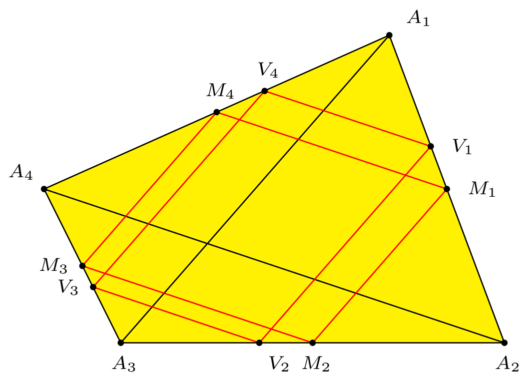


Figure 12.

*Proof.* In fact, since the triangles $A_1V_1V_4$ and $A_1M_1M_4$ are similar, as are triangles $A_3V_2V_3$ and $A_3M_2M_3$ (see Figure 12), we have

$$\frac{A_1V_1}{A_1M_1} = \frac{A_1V_4}{A_1M_4} = \frac{V_1V_4}{M_1M_4}, \qquad \frac{A_3V_2}{A_3M_2} = \frac{A_3V_3}{A_3M_3} = \frac{V_2V_3}{M_2M_3}.$$

Since $V_1V_4 = V_2V_3$ and $M_1M_4 = M_2M_3$, (7) holds.

Analogously, since the triangles $A_2V_1V_2$ and $A_2M_1M_2$ are similar, as are $A_4V_3V_4$ and $A_4M_3M_4$, (8) also holds. □

**Theorem 11.** *If* $\mathbf{Q}$ *is cyclic, the diagonals of the quadrilateral of the vaxes and those of the quadrilateral of the valtitudes are parallel to the diagonals of* $\mathbf{Q}$.

*Proof.* Let $O$ be the circumcenter of $\mathbf{Q}$ (see Figure 13). Let $C_4'$ and $C_4''$ be the common points of the line $A_1O$ with the vaxes $k_1$ and $k_4$ respectively. Since the triangles $A_1V_1C_4'$ and $A_1M_1O$ are similar, as are triangles $A_1V_4C_4''$ and $A_1M_4O$, we have

$$\frac{A_1V_1}{A_1M_1} = \frac{A_1C_4'}{A_1O}, \qquad \frac{A_1V_4}{A_1M_4} = \frac{A_1C_4''}{A_1O}.$$

From (7), we have $\frac{A_1C_4'}{A_1O} = \frac{A_1C_4''}{A_1O}$. Therefore, $C_4'' = C_4' = C_4$, and $C_4$ lies on the line $A_1O$. Moreover,

$$\frac{A_1C_4}{A_1O} = \frac{A_1V_1}{A_1M_1} = \frac{A_1V_4}{A_1M_4}. \tag{9}$$

Analogously, it is possible to prove that $C_2$ lies on the line $A_3O$ and

$$\frac{A_3C_2}{A_3O} = \frac{A_3V_2}{A_3M_2} = \frac{A_3V_3}{A_3M_3}. \tag{10}$$

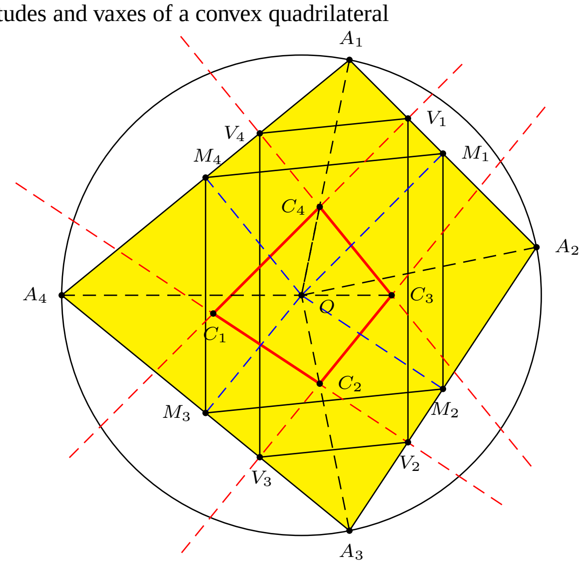


Figure 13.

From (9), (7) and (10), it follows that

$$\frac{A_1C_4}{A_1O} = \frac{A_3C_2}{A_3O}.$$

Thus, the triangles $OC_2C_4$ and $OA_1A_3$ are similar, and the diagonal $C_2C_4$ of $\mathbf{Q}_k$ is parallel to the diagonal $A_1A_3$ of $\mathbf{Q}$.

Analogously, by using (8), it is possible to prove that the triangles $OC_1C_3$ and $OA_2A_4$ are similar, and the diagonal $C_1C_3$ of $\mathbf{Q}_k$ is parallel to the diagonal $A_2A_4$ of $\mathbf{Q}$. Since, from Theorem 3, $\mathbf{Q}_k$ and $\mathbf{Q}_h$ are symmetric with respect to a point, the diagonals of $\mathbf{Q}_h$ are parallel to the diagonals of $\mathbf{Q}_k$ and thus they are parallel to the diagonals of $\mathbf{Q}$. □

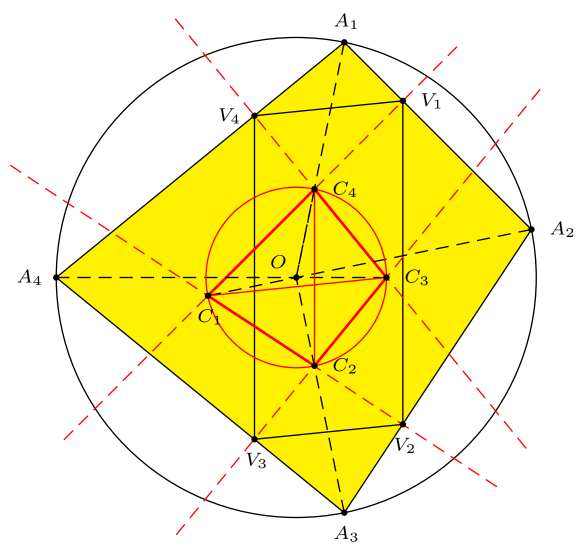


Figure 14.

**Theorem 12.** *If* $\mathbf{Q}$ *is cyclic and* $\mathbf{V}$ *is a v-parallelogram of* $\mathbf{Q}$*, the quadrilateral of the vaxes relative to* $\mathbf{V}$ *has the same circumcenter of* $\mathbf{Q}$.

*Proof.* From Theorem 8, $\mathbf{Q}_k$ is cyclic. The axes of segments $C_2C_4$ and $C_1C_3$ meet at the circumcenter of $\mathbf{Q}_k$. The triangles $OC_2C_4$ and $OA_1A_3$ are correspondent in a homothetic transformation with center the circumcenter $O$ of $\mathbf{Q}$, because, from theorem 11, the lines $C_2C_4$ and $A_1A_3$ are parallel (see Figure 14). It follows that the axes of segments $C_2C_4$ and $A_1A_3$ coincide. Analogously, the axes of segments $C_1C_3$ and $A_2A_4$ coincide. Then it follows that $O$ is the circumcenter of $\mathbf{Q}_k$. □

**Theorem 13.** *If* $\mathbf{Q}$ *is cyclic, all the quadrilaterals of the vaxes of* $\mathbf{Q}$ *have the same Euler line.*

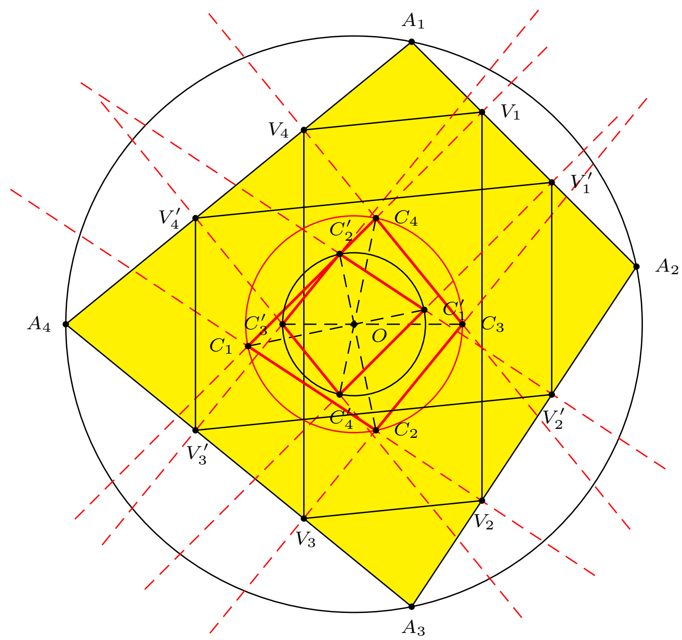


Figure 15.

*Proof.* Consider two v-parallelograms $\mathbf{V}$ and $\mathbf{V}'$ and their quadrilaterals of the vaxes $\mathbf{Q}_k$ and $\mathbf{Q}_k'$ respectively (see Figure 15). The vertices $C_i$ and $C_i'$ of $\mathbf{Q}_k$ and $\mathbf{Q}_k'$ respectively lie on the line $OA_{i+1}$, and the ratio between $OC_i$ and $OC_i'$ is equal to the ratio between the circumradii of $\mathbf{Q}_k$ and $\mathbf{Q}_k'$. Then, $\mathbf{Q}_k$ and $\mathbf{Q}_k'$ are correspondent in a homothetic transformation with center $O$. From Theorem 12, the Euler line of $\mathbf{Q}_k$ passes through $O$, therefore it is fixed in the homothetic transformation. It follows that $\mathbf{Q}_k$ and $\mathbf{Q}_k'$ have the same Euler line. □

We call the *k-line* of $\mathbf{Q}$ (cyclic) the Euler line of all the quadrilaterals of the vaxes of $\mathbf{Q}$.

**Theorem 14.** *If* $\mathbf{Q}$ *is cyclic and* $\mathbf{V}$ *is a v-parallelogram of* $\mathbf{Q}$*, the quadrilateral of the valtitudes relative to* $\mathbf{V}$ *has the same anticenter of* $\mathbf{Q}$.

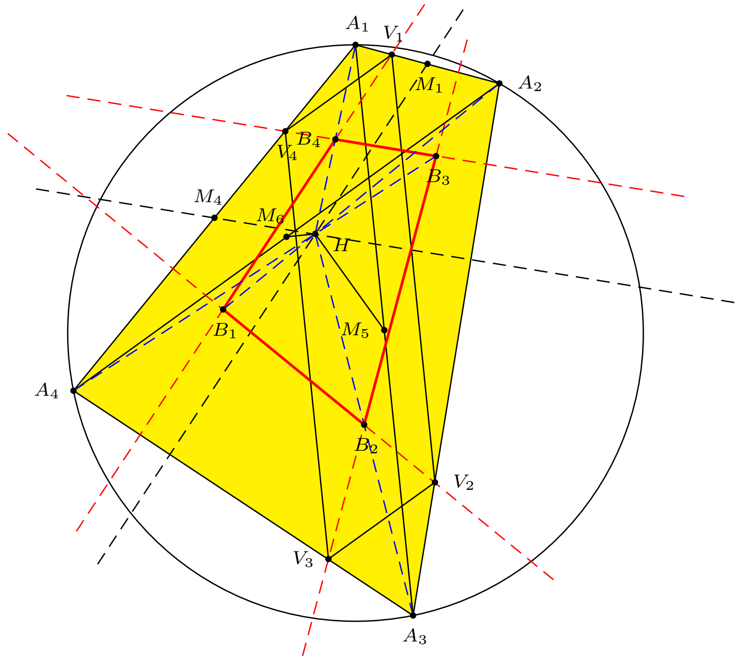


Figure 16.

*Proof.* Let $H$ be the anticenter of $\mathbf{Q}$. Let $B_4'$ and $B_4''$ be the common points of the line $A_1H$ with the valtitudes $h_1$ and $h_4$, respectively (see Figure 16).

Since the triangles $A_1V_1B_4'$ and $A_1M_1H$ are similar, as are $A_1V_4B_4''$ and $A_1M_4H$, we have

$$\frac{A_1V_1}{A_1M_1} = \frac{A_1B_4'}{A_1H}, \qquad \frac{A_1V_4}{A_1M_4} = \frac{A_1B_4''}{A_1H}.$$

From (7) it follows that

$$\frac{A_1B_4'}{A_1H} = \frac{A_1B_4''}{A_1H}.$$

Therefore, $B_4' = B_4'' = B_4$ and $B_4$ lies on the line $A_1H$. Analogously it is possible to prove that $B_2$ lies on the line $A_3H$.

Now consider the third bimedian $M_5M_6$ of $\mathbf{Q}$, with $M_5$ and $M_6$ the midpoints of the diagonals $A_1A_3$ and $A_2A_4$ of $\mathbf{Q}$ respectively. Let $h_5$ be the perpendicular to the line $A_2A_4$ through the point $M_5$ and let $h_6$ be the perpendicular to the line $A_1A_3$ through $M_6$. The lines $h_5$ and $h_6$ pass through $H$ (see §2). The triangles $HB_2B_4$ and $HA_1A_3$ are correspondent in a homothetic transformation with center $H$, because, from Theorem 11, $B_2B_4$ and $A_1A_3$ are parallel. It follows that $h_5$ passes through the midpoint of $B_2B_4$ and it is perpendicular to $B_1B_3$, then it passes through the anticenter of $\mathbf{Q}_h$. Analogously, $h_6$ passes through the anticenter of $\mathbf{Q}_h$ as well, then $H$ is the anticenter of $\mathbf{Q}_h$. □

**Theorem 15.** *If* $\mathbf{Q}$ *is cyclic, all the quadrilaterals of the valtitudes of* $\mathbf{Q}$ *have the same Euler line.*

*Proof.* Given a v-parallelogram $\mathbf{V}$ and the quadrilaterals $\mathbf{Q}_k$ and $\mathbf{Q}_h$ relative to it, from Theorem 3, the Euler line of $\mathbf{Q}_h$ is the symmetric of the Euler line of $\mathbf{Q}_k$ with respect to the point $G'$, common point to the diagonals of $\mathbf{V}$. Then, the theorem follows from Theorem 13. □

We call the *h-line* of $\mathbf{Q}$ (cyclic) the Euler line of all the quadrilaterals of the valtitudes of $\mathbf{Q}$.

**Theorem 16.** *If $\mathbf{Q}$ is cyclic, the h-line and the k-line of $\mathbf{Q}$ are parallel and are symmetric with respect to the line containing the third bimedian of $\mathbf{Q}$.*

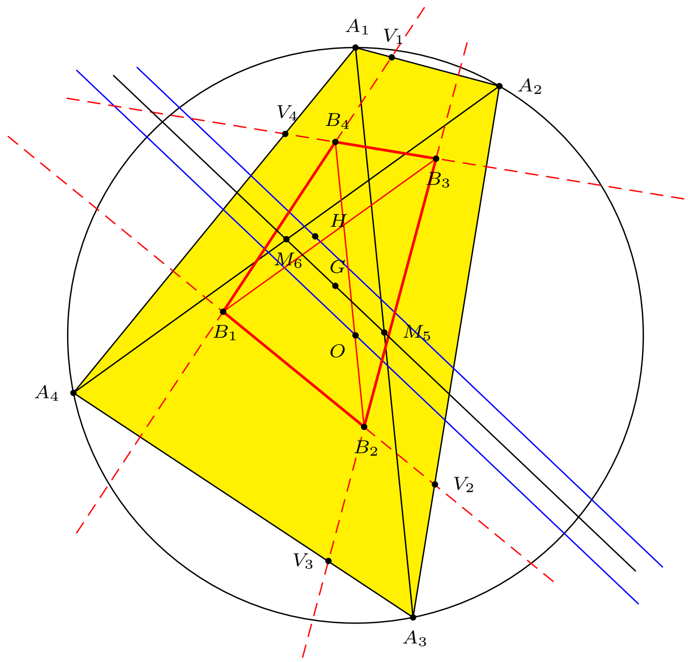


Figure 17.

*Proof.* From Theorems 3, 13 and 15 it follows that the h-line and the k-line of $\mathbf{Q}$ are symmetric with respect to $G'$, common point of the diagonals of any v-parallelogram of $\mathbf{Q}$. Therefore, in particular, they are parallel. Moreover, from Theorem 1, the points $G'$ lie on the third bimedian of $\mathbf{Q}$, then the h-line and the k-line of $\mathbf{Q}$ are symmetric with respect to the line containing the third bimedian of $\mathbf{Q}$ (see Figure 17). □

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