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A New Proof of Yun's Inequality for Bicentric Quadrilaterals

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Abstract. We give a new proof of a recent inequality for bicentric quadrilaterals that is an extension of the Euler-like inequality $R \geq \sqrt{2}r$.

A *bicentric quadrilateral* $ABCD$ is a convex quadrilateral that has both an incircle and a circumcircle. In [6], Zhang Yun called these "double circle quadrilaterals" and proved that

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \leq 1$$

where r and R are the inradius and circumradius respectively. While his proof mainly focused on the angles of the quadrilateral and how they are related to the two radii, our proof is based on calculations with the sides.

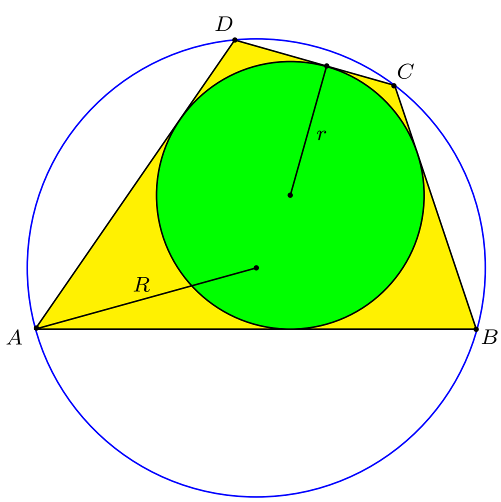

Figure 1. A bicentric quadrilateral with its inradius and circumradius

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In [4, p.156] we proved that the half angles of tangent in a bicentric quadrilateral $ABCD$ with sides a, b, c, d are given by

$$\tan\frac{A}{2} = \sqrt{\frac{bc}{ad}} = \cot\frac{C}{2},$$
$$\tan\frac{B}{2} = \sqrt{\frac{cd}{ab}} = \cot\frac{D}{2}.$$

We need to convert these into half angle formulas of sine and cosine. The trigonometric identities

$$\sin\frac{x}{2} = \frac{\tan\frac{x}{2}}{\sqrt{\tan^2\frac{x}{2}+1}},$$
$$\cos\frac{x}{2} = \frac{1}{\sqrt{\tan^2\frac{x}{2}+1}}$$

yields

$$\sin\frac{A}{2} = \sqrt{\frac{bc}{ad+bc}} = \cos\frac{C}{2}, \tag{1}$$
$$\cos\frac{A}{2} = \sqrt{\frac{ad}{ad+bc}} = \sin\frac{C}{2} \tag{2}$$

and

$$\sin\frac{B}{2} = \sqrt{\frac{cd}{ab+cd}} = \cos\frac{D}{2}, \tag{3}$$
$$\cos\frac{B}{2} = \sqrt{\frac{ab}{ab+cd}} = \sin\frac{D}{2}. \tag{4}$$

From the formulas for the inradius and circumradius in a bicentric quadrilateral (these where also used by Yun, but in another way)

$$r = \frac{2\sqrt{abcd}}{a+b+c+d},$$
$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$$

we have

$$\begin{aligned}\frac{r\sqrt{2}}{R} &= \frac{8\sqrt{2}abcd}{(a+b+c+d)\sqrt{(ab+cd)(ac+bd)(ad+bc)}} \\ &\le \frac{8\sqrt{2}abcd}{4\sqrt[4]{abcd}\sqrt{(ab+cd)(ad+bc)}\sqrt{2\sqrt{acbd}}} \\ &= \frac{2\sqrt{abcd}}{\sqrt{(ab+cd)(ad+bc)}}\end{aligned}$$

where we used the AM-GM inequality twice.

Let us for the sake of brevity denote the trigonometric expression in the parenthesis in Yun's inequality by Σ. Thus

$$\Sigma = \sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}$$

and the half angle formulas (1), (2), (3) and (4) yields

$$\Sigma = \frac{\sqrt{ab^2c} + \sqrt{bc^2d} + \sqrt{acd^2} + \sqrt{a^2bd}}{\sqrt{(ab+cd)(ad+bc)}} = \frac{(\sqrt{ab}+\sqrt{cd})(\sqrt{ad}+\sqrt{bc})}{\sqrt{(ab+cd)(ad+bc)}}.$$

Using the AM-GM inequality again,

$$(\sqrt{ab}+\sqrt{cd})(\sqrt{ad}+\sqrt{bc}) \geq 2\sqrt{\sqrt{ab}\sqrt{cd}}\cdot 2\sqrt{\sqrt{ad}\sqrt{bc}} = 4\sqrt{abcd}.$$

Hence

$$\frac{r\sqrt{2}}{R} \leq \frac{2\sqrt{abcd}}{\sqrt{(ab+cd)(ad+bc)}} \leq \frac{1}{2}\Sigma.$$

This proves the left hand side of Yun's inequality.

For the right hand side we need to prove that

$$\frac{(\sqrt{ab}+\sqrt{cd})(\sqrt{ad}+\sqrt{bc})}{\sqrt{(ab+cd)(ad+bc)}} \leq 2.$$

By symmetry it is enough to prove the inequality

$$\frac{\sqrt{ab}+\sqrt{cd}}{\sqrt{ab+cd}} \leq \sqrt{2}.$$

Since both sides are positive, we can rewrite this as

$$(\sqrt{ab}+\sqrt{cd})^2 \leq 2(ab+cd) \quad \Leftrightarrow \quad 2\sqrt{abcd} \leq ab+cd$$

which is true according to the AM-GM inequality.

This completes our proof of Yun's inequality for bicentric quadrilaterals. From the calculations with the AM-GM inequality we see that there is equality on the left hand side only when all the sides are equal since we used $a+b+c+d \geq 4\sqrt[4]{abcd}$, with equality only if $a = b = c = d$. On the right hand side we have equality only if $ab = cd$ and $ad = bc$, which is equivalent to $a = c$ and $b = d$. Since it is a bicentric quadrilateral, we have equality on either side if and only if it is a square.

It can be noted that since opposite angles in a bicentric quadrilateral are supplementary angles, Yun's inequality can also (after rearranging the terms) be stated as either

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{D}{2} + \sin\frac{D}{2}\sin\frac{A}{2}\right) \leq 1$$

or

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2} + \cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{D}{2} + \cos\frac{D}{2}\cos\frac{A}{2}\right) \leq 1.$$

We conclude this note by a few comments on the simpler inequality $R \geq \sqrt{2}r$. According to [2, p.132] it was proved by Gerasimov and Kotii in 1964. The next year, the American mathematician Carlitz published a paper [3] where he derived a generalization of Euler's triangle formula to a bicentric quadrilateral. His formula gave $R \geq \sqrt{2}r$ as a special case. Another proof can be based on Fuss' theorem, see [5]. The inequality also directly follows from the fact that the area K of a bicentric quadrilateral satisfies $2R^2 \geq K \geq 4r^2$, which was proved in [1].

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