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Three Conics Derived from Perpendicular Lines

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Abstract. Given a triangle ABC and a generic point P on its plain, we consider the rectangular hyperbola $\mathcal{H}$ which is the isogonal conjugate of the line OP where O is the circumcenter of the triangle. We also consider the line $\mathcal{L}$ perpendicular to OP at the point P, the conic $\mathcal{E}$ which is the isogonal conjugate of this line and the inscribed parabola $\mathcal{P}$, tangent to the line $\mathcal{L}$. We discuss some relations between this three conics.

Let ABC be a triangle with sides a, b and c. Let P be a generic point with homogenous barycentric coordinates $(u : v : w)$ and

$$O = (a^2 S_A : b^2 S_B : c^2 S_C),$$

the circumcenter of the triangle ABC. The line OP is given by

$$\sum_{\text{cyclic}} (c^2 S_C v - b^2 S_B w)x = 0. \tag{1}$$

Let us define

$$p_a = -u + v + w, \quad p_b = u - v + w, \quad p_c = u + v - w,$$

and

$$\lambda_a = p_b\, S_B - p_c\, S_C, \quad \lambda_b = p_c\, S_C - p_a\, S_A, \quad \lambda_c = p_a\, S_A - p_b\, S_B.$$

Lemma 1. *In terms of these expressions,*
(a) *the line OP can be expressed as*

$$\sum_{\text{cyclic}} (b^2 \lambda_c + c^2 \lambda_b)x = 0, \tag{2}$$

(b) *the point at infinity of the line OP is given by*

$$I_{OP} = (\lambda_b\, S_B - \lambda_c\, S_C : \lambda_c\, S_C - \lambda_a\, S_A : \lambda_a\, S_A - \lambda_b\, S_B)\,, \tag{3}$$

(c) *and the infinite point of perpendicular lines to OP is given by*

$$I_{\mathcal{L}} = (\lambda_a : \lambda_b : \lambda_c). \tag{4}$$

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Equations (2), (3) and (4) follow easily from (1) and the definitions.

Let $\mathcal{L}$ be the line perpendicular to the line OP at the point P, with equation

$$\mathcal{L}: \qquad (\lambda_c\, v - \lambda_b\, w)\, x + (\lambda_a\, w - \lambda_c\, u)\, y + (\lambda_b\, u - \lambda_a\, v)\, z = 0.$$

Next we shall consider the isogonal conjugates of the lines OP and $\mathcal{L}$. The isogonal conjugate of the line OP is the rectangular hyperbola

$$\mathcal{H}: \qquad \sum_{\text{cyclic}} a^2 \left(b^2\lambda_c + c^2\lambda_b\right) y\, z = 0.$$

The fourth point of intersection of the hyperbola $\mathcal{H}$ with the circumcircle is the isogonal conjugate of the point I_{OP}:

$$H' = \left(\frac{a^2}{\lambda_b\, S_B - \lambda_c\, S_C} : \frac{b^2}{\lambda_c\, S_C - \lambda_a\, S_A} : \frac{c^2}{\lambda_a\, S_A - \lambda_b\, S_B}\right).$$

The center M of $\mathcal{H}$ (on the nine point circle) is the midpoint of the points H and H', where H is the orthocenter of the triangle ABC,

$$M = \left(\left(b^2\lambda_c + c^2\lambda_b\right)\lambda_a : \left(c^2\lambda_a + a^2\lambda_c\right)\lambda_b : \left(a^2\lambda_b + b^2\lambda_a\right)\lambda_c\right).$$

The circumconic $\mathcal{E}$ is the isogonal conjugate of $\mathcal{L}$:

$$\mathcal{E}: \qquad \sum_{\text{cyclic}} a^2\, (\lambda_c\, v - \lambda_b\, w)\, yz = 0.$$

The center of the circumconic $\mathcal{E}$ is the point

$$N = \left(a^2\, (\lambda_c\, v - \lambda_b\, w)\left(b^2\lambda_c w - c^2\lambda_b v + \lambda_b\lambda_c\right) : \cdots : \cdots\right).$$

The fourth intersection of $\mathcal{E}$ with the circumcircle is the isogonal conjugate of the point $I_{\mathcal{L}}$

$$E = \left(a^2\lambda_b\lambda_c : b^2\lambda_c\lambda_a : c^2\lambda_a\lambda_b\right).$$

The points H' and E are antipodes in circumcenter being the isogonal conjugates of points at infinity on perpendicular lines.

Finally we will consider the inscribed parabola tangent to the line $\mathcal{L}$. This is the parabola

$$\mathcal{P}: \sum_{\text{cyclic}} \left(\lambda_a^2\, (\lambda_c\, v - \lambda_b\, w)^2\, x^2 - 2\lambda_b\, \lambda_c\, (\lambda_a\, w - \lambda_c\, u)\, (\lambda_b\, u - \lambda_a\, v)\, y\, z\right) = 0.$$

The center of the parabola $\mathcal{P}$ is the infinite point

$$J = ((\lambda_c\, v - \lambda_b\, w)\, \lambda_a : (\lambda_a\, w - \lambda_c\, u)\, \lambda_b : (\lambda_b\, u - \lambda_a\, v)\, \lambda_c).$$

The focus of $\mathcal{P}$ is the isogonal conjugate of J

$$F = \left(\frac{a^2\lambda_b\, \lambda_c}{\lambda_c\, v - \lambda_b\, w} : \frac{b^2\lambda_c\, \lambda_a}{\lambda_a\, w - \lambda_c\, u} : \frac{c^2\lambda_a\, \lambda_b}{\lambda_b\, u - \lambda_a\, v}\right),$$

and the perspector of $\mathcal{P}$, on the Steiner circumellipse $\mathcal{E}_0$, is the isotomic conjugate of J:

$$Q = \left(\frac{\lambda_b\, \lambda_c}{\lambda_c\, v - \lambda_b\, w} : \frac{\lambda_c\, \lambda_a}{\lambda_a\, w - \lambda_c\, u} : \frac{\lambda_a\, \lambda_b}{\lambda_b\, u - \lambda_a\, v}\right).$$

The point of contact between $\mathcal{P}$ and $\mathcal{L}$ is the point

$$T = \left(\frac{\lambda_a}{\lambda_c\, v - \lambda_b\, w} : \frac{\lambda_b}{\lambda_a\, w - \lambda_c\, u} : \frac{\lambda_c}{\lambda_b\, u - \lambda_a\, v} \right).$$

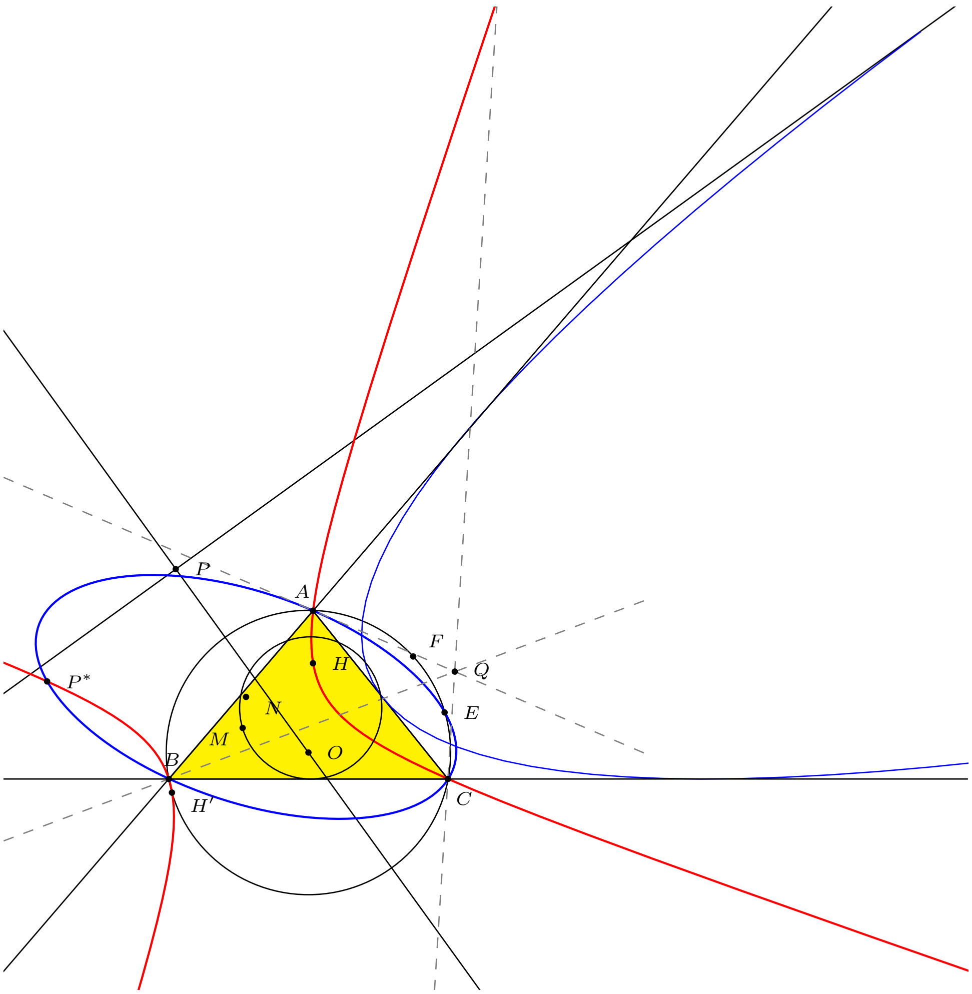

Figure 1. Three conics

Theorem 2. *The tangent to $\mathcal{E}$ at E*
(a) *passes through the focus F of $\mathcal{P}$;*
(b) *is parallel to the tangent to $\mathcal{E}$ at P^*, the isogonal conjugate of the point P;*
(c) *has as its pole K with respect to $\mathcal{P}$ on $\mathcal{H}$.*

Proof. (a) The tangent $\mathcal{T}$ to $\mathcal{E}$ at the point E has the equation

$$\frac{(\lambda_c\, v - \lambda_b\, w)\, \lambda_a^2}{a^2} x + \frac{(\lambda_a\, w - \lambda_c\, u)\, \lambda_b^2}{b^2} y + \frac{(\lambda_b\, u - \lambda_a\, v)\, \lambda_c^2}{c^2} z = 0. \qquad (5)$$

If $(x:y:z)$ are the coordinates of the point F, the left hand side of the above expression simplifies to a constant multiplied by $\lambda_a + \lambda_b + \lambda_c$. But this sum is equal to zero, verifying that the point F is on the tangent $\mathcal{T}$.

(b) The tangent to $\mathcal{E}$ at the point P^* is given by

$$\frac{(\lambda_c\, v - \lambda_b\, w)\, u^2}{a^2} x + \frac{(\lambda_a\, w - \lambda_c\, u)\, v^2}{b^2} y + \frac{(\lambda_b\, u - \lambda_a\, v)\, w^2}{c^2} z = 0.$$

The point of intersection of this line with the line $\mathcal{T}$ may be written as

$$\left((\lambda_c\, v + \lambda_b\, w)\, a^2 : (\lambda_a\, w + \lambda_c\, u)\, b^2 : (\lambda_b\, u + \lambda_a\, v)\, c^2\right)$$

The sum of this coordinates gives

$$\left(b^2\lambda_c + c^2\lambda_b\right) u + \left(c^2\lambda_a + a^2\lambda_c\right) v + \left(a^2\lambda_b + b^2\lambda_a\right) w.$$

The sum is equal to zero because this is the condition that the point P is on the line OP (2). This shows that the tangents to $\mathcal{E}$ at E and P^* are parallel.

(c) The polar K of the line $\mathcal{T}$ with respect to the parabola is given by

$$K = \left(\frac{\left(b^2\lambda_c + c^2\lambda_b\right) a^2}{(\lambda_c\, v - \lambda_b\, w)\, \lambda_a} : \frac{\left(c^2\lambda_a + a^2\lambda_c\right) b^2}{(\lambda_a\, w - \lambda_c\, u)\, \lambda_b} : \frac{\left(a^2\lambda_b + b^2\lambda_a\right) c^2}{(\lambda_b\, u - \lambda_a\, v)\, \lambda_c} \right).$$

Inserting the coordinates of the point K in the left hand side of the equation of $\mathcal{H}$, simplifies to

$$\left(\prod_{\text{cyclic}} \frac{\left(b^2\lambda_c + c^2\lambda_b\right) a^2}{(\lambda_c\, v - \lambda_b\, w)\, \lambda_a} \right) \sum_{\text{cyclic}} \left((\lambda_c\, v - \lambda_b\, w)\, \lambda_a\right).$$

But the sum is zero the as it represent the fact that the point $(\lambda_a : \lambda_b : \lambda_c)$ is on the line $\mathcal{L}$. This shows that the point K is on the hyperbola $\mathcal{H}$. □

Corollary 3. *The center N of the conic $\mathcal{E}$ is the midpoint of the points P^* and E.*

Corollary 4. *The directrix of the parabola is the line HK.*

Let R be the fourth intersection of the hyperbola $\mathcal{H}$ with the Steiner circum-ellipse.

Theorem 5. *The lines FH', EP^* and QR concur at the point K on $\mathcal{H}$.*

Proof. The equations of the lines FH' and EP^* are given by

$$FH' : \qquad \sum_{\text{cyclic}} \frac{\lambda_a}{a^2} (\lambda_b\, S_B - \lambda_c\, S_C)\, (\lambda_c\, v - \lambda_b\, w)\, x = 0$$

and

$$EP^* : \qquad \sum_{\text{cyclic}} \frac{\lambda_a}{a^2} (\lambda_c\, v - \lambda_b\, w)\, u\, x = 0.$$

It is easy to verify that the cross product of the line coordinates of this lines are proportional to the coordinates of the point K. The constant of proportionality is

$$\frac{\lambda_a\lambda_b\lambda_c}{2a^2b^2c^2} (u+v+w)\, (\lambda_c\, v - \lambda_b\, w)\, (\lambda_a\, w - \lambda_c\, u)\, (\lambda_b\, u - \lambda_a\, v).$$

On the other hand, the equation of the line QR is given by

$$\sum_{\text{cyclic}} a^2\lambda_a \left(b^2\lambda_c - c^2\lambda_b\right)\left(\lambda_c\, v - \lambda_b\, w\right) x = 0.$$

Inserting the coordinates of the point K gives

$$a^4\left(b^4\lambda_c^2 - c^4\lambda_b^2\right) + b^4\left(a^4\lambda_c^2 - c^4\lambda_a^2\right) + c^4\left(a^4\lambda_b^2 + b^4\lambda_a^2\right),$$

which is clearly equal to zero. □

Let D be the fourth intersection of the conic $\mathcal{E}$ with the Steiner circum-ellipse $\mathcal{E}_0$,

$$D = \left(\frac{1}{\left(\lambda_a\, w - \lambda_c\, u\right) b^2 + \left(\lambda_a\, v - \lambda_b\, u\right) c^2} : \cdots : \cdots\right).$$

Theorem 6. *The point D is on the line EQ.*

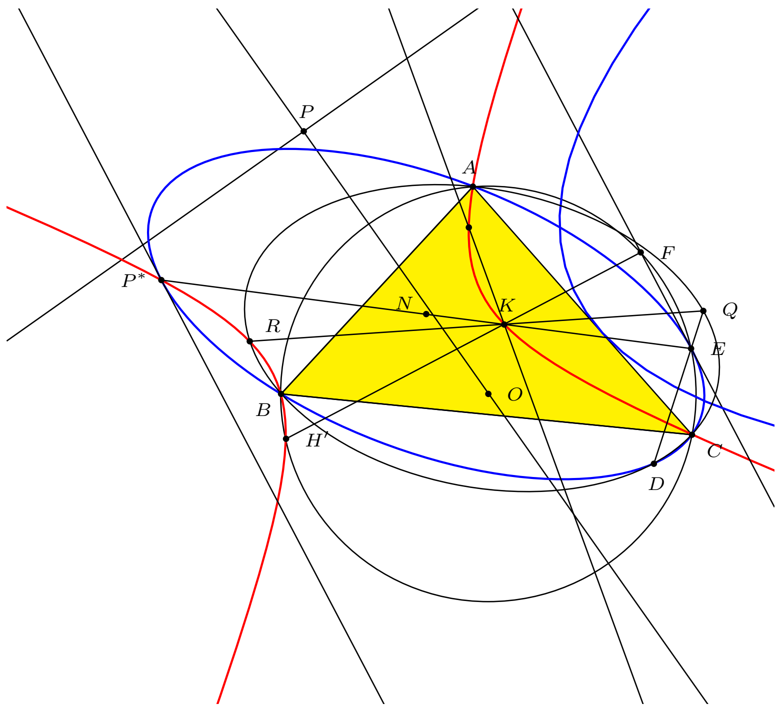

Figure 2. Collinearities

Proof. The line EQ can be written as

$$\sum_{\text{cyclic}} \lambda_a \left(\left(\lambda_a\, w - \lambda_c\, u\right) b^2 + \left(\lambda_a\, v - \lambda_b\, u\right) c^2\right)\left(\lambda_c\, v - \lambda_b\, w\right) x = 0$$

A direct calculation shows that, inserting the coordinates of the point D in this equation, simplifies to zero. □

Theorem 7. *The following pairs of* (*perpendicular*) *lines are parallel to the asymptotes of* $\mathcal{H}$*:*
(a) *the axes of* $\mathcal{E}$*,*
(b) *the tangents from* K *to the parabola* $\mathcal{P}$*.*

Proof. Let us denote with L_1 and L_2 the points of intersection of the line OP with the circumcircle of the triangle

$$L_1 = \left(a\,b\,c\,(\lambda_b\,S_B - \lambda_c\,S_C) + a^2 S_A\mu : \cdots : \cdots\right),$$
$$L_2 = \left(a\,b\,c\,(\lambda_b\,S_B - \lambda_c\,S_C) - a^2 S_A\mu : \cdots : \cdots\right),$$

where $\mu = \sqrt{\lambda_a^2\,S_A + \lambda_b^2\,S_B + \lambda_c^2\,S_C}$.

(a) The isogonal conjugates L_1^* and L_2^*, are the points where the asymptotes of the hyperbola $\mathcal{H}$ meet the line at infinity. The polars of L_1^* and L_2^* with respect to the conic $\mathcal{E}$ are diameters of the conic. If this diameters are conjugate with respect to $\mathcal{E}$, then they are orthogonal and are the axis of the said conic [1, page 220, §297]. But the polar of a point is conjugate to the one of another point if this last point is on the polar of the first point. The polar of the point L_1^* is the line

$$\sum_{\text{cyclic}} \left(\frac{b^2c^2\,(\lambda_b\,u - \lambda_a\,v)}{abc\,(\lambda_c\,S_C - \lambda_a\,S_A) + b^2 S_B\mu} + \frac{b^2c^2\,(\lambda_a\,w - \lambda_c\,u)}{abc\,(\lambda_a\,S_A - \lambda_b\,S_B) + c^2 S_C\mu} \right) x = 0$$

and a (not so short) calculation shows that, indeed L_2^* is on this polar. Thus the diameters are orthogonal and conjugate, and are the axis of the conic $\mathcal{E}$.

(b) As the point K lies on the directrix of $\mathcal{P}$ the tangents from K to $\mathcal{P}$ are perpendicular. Thus it suffice to show that the line KL_1^* is tangent to $\mathcal{P}$. The line KL_1^* can be expressed as

$$\sum_{\text{cyclic}} \left(\frac{b^2c\,\left(c^2\lambda_a + a^2\lambda_c\right)}{\lambda_b\,(\lambda_a\,w - \lambda_c\,u)\,f(c,a,b)} - \frac{bc^2\,\left(a^2\lambda_b + b^2\lambda_a\right)}{\lambda_c\,(\lambda_b\,u - \lambda_a\,v)\,f(b,c,a)} \right) x = 0$$

where $f(a,b,c) = b\,c\,(\lambda_b\,S_B - \lambda_c S_C) + a\,S_A\mu$. A long calculation shows that the line KL_1^* is tangent to $\mathcal{P}$. □

Let S be the second intersection of the line EP^* with the circumcircle,

$$S = \left(\frac{a^2}{(\lambda_c\,v - \lambda_b\,w)\,u} : \frac{b^2}{(\lambda_a\,w - \lambda_c\,u)\,v} : \frac{c^2}{(\lambda_b\,u - \lambda_a\,v)\,w} \right).$$

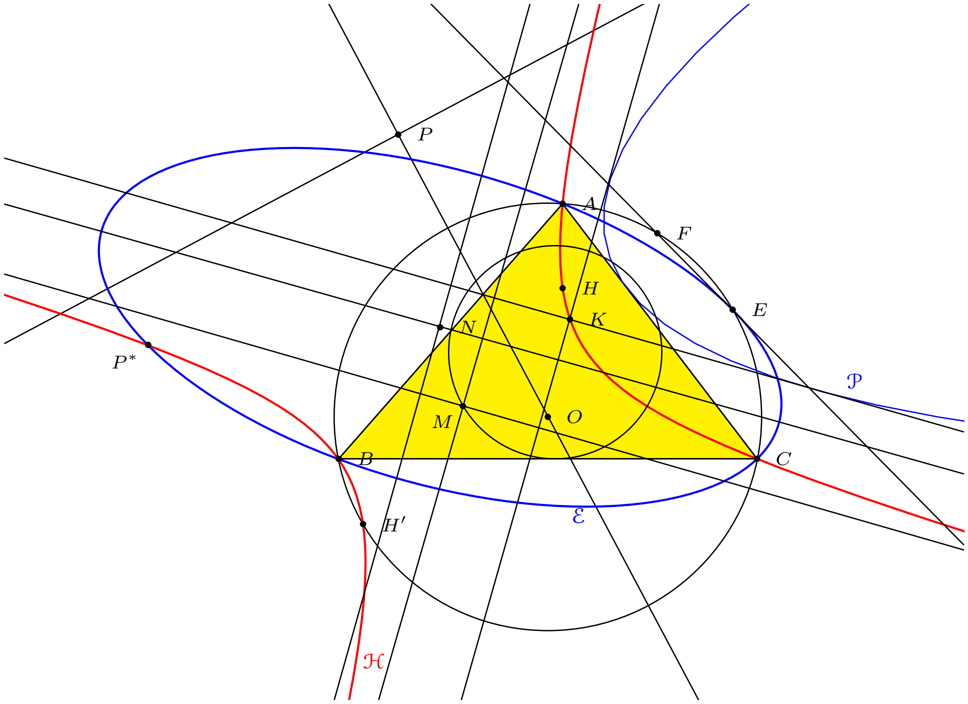

Figure 3. Asymptotes, axis and tangents

Theorem 8. *The pole P' of the line $\mathcal{L}$ is on the line FS.*

Proof. The line FS is given by

$$\frac{\lambda_a\,(\lambda_c\,v-\lambda_b\,w)^2\,u}{a^2}x+\frac{\lambda_b\,(\lambda_a\,w-\lambda_c\,u)^2\,v}{b^2}y+\frac{\lambda_c\,(\lambda_b\,u-\lambda_a\,v)^2\,w}{c^2}z=0,$$

and the point P' by

$$P'=\left((\lambda_c\,v-\lambda_b\,w)\,a^2-(\lambda_a\,w-\lambda_c\,u)\,b^2-(\lambda_b\,u-\lambda_a\,v)\,c^2:\cdots:\cdots\right).$$

Inserting the coordinates of P' in the equation of the line FS simplifies to

$$\left(\prod_{\text{cyclic}}(\lambda_c\,v-\lambda_b\,w)\right)\sum_{\text{cyclic}}\left(b^2\lambda_c+c^2\lambda_b\right)u$$

and, as already seen, the sum is equal to zero. □

P' is also the inverse in circumcircle of the point P. If T, on the line $\mathcal{L}$, is the pole of the line FS it follows that points O, P, F, S, and T are concyclic.

The point T can be expressed as

$$T=\left(\frac{(\lambda_c\,v+\lambda_b\,w)\,a^2}{(\lambda_c\,v-\lambda_b\,w)}:\frac{(\lambda_a\,w+\lambda_c\,u)\,b^2}{(\lambda_a\,w-\lambda_c\,u)}:\frac{(\lambda_b\,u+\lambda_a\,v)\,c^2}{(\lambda_b\,u-\lambda_a\,v)}\right).$$

The point T is also the center of a circle $\mathcal{C}$ through the points F and S. The circle $\mathcal{C}$ is orthogonal to the circumcircle.

Theorem 9. *Points on $\mathcal{C}$ are*
(a) *the point K,*
(b) *the intersections of the line $\mathcal{L}$ with the tangents from the point K to the parabola $\mathcal{P}$.*

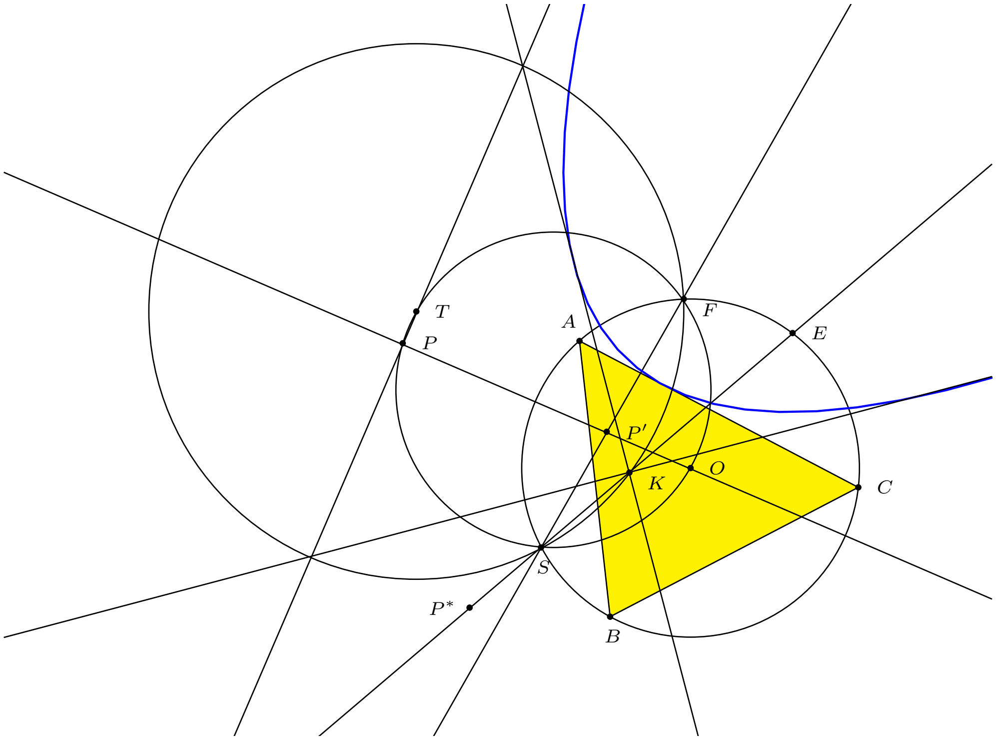

Figure 4. Circles

Proof. (a) A long calculation allows one to show that indeed, the point T is equidistant to the points F and K. [1] The common distance of the point T to the points F and S can be expressed as $d_1/(d_2 d_3)$ where

$$d_1 = \sum_{\text{cyclic}} a^4 S_A \left(b^2 w^2 \nu_c^2 - c^2 v^2 \nu_b^2\right)^2,$$

$$d_2 = (a^2 \nu_b\, \nu_c\, v\, w + b^2 \nu_c\, \nu_a\, w\, u + c^2 \nu_a\, \nu_b\, u\, v)^2,$$

$$d_3 = \left(\sum_{\text{cyclic}} \frac{a^2\,(w\lambda_b + v\lambda_c)}{\nu_a}\right)^2,$$

and

$$\nu_a = \lambda_c\, v - \lambda_b\, w, \quad \nu_b = \lambda_a\, w - \lambda_c\, u, \quad \nu_c = \lambda_b\, u - \lambda_a\, v.$$

[1] For an equation of the distance of two points in barycentric coordinates see [2, Chapter 7].

(b) Consider the triangle whose sides are the line $\mathcal{L}$ and the tangents to the parabola from the point K. The three sides of this triangle are tangent to the parabola. Thus the focus F is on the circumcircle of this triangle and the center of this circle is on the line $\mathcal{L}$. But by part (a) of the proof, the only circle through the points F and K with center on $\mathcal{L}$ is the circle $\mathcal{C}$. □

Interesting examples of the relations shown in this work arise if one takes the point P as the inverse in circumcircle of the symmedian point of the triangle[2], the inverse in circumcircle of the orthocenter, or when P is the intersection of the line OI, where I is the incenter, with the radical axis of the circumcircle and the incircle.

References

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[2]In this case the points E and Q are the same and there is no point D, the conics $\mathcal{E}$ and $\mathcal{E}_0$ coincide.