Forum Geometricorum
Volume 12 (2012) 191–192.

FORUM GEOM
ISSN 1534-1178

A Highway from Heron to Brahmagupta

Albrecht Hess

Abstract. We give a simple derivation of Brahmagupta's area formula for a cyclic quadrilateral from Heron's formula for the area of a triangle.

Brahmagupta's formula

$$A = \frac{1}{4}\sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}$$

for the area of a cyclic quadrilateral is very similar to Heron's formula

$$\Delta = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

for the area of a triangle, which is itself a consequence of Brahmagupta's formula for $d = 0$. Although I have searched extensively ([1, §3], [2, §9], [3], [4, Theorem 3.22], [5, Theorem 109]), the following derivation of the area of a cyclic quadrilateral from Heron's formula seems to be unknown.

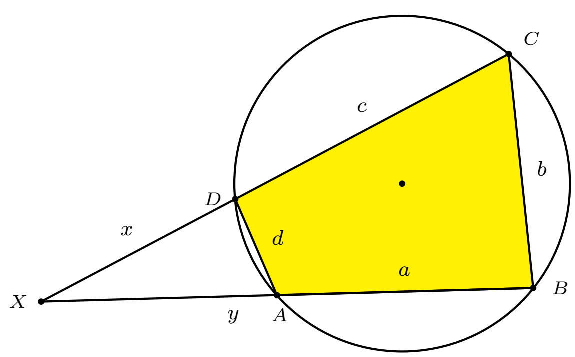

Figure 1

Let $ABCD$ be a cyclic quadrilateral with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$. Brahmagupta's formula is obvious if both pairs of opposite sides are parallel. We may assume that AB and CD intersect at point X and that $XD = x$, $XB = y$. Let S_1, S_2, S_3, S_4 be the four factors under the radical in Heron's

Publication Date: June 6, 2012. Communicating Editor: Paul Yiu.

formula for the area of triangle XBC. Note that from the similarity of triangles XBC and XDA (with ratio λ),

$$\begin{aligned} 4A &= 4\Delta(XBC) - 4\Delta(XDA) \\ &= \sqrt{S_1S_2S_3S_4} - \sqrt{(\lambda S_1)(\lambda S_2)(\lambda S_3)(\lambda S_4)} \\ &= \sqrt{(S_1 - \lambda S_1)(S_2 - \lambda S_2)(S_3 + \lambda S_3)(S_4 + \lambda S_4)}. \end{aligned}$$

Upon simplification, x and y vanish in these factors:

$$\begin{aligned} S_1 - \lambda S_1 &= (b + (c + x) + y) - (d + (y - a) + x) = a + b + c - d, \\ S_2 - \lambda S_2 &= (-b + (c + x) + y) - (-d + (y - a) + x) = a - b + c + d, \\ S_3 + \lambda S_3 &= (b - (c + x) + y) + (d - (y - a) + x) = a + b - c + d, \\ S_4 + \lambda S_4 &= (b + (c + x) - y) + (d + (y - a) - x) = -a + b + c + d, \end{aligned}$$

and Brahmagupta's formula appears.

References

[1] C. A. Bretschneider, Trigonometrische Relationen zwischen den Seiten und Winkeln zweier beliebiger ebener oder sphärischer Dreiecke, *Archiv der Math.*, 2 (1842) 132–145.
[2] C. A. Bretschneider, Untersuchung der trigonometrischen Relationen des geradlinigen Viereckes, *Archiv der Math.*, 2 (1842) 225–261.
[3] J. L. Coolidge, A historically interesting formula for the area of a quadrilateral, *Amer. Math. Monthly*, 46 (1939) 345–347.
[4] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. Amer. 1967.
[5] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.

Albrecht Hess: Deutsche Schule Madrid, Avenida Concha Espina 32, 28016 Madrid, Spain
E-mail address: albrecht.hess@gmail.com