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Non-Euclidean Versions of Some Classical Triangle Inequalities

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Abstract. In this paper we recall with short proofs of some classical triangle inequalities, and prove corresponding non-Euclidean, *i.e.*, spherical and hyperbolic versions of these inequalities. Among them are the well known Euler's inequality, Rouché's inequality (also called "the fundamental triangle inequality"), Finsler–Hadwiger's inequality, isoperimetric inequality and others.

1. Introduction

As it is well known, the Euclid's Fifth Postulate (through any point in a plane outside of a given line there is only one line parallel to that line) has many equivalent formulations. Recall some of them: sum of the angles of a triangle is π (or 180°), there are similar (non-congruent) triangles, there is the area function (with usual properties), every triangle has unique circumcircle, Pythagoras' theorem and its equivalent theorems such as the law of cosines, the law of sines, Heron's formula and many more.

The negations of the Fifth Postulate lead to spherical and hyperbolical geometries. So, negations of some equalities characteristic for the Euclidean geometry lead to inequalities specific for either spherical or hyperbolic geometry. For example, for a triangle in the Euclidean plane we have the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

where we stick with standard notations (that is a, b and c are the side lengths and $A,\ B$ and C are the angles opposite, respectively to the sides a, b and c).

It can be proved that the following Pythagoras' inequalities hold. In spherical geometry one has the inequality

$$c^2 < a^2 + b^2 - 2ab\cos C,$$

and in the hyperbolic geometry the opposite inequality

$$c^2 > a^2 + b^2 - 2ab\cos C.$$

In fact, in the hyperbolic case we have

$$a^2 + b^2 - 2ab\cos C < c^2 < a^2 + b^2 + 2ab\cos(A + B).$$

See [13] for details.

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On the other hand, there are plenty of interesting inequalities in (ordinary or Euclidean) triangle geometry relating various triangle elements. In this paper we prove some of their counterparts in non-Euclidean cases.

Let us fix (mostly standard) notations. For a given triangle $\triangle ABC$, let a, b, c denote the side lengths (a opposite to the vertex A, etc.), A, B, C the corresponding angles, $2s = a + b + c$ the perimeter, S its area, R the circumradius, r the inradius, and r_a, r_b, r_c the radii of excircles.

We use the symbols of cyclic sums and products such as:

$$\begin{aligned}\sum f(a) =&\ f(a) + f(b) + f(c),\\ \sum f(A) =&\ f(A) + f(B) + f(C),\\ \sum f(a,b) =&\ f(a,b) + f(b,c) + f(c,a),\\ \prod f(a) =&\ f(a)f(b)f(c),\\ \prod f(x) =&\ f(x)f(y)f(z).\end{aligned}$$

2. Euler's inequality

In 1765, Euler proved that the triangle's circumradius R is at least twice as big as its inradius r, *i.e.*,

$$R \geq 2r,$$

with equality if and only if the triangle is equilateral.Here is a short proof. $R \geq 2r \Leftrightarrow \frac{abc}{4S} \geq \frac{2S}{s} \Leftrightarrow sabc \geq 8S^2 = 8s\underbrace{(s-a)}_{=x}\underbrace{(s-b)}_{=y}\underbrace{(s-c)}_{=z} \Leftrightarrow \prod(s-x) \geq 8\prod x \Leftrightarrow s\sum xy - \prod x \geq 8\prod x \Leftrightarrow \sum x \cdot \sum xy \geq 9\prod x \Leftrightarrow \sum x^2y \geq 6\prod x \overset{A-G}{\Longleftrightarrow} \sum x^2y \geq 6(\prod x^2y)^{\frac{1}{6}} = 6\prod x$. [1] The equality case is clear.

The inequality $8S^2 \leq sabc$ (equivalent to Euler's) can also be easily obtained as a consequence (via $A-G$) of the "isoperimetric triangle inequality":

$$S \leq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}},$$

which we shall prove in §4.

The Euler inequality has been improved and generalized (*e.g.*, for simplices) many times. A recent and so far the best improvement of Euler's inequality is given by (see [11], [14]) (and it improves [17]):

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 1 \geq \frac{2}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq 2.$$

Now we turn to the non-Euclidean versions of Euler's inequality. Let k be the (constant) curvature of the hyperbolic plane in which a hyperbolic triangle $\triangle ABC$ sits. Let $\delta = \pi - (A + B + C)$ be the triangle's defect. The area of the hyperbolic triangle is given by $S = k^2\delta$.

[1] Yet another way to prove the last inequality: $x^2y + yz^2 = y(x^2 + z^2) \geq 2xyz$, and add such three similar inequalities.

Theorem 1 (Hyperbolic Euler's inequality)**.** *Suppose a hyperbolic triangle has a circumcircle and let R be its radius. Let r be the radius of the triangle's incircle. Then*

$$\tanh\frac{R}{k} \geq 2\tanh\frac{r}{k}. \tag{1}$$

The equality is achieved for an equilateral triangle for any fixed defect.

Proof. Recall that the radius R of the circumcircle of a hyperbolic triangle (if it exists) is given by

$$\tanh\frac{R}{k} = \sqrt{\frac{\sin\frac{\delta}{2}}{\prod\sin(A+\frac{\delta}{2})}} = \frac{2\prod\sinh\frac{a}{2k}}{\sqrt{\sinh\frac{s}{k}\prod\sinh\frac{s-a}{k}}} \tag{2}$$

Also, the radius of the incircle (radius of the inscribed circle) r of the hyperbolic triangle is given by

$$\tanh\frac{r}{k} = \sqrt{\frac{\prod\sinh\frac{s-a}{k}}{\sinh\frac{s}{k}}} \tag{3}$$

See, *e.g.*, [5], [6], [7], [8], [9]. We can take $k = 1$ in the above formulas. Then it is easy to see that (1) is equivalent to

$$\prod\sinh(s-a) \leq \prod\sinh\frac{a}{2},$$

or, by putting (as in the Euclidean case) $x = s-a$, $y = s-b$, $z = s-c$, to

$$\prod\sinh x \leq \prod\sinh\frac{s-x}{2}. \tag{4}$$

By writing $2x$ instead of x etc., (4) becomes

$$\prod\sinh 2x \leq \prod\sinh(s-x) = \prod\sinh(y+z).$$

Now by the double formula and addition formula for $\sinh$, after multiplications we get

$$8\prod\sinh x\cdot\prod\cosh x \leq \sum\sinh^2 x\ \sinh y\ \cosh y\ \cosh^2 z + 2\prod\sinh x\prod\cosh x.$$

Hence,

$$6\prod\sinh x\cdot\prod\cosh x \leq \sum\sinh^2 x\ \sinh y\ \cosh y\ \cosh^2 z. \tag{5}$$

However, (5) is simply the $A-G$ inequality for the six (nonnegative) numbers $\sinh x$, $\cosh x$, $\ldots$, $\cosh z$. The equality case follows easily. This proves the hyperbolic Euler's inequality. $\square$

Note also that (5) can be proved alternatively in the following way, using three times the simplest $A-G$ inequality:

$$\begin{aligned}
&\sinh^2 x\ \sinh y\ \cosh y\ \cosh^2 z + \cosh^2 x\ \sinh y\ \cosh y\ \sinh^2 z \\
&= \sinh y\ \cosh y[(\sinh x\ \cosh z)^2 + (\cosh x\ \sinh z)^2] \\
&\geq 2\sinh y\ \cosh y\ \sinh x\ \cosh z\ \cosh x\ \sinh z.
\end{aligned}$$

In the spherical case the analogous formula to (2) and (3) and similar reasoning to the previous proof boils down to proving analogous inequality to (4):

$$\prod \sin x \leq \prod \sin \frac{s-x}{2} \tag{6}$$

But (6) follows in the same manner as above. So, we have the following.

Theorem 2 (Spherical Euler's inequality)**.** *The circumradius R and the inradius r of a spherical triangle on a sphere of radius ρ are related by*

$$\tan \frac{R}{\rho} \geq 2 \tan \frac{r}{\rho}. \tag{7}$$

The equality is achieved for an equilateral triangle for any fixed spherical excess $\varepsilon = (A+B+C) - \pi$.

Remark. At present, we do not know how to improve these non-Euclidean Euler inequalities in the sense of the previous discussions in the Euclidean case. It would also be of interest to have the non-Euclidean analogues of the Euler inequality $R \geq 3r$ for a tetrahedron (and simplices in higher dimensions).

3. Finsler–Hadwiger's inequality

In 1938, Finsler and Hadwiger [3] proved the following sharp upper bound for the area S in terms of side lengths a, b, c of a Euclidean triangle (improving upon Weitzenboeck's inequality):

$$\sum a^2 \geq \sum (b-c)^2 + 4\sqrt{3}S. \tag{8}$$

Here are two short proofs of (8). First proof ([10]): Start with the law of cosines $a^2 = b^2 + c^2 - 2bc\cos A$, or equivalently $a^2 = (b-c)^2 + 2bc(1-\cos A)$. From the area formula $2S = bc\sin A$, it then follows $a^2 = (b-c)^2 + 4S\tan\frac{A}{2}$. By adding all three such equalities we obtain

$$\sum a^2 = \sum (b-c)^2 + 4S \sum \tan \frac{A}{2}.$$

By applying Jensen's inequality to the sum $\sum \tan\frac{A}{2}$ (*i.e.*, using convexity of $\tan\frac{x}{2}$, $0 < x < \pi$) and the equality $A + B + C = \pi$, (8) follows at once.

Second proof ([8]): Put $x = s - a$, $y = s - b$, $z = s - c$. Then

$$\sum [a^2 - (b-c)^2] = 4 \sum xy.$$

On the other hand, Heron's formula can be written as $4\sqrt{3}S = 4\sqrt{3\sum x \prod x}$. Then (8) is equivalent to $\sqrt{3\sum x \cdot \prod x} \leq \sum xy$, and this is equivalent to $\sum x^2yz \leq \sum (xy)^2$, which in turn is equivalent to $\sum [x(y-z)]^2 \geq 0$, and this is obvious.

Remark. The seemingly weaker Weitzenboeck's inequality

$$\sum a^2 \geq 4\sqrt{3}S$$

is, in fact, equivalent to (8) (see [17]).

There are many ways to rewrite Finsler–Hadwiger's inequality. For example, since

$$\sum [a^2 - (b-c)^2] = 4r(r+4R),$$

it follows that (8) is equivalent to

$$r(r+4R) \geq \sqrt{3}S,$$

or, since $S = rs$, it is equivalent to

$$s\sqrt{3} \leq r + 4R.$$

There are also many generalizations, improvements and strengthening of (8) (see [4]). Let us mention here only two recent ones. One is (see [1]):

$$\sum (b+c) \cdot \sum \frac{1}{b+c} \leq 10 - \frac{r}{s^2}[s\sqrt{3} + 2(r+4R)],$$

and the other one is (see [15])

$$\sum a^2 \geq 4\sqrt{3}S + \sum (a-b)^2 + \sum [\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}]^2.$$

The opposite inequality of (8) is (see [17]):

$$\sum a^2 \leq 4\sqrt{3}S + 3\sum (b-c)^2.$$

Note that all these inequalities are sharp in the sense that equalities hold if and only if the triangles are equilateral (regular).

For the hyperbolic case, we need first an analogue of the area formula $2S = bc \sin A$. It is not common in the literature, so for the reader's convenience we provide its short proof (see *e.g.*, [5]).

Lemma 3 (Cagnolli's first formula)**.** *The area $S = k^2\delta$ of a hyperbolic triangle ABC is given by*

$$\sin \frac{S}{2k^2} = \frac{\sinh \frac{a}{2k} \sinh \frac{b}{2k} \sin C}{\cosh \frac{c}{2k}} \tag{9}$$

Proof. From the well known second (or "polar") law of cosines in elementary hyperbolic geometry

$$\cosh \frac{a}{k} = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

we get

$$\cosh \frac{a}{2k} = \sqrt{\frac{\sin\left(B + \frac{\delta}{2}\right) \sin\left(C + \frac{\delta}{2}\right)}{\sin B \sin C}}, \quad \sinh \frac{a}{2k} = \sqrt{\frac{\sin\left(\frac{\delta}{2}\right) \sin\left(A + \frac{\delta}{2}\right)}{\sin B \sin C}}. \tag{10}$$

By multiplying two expressions $\sinh \frac{a}{2k} \cdot \sinh \frac{b}{2k}$, and using (10) we get

$$\sinh \frac{a}{2k} \cdot \sinh \frac{b}{2k} = \frac{\sin \frac{\delta}{2}}{\sin C} \cosh \frac{c}{2k}.$$

This implies (9). □

Theorem 4 (Hyperbolic Finsler–Hadwiger's inequality)**.** *For a hyperbolic triangle ABC we have:*

$$\sum \cosh \frac{a}{k} \geq \sum \cosh \frac{b-c}{k} + 12 \sin \frac{S}{2k^2} \prod \cosh \frac{a}{2k} \tan \frac{\pi - \delta}{6} \tag{11}$$

The equality in (11) *holds if and only if for any fixed defect δ, the triangle is equilateral.*

Proof. The idea is to try to mimic (as much as possible) the first proof of (8). Start with the hyperbolic law of cosines

$$\cosh \frac{a}{k} = \cosh \frac{b}{k} \cosh \frac{c}{k} - \sinh \frac{b}{k} \sinh \frac{c}{k} \cos A.$$

By adding and subtracting $\sinh \frac{b}{k} \sinh \frac{c}{k}$, we obtain

$$\begin{aligned} \cosh \frac{a}{k} &= \cosh \frac{b-c}{k} + \sinh \frac{b}{k} \sinh \frac{c}{k} - \sinh \frac{b}{k} \sinh \frac{c}{k} \cos A \\ &= \cosh \frac{b-c}{k} + \sinh \frac{b}{k} \sinh \frac{c}{k} \cdot 2 \sin^2 \frac{A}{2} \\ &= \cosh \frac{b-c}{k} + 4 \sinh \frac{b}{2k} \sinh \frac{c}{2k} \cosh \frac{b}{2k} \cosh \frac{c}{2k} \cdot 2 \sin^2 \frac{A}{2}. \end{aligned}$$

By Cagnolli's formula (9), substitute here the part $\sinh \frac{b}{2k} \sinh \frac{c}{2k}$ to obtain

$$\cosh \frac{a}{k} = \cosh \frac{b-c}{k} + 4 \cosh \frac{a}{2k} \cosh \frac{b}{2k} \cosh \frac{c}{2k} \sin \frac{S}{2k^2} \tan \frac{A}{2}. \tag{12}$$

Apply to both sides of (12) the cyclic sum operator $\sum$, and (again) apply Jensen's inequality (*i.e.*, convexity of $\tan \frac{x}{2}$):

$$\frac{1}{3} \sum \tan \frac{A}{2} \geq \tan \left(\frac{1}{3} \sum \frac{A}{2} \right) = \tan \frac{\pi - \delta}{6}.$$

This implies (11). The equality claim is also clear from the above argument. $\square$

The corresponding spherical Finsler–Hadwiger inequality can be obtained mutatis mutandis from the hyperbolic case. The area S of a spherical triangle ABC on a sphere of radius ρ is given by $S = \rho^2 \varepsilon$, where $\varepsilon = A + B + C - \pi$ is the triangle's excess. The spherical Cagnolli formula (like 9) reads as follows:

$$\sin \frac{S}{2\rho^2} = \frac{\sin \frac{a}{2\rho} \sin \frac{b}{2\rho} \sin C}{\cos \frac{c}{2\rho}}. \tag{13}$$

So, starting with the spherical law of cosines, using (13) and Jensen's inequality, one can show the following.

Theorem 5 (Spherical Finsler–Hadwiger's inequality)**.** *For a spherical triangle ABC on a sphere of radius ρ we have*

$$\sum \cos \frac{a}{\rho} \geq \sum \cos \frac{b-c}{\rho} + 12 \sin \frac{S}{2\rho^2} \cos \frac{a}{2\rho} \cos \frac{b}{2\rho} \cos \frac{c}{2\rho} \tan \frac{\varepsilon - \pi}{6}. \tag{14}$$

The equality in (14) *holds if and only if for any fixed ε, the triangle is equilateral.*

Remark. Note that both hyperbolic and spherical inequalities (11) and (14) reduce to Finsler–Hadwiger's inequality (8) when $k \to \infty$ in (11), or $\rho \to \infty$ in (14). This is immediate from the power sum expansions of trigonometric or hyperbolic functions.

4. Isoperimetric triangle inequalities

In the Euclidean case, if we multiply all three area formulas, one of which is $S = \frac{1}{2}bc\sin A$, we obtain a symmetric formula for the triangle area

$$S^3 = \frac{1}{8}(abc)^2 \sin A \ \sin B \ \sin B. \tag{15}$$

By using the $A-G$ inequality and the concavity of the function $\sin x$ on $[0,\pi]$ (or, Jensen's inequality again), we have:

$$\begin{aligned}\sin A \ \sin B \ \sin C \leq \ & \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \\ \leq \ & \left(\sin\frac{A+B+C}{3}\right)^3 = \sin^3\frac{\pi}{3} = \frac{3\sqrt{3}}{8}.\end{aligned}$$

This and (15) imply the so called "isoperimetric inequality" for a triangle:

$$S^3 \leq \frac{3\sqrt{3}}{64}(abc)^2, \text{or in a more appropriate form}$$

$$S \leq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}. \tag{16}$$

Inequality (16) and $A-G$ imply that $S \leq \frac{\sqrt{3}}{36}(a+b+c)^2$, and this is why we call it the "isoperimetric inequality".

By Heron's formula we have $(4S)^2 = 2sd_3(a,b,c)$, where $2s = a+b+c$ and $d_3(a,b,c) := (a+b-c)(b+c-a)(c+a-b)$. By [11, Cor. 6.2], we have a sharp inequality

$$d_3(a,b,c) \leq \frac{(2abc)^2}{a^3+b^3+c^3+abc}. \tag{17}$$

From Heron's formula and (17) it easily follows

$$S \leq \frac{1}{2}abc\sqrt{\frac{a+b+c}{a^3+b^3+c^3+abc}}. \tag{18}$$

We claim that (18) improves the "isoperimetric inequality" (16). Namely, we claim

$$\frac{1}{2}abc\sqrt{\frac{a+b+c}{a^3+b^3+c^3+abc}} \leq \frac{\sqrt{3}}{4}\sqrt[3]{(abc)^2}. \tag{19}$$

But (19) is equivalent to

$$\left(\frac{a^3+b^3+c^3+abc}{4}\right)^3 \geq (abc)^2\left(\frac{a+b+c}{3}\right)^3. \tag{20}$$

To prove (20) we can take $abc = 1$ and prove

$$\frac{a^3+b^3+c^3+1}{4} \geq \frac{a+b+c}{3}. \tag{21}$$

Instead, we prove an even stronger inequality

$$\frac{a^3+b^3+c^3+1}{4} \geq \sqrt[3]{\frac{a^3+b^3+c^3}{3}}. \tag{22}$$

Inequality (22) is stronger than (21) because the means are increasing, *i.e.*,

$$M_p(a,b,c) \leq M_q(a,b,c) \quad \text{for } a,b,c > 0 \text{ and } 0 \leq p \leq q,$$

where $M_p(a,b,c) = \left[\frac{(a^p+b^p+c^p)}{3}\right]^{\frac{1}{p}}$. To prove (22), denote $x = a^3+b^3+c^3$ and consider the function

$$f(x) = \left(\frac{x+1}{4}\right)^3 - \frac{x}{3}.$$

Since (by $A-G$) $\frac{x}{3} \geq abc = 1$, *i.e.*, $x \geq 3$, we consider $f(x)$ only for $x \geq 3$. Since $f(3) = 0$ and the derivative $f'(x) \geq 0$ for $x \geq 3$, we conclude $f(x) \geq 0$ for $x \geq 3$ and hence prove (19).

Putting all together, we finally have a chain of inequalities for the triangle area S symmetrically expressed in terms of the side lengths a, b, c.

Theorem 6 (Improved Euclidean isoperimetric triangle inequalities)**.**

$$S \leq \frac{1}{2}abc\sqrt{\frac{a+b+c}{a^3+b^3+c^3+abc}} \leq \frac{1}{4}\sqrt[6]{\frac{3(a+b+c)^3(abc)^4}{a^3+b^3+c^3}} \leq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}} \tag{23}$$

We shall now make an analogue of the "isoperimetric inequality" (16) in the hyperbolic case.

Start with Cagnolli's formula (9) and multiply all such three formulas to get (since $S = \delta k^2$):

$$\sin^3\frac{\delta}{2} = \prod \sinh\frac{a}{2k}\prod\tanh\frac{a}{2k}\prod\sin A. \tag{24}$$

As in the Euclidean case we have

$$\prod \sin A \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \leq \left(\sin\frac{A+B+C}{3}\right)^3 = \left(\sin\frac{\pi-\delta}{3}\right)^3$$

So, this inequality together with (24) implies the following.

Theorem 7. *The area $S = \delta k^2$ of a hyperbolic triangle with side lengths a, b, c satisfies the following inequality*

$$\left(\frac{\sin\frac{\delta}{2}}{\sin\frac{\pi-\delta}{3}}\right)^3 \leq \prod\sinh\frac{a}{2k}\cdot\prod\tanh\frac{a}{2k}. \tag{25}$$

For an equilateral triangle ($a = b = c, A = B = C$) *and any fixed defect δ, the inequality* (25) *becomes an equality* (by Cagnolli's formula (9)).

The corresponding isoperimetric inequality can be obtained for a spherical triangle:

$$\left(\frac{\sin\frac{\varepsilon}{2}}{\sin\frac{\varepsilon-\pi}{3}}\right)^3 \le \prod \sin\frac{a}{2\rho}\cdot\prod \tan\frac{a}{2\rho}. \tag{26}$$

Remark. In the 3–dimensional case we have a well known upper bound of the volume V of a (Euclidean) tetrahedron in terms of product of lengths of its edges (like (16)) :

$$V \le \frac{\sqrt{2}}{12}\sqrt{abcdef}$$

with equality if and only if the tetrahedron is regular (and similarly in any dimension); see [12].

Non–Euclidean tetrahedra (and simplices) lack good volume formulas of Heron's type, except the Cayley–Menger determinant formulas in all three geometries. Kahan's formula [2] for volume of a Euclidean tetrahedron is known only for the Euclidean case. There are some volume formulas for tetrahedra in all three geometries now available on Internet, but they are rather involved. We don't know at present how to use them to obtain a good and simple enough upper bound.

In dimension 2, Heron's formula in all three geometries can very easily be deduced. A very short proof of Heron's formula is as follows. Start with the triangle area $4S = 2ab\sin C$ and the law of cosines $a^2+b^2-c^2 = 2ab\cos C$. Now square and add them. The result is a form of the Heron's formula $(4S)^2+(a^2+b^2-c^2)^2 = (2ab)^2$. In a similar way one can get triangle area formulas in the non-Euclidean case by starting with Cagnolli's formula ((9) or (13)) and the appropriate law of cosines.

The result in the hyperbolic geometry is the formula

$$\left(4\sin\frac{\delta}{2}\prod\cosh\frac{a}{2k}\right)^2 + \left(\cosh\frac{a}{k}\cosh\frac{b}{k} - \cosh\frac{c}{k}\right)^2 = \left(\sinh\frac{a}{k}\sinh\frac{b}{k}\right)^2$$

or

$$\left(4\sin\frac{\delta}{2}\prod\cosh\frac{a}{2k}\right)^2 + \sum\cosh^2\frac{a}{k} = 1 + 2\prod\cosh\frac{a}{k}.$$

Remark. In order to improve the non-Euclidean 2–dimensional isoperimetric inequality analogous to (23) we would need an analogue of the function $d_3(a,b,c)$ and a corresponding inequality like (17). This inequality was proved in [11] as a consequence of the inequality $d_3(a^2,b^2,c^2) \le d_3^2(a,b,c)$, and this follows from an identity expressing the difference $d_3^2(a,b,c) - d_3(a^2,b^2,c^2)$ as a sum of four squares. But at present we do not know the right hyperbolic analogue $d_3^H(a,b,c)$ or spherical analogue $d_3^S(a,b,c)$ of the function $d_3(a,b,c)$.

[2] see www.cs.berkeley.edu/~wkahan/VtetLang.pdf, 2001.

5. Rouché's inequality and Blundon's inequality

The following inequality is a necessary and sufficient condition for the existence of an (Euclidean) triangle with elements R, r and s (see [4]):

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \leq s^2$$
$$\leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}. \tag{27}$$

This inequality (sometimes called "the fundamental triangle inequality") was first proved by É. Rouché in 1851, answering a question of Ramus. It was recently improved in [16].

A short proof of (27) is as follows. Let r_a, r_b, r_c be the excircle radii of the triangle ABC. It is well known (and easy to check) that $\sum r_a = 4R+r$, $\sum r_a r_b = s^2$ and $r_a r_b r_c = rs^2$. Hence r_a, r_b, r_c are the roots of the cubic

$$x^3 - (4R+r)x^2 + s^2 x - rs^2 = 0. \tag{28}$$

Now consider the discriminant of this cubic, *i.e.*, $D = \prod (r_a - r_b)^2$.
In terms of the elementary symmetric functions e_1, e_2, e_3 in the variables r_a, r_b, r_c,

$$D = e_1^2 e_2^2 - 4e_2^3 - 4e_1^3 e_3 + 18 e_1 e_2 e_3 - 27 e_3^2. \tag{29}$$

Since $e_1 = \sum r_a = 4R + r$, $e_2 = \sum r_a r_b = s^2$, $e_3 = \prod r_a = rs^2$, we have

$$D = s^2[(4R+r)^2 s^2 - 4s^4 - 4(4R+r)^3 r + 18(4R+r)rs^2 - 27r^2 s^2].$$

From $D \geq 0$, (27) follows easily. In fact, the inequality $D \geq 0$ reduces to the quadratic inequality in s^2:

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + (4R+r)^3 r \leq 0. \tag{30}$$

The "fundamental" inequality (27) implies a sharp linear upper bound of s in terms of r and R, known as Blundon's inequality [2]:

$$s \leq (3\sqrt{3} - 4)r + 2R. \tag{31}$$

To prove (31), it is enough to prove that

$$2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr} \leq [(3\sqrt{3} - 4)r + 2R]^2.$$

A little computation shows that this is equivalent to the following cubic inequality (with $x = R/r$):

$$f(x) := 4(3\sqrt{3}-5)x^3 - 3(60\sqrt{3}-103)x^2 + 12(48\sqrt{3}-83)x + 4(229-132\sqrt{3}) \geq 0.$$

By Euler's inequality $x \geq 2$, $f(2) = 0$ and hence clearly $f(x) \geq 0$ for $x \geq 2$.

Yet another (standard) way to prove Blundon's inequality (31) is to use the convexity of the biquadratic function on the left hand side of the inequality (30).

Blundon's inequality is also sharp in the sense that equality holds in (31) if and only if the triangle is equilateral. (Recall by the way that a triangle is a right triangle if and only if $s = r + 2R$).

Let us turn to non-Euclidean versions of the "fundamental triangle inequality".

Suppose a hyperbolic triangle has a circumscribed circle. As before, denote by R, r, and r_a, r_b, r_c, respectively, the radii of the circumscribed, inscribed and escribed circles of the triangle. Then by (2) and (3) we know R and r, while r_a (and similarly r_b and r_c) is given by

$$\tanh\frac{r_a}{k} = \sinh\frac{s}{k}\tan\frac{A}{2}, \tag{32}$$

and by using

$$\tan\frac{A}{2} = \sqrt{\frac{\sinh\frac{s-b}{k}\sinh\frac{s-c}{k}}{\sinh\frac{s}{k}\sinh\frac{s-a}{k}}}. \tag{33}$$

The combination of these two expresses r_a in terms of a, b, and c. In order to obtain for the hyperbolic triangle the analogue of the cubic equation (28) whose roots are $x_1 = \tanh\frac{r_a}{k}$, $x_2 = \tanh\frac{r_b}{k}$, $x_3 = \tanh\frac{r_c}{k}$, we have to compute the elementary symmetric functions e_1, e_2, e_3 in the variables x_1, x_2, x_3. We compute first (the easiest) e_3. Equations (32), (33) and (3) yield

$$e_3 = \prod\tanh\frac{r_a}{k} = \sinh^2\frac{s}{k}\tanh\frac{r}{k}. \tag{34}$$

Next, by (32) and (33):

$$e_2 = \sum\tanh\frac{r_a}{k}\cdot\tanh\frac{r_b}{k} = \sinh^2\frac{s}{k}\sum\tan\frac{A}{2}\tan\frac{B}{2} = \sinh\frac{s}{k}\sum\sinh\frac{s-a}{k}.$$

Applying the identity

$$\sinh(x+y+z) - (\sinh x + \sinh y + \sinh z) = 4\sinh\frac{y+z}{2}\sinh\frac{z+x}{2}\sinh\frac{x+y}{2},$$

with $x = \frac{s-a}{2}, y = \frac{s-b}{2}, z = \frac{s-c}{2}$, we obtain

$$\sinh\frac{s}{k} - \sum\sinh\frac{s-a}{k} = 4\prod\sinh\frac{a}{2k}. \tag{35}$$

And now from (2) and (3) we get

$$e_2 = \sinh^2\frac{s}{k}\left(1 - 2\tanh\frac{r}{k}\tanh\frac{R}{k}\right). \tag{36}$$

Finally, to compute e_1, we use the identity

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan y\tan z - \tan z\tan x}. \tag{37}$$

By (32), $e_1 = \sinh\frac{s}{k}\sum\tan\frac{A}{2}$. Now from (37):

$$\sum\tan\frac{A}{2} = \tan\frac{A+B+C}{2}\left(1 - \sum\tan\frac{A}{2}\tan\frac{B}{2}\right) + \prod\tan\frac{A}{2},$$

$$\tan\frac{A+B+C}{2} = \tan\frac{\pi-\delta}{2} = \cot\frac{\delta}{2}.$$

From (3), we have $\prod\tan\frac{A}{2} = \frac{\tanh\frac{r}{k}}{\sinh\frac{s}{k}}$.

By (33), (35), and (2), (3) it follows easily

$$1 - \sum\tan\frac{A}{2}\tan\frac{B}{2} = 2\tanh\frac{r}{k}\tanh\frac{R}{k}\sinh\frac{s}{k}.$$

Finally, putting all together yields

$$e_1 = \tanh \frac{r}{k} \left(1 + 2 \tanh \frac{R}{k} \sinh \frac{s}{k} \cot \frac{\delta}{2}\right). \tag{38}$$

Equations (34), (36) and (38) yield via $x^3 - e_1 x^2 + e_2 x - e_3 = 0$ the cubic equation

$$\begin{aligned} & x^3 - \tanh \frac{r}{k} \left(1 + 2 \tanh \frac{R}{k} \sinh \frac{s}{k} \cot \frac{\delta}{2}\right) x^2 \\ & + \sinh^2 \frac{s}{k} \left(1 - 2 \tanh \frac{r}{k} \tanh \frac{R}{k}\right) x - \sinh^2 \frac{s}{k} \tanh \frac{r}{k} = 0. \end{aligned} \tag{39}$$

This cubic (with roots $\tanh \frac{r_a}{k}$ etc.) reduces to the cubic (28) by letting $k \to \infty$. This follows from the identity

$$\frac{\sinh \frac{s}{k} \cdot \tanh \frac{r}{k}}{\sin \frac{\delta}{2}} = 2 \prod \cosh \frac{a}{2k}.$$

If $k \to \infty$, then the right hand side tends to 2 and therefore the coefficient by x^2 in (39) goes to $r + 4R$ which appears in (28); similarly for the other coefficients.

Consider the discriminant of (39)

$$D = \prod \left(\tanh \frac{r_a}{k} - \tanh \frac{r_b}{k}\right)^2.$$

Now, by applying (29) and (34), (36) and (38) we obtain the quartic polynomial (in fact degree 6) in $\sinh \frac{s}{k}$ for an expression D. By the following legend

$$\begin{array}{ll} r \longleftrightarrow \tanh \frac{r}{k} & \delta \longleftrightarrow \cot \frac{\delta}{2} \\ R \longleftrightarrow \tanh \frac{R}{k} & s \longleftrightarrow \sinh \frac{s}{k} \end{array} \tag{40}$$

we can write D as follows (after some computation); note that it has almost double number of terms than the corresponding Euclidean discriminant

$$\begin{aligned} D = \;& s^2[(r^2R^2\delta^2 + 4r^4R^4\delta^2 - 4r^3R^3\delta^2 - 1 + 6rR - 12r^2R^2 + 8r^3R^3)s^4 \\ & + r^2R\delta(1 - 4rR + 4r^2R^2\delta - 8r^2R^2\delta^2 + 9\delta + 18rR\delta)s^3 \\ & + r^2(r^2R^2 - 10rR - 12r^2R^2\delta^2 - 2)s^2 \\ & - 6r^4R\delta s - r^4]. \end{aligned} \tag{41}$$

By definition $D \geq 0$, so the quartic polynomial in s (in fact in $\sinh \frac{s}{k}$), *i.e.*, the polynomial in brackets in (41) is ≥ 0.

So the hyperbolic analogue of the “fundamental triangle inequality” (27), or rather degree–four polynomial inequality (30) is the quartic (in s) polynomial inequality $\frac{D}{s^2} \geq 0$.

Theorem 8 (Hyperbolic “fundamental triangle inequality”)**.** *For a hyperbolic triangle that has a circumcircle of radius R, incircle of radius r, semiperimeter s, and excess δ, we have*

$$\frac{D}{s^2} \geq 0, \tag{42}$$

where D is given by (41) *together with the legend* (40)*. When $k \to \infty$,* (42) *reduces to* (30).

Blundon's hyperbolic inequality can also be derived as a corollary of Theorem 8.

The spherical version of the "fundamental inequality" as well as the corresponding spherical Blundon's inequality can also be obtained, but we omit them here.

In conclusion, we may say that all these triangle inequalities give more information and better insight to the geometry of 2– and 3– manifolds.

References

[1] S. J. Bilichev and P. M. Vlamos, About some improvements of the Finsler–Hadwiger's inequality, *Geometry and Math. Competitions, 4th. Congress of the World Fed. of Mat., Math. Comp.*, Melbourne, 2002; 19–36.

[2] W. J. Blundon, Inequalities associated with the triangle, *Canad. Math. Bull.*, 8 (1965) 615–626.

[3] P. Finsler and H. Hadwiger, Einige Relationen im Dreieck, *Comment. Math. Helv.*, 10 (1938) 316–326.

[4] D. S. Mitrinović, J. E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluver Acad. Publ., Amsterdam, 1989.

[5] N. M. Nestorovich, *Geometricheskie postroenija v ploskosti Lobachevskogo*, (in Russian) M.–L.:GITTL, Leningrad, 1951.

[6] V. V. Prasolov, V. M. Tikhomirov, *Geometry*, Translations of Mathematical Monographs, AMS, Providence, R.I., 2001.

[7] V. V. Prasolov, *Geometrija Lobachevskogo*, (in Russian) MCNMO, Moscow, 2004.

[8] V. V. Prasolov, *Problems in Planimetry*, (in Russian) MCNMO, OAO, "Moscow textbooks", Moscow, 2006.

[9] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, GTM, Springer Verlag, New York, 1994.

[10] J. M. Steele, *The Cauchy–Schwarz Master Class*, MAA, Cambridge University Press, Cambridge, 2004.

[11] D. Svrtan, I. Urbiha, Verification and strengthening of the Atiyah–Sutcliffe conjectures for the sine theorem and several types of configurations, arXiv:math/0609174

[12] D. Veljan, Inequalities for volumes of simplices and determinants, *Lin. Alg. and its Appl.*, 219 (1995) 79–91.

[13] D. Veljan, Geometry and convexity of $\cos\sqrt{x}$, *Amer. Math. Monthly*, 111 (2004) 592–595.

[14] D. Veljan, S. Wu, Parametrized Klamkin's inequality and improved Euler's inequality, *Math. Inequalities Appl.*, 11 (2008) 729–737.

[15] Sh.–H. Wu, Generalizations and sharpness of Finsler–Hadwiger's inequality and its applications, *Math. Inequalities Appl.*, 9 (2006) 421–426.

[16] S. Wu, A sharpened version of the fundamental triangle inequality, *Math. Inequalities Appl.*, 11 (2008) 477–482.

[17] Sh.–H. Wu, Zh.–H. Zhang, and Zh.–G. Xiao, On Weitzenboeck's inequality and its generalizations, *RGMIA Research Report Collection*, 6(4), Article 14, 2003.

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