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The Spheres Tangent Externally to the Tritangent Spheres of a Triangle

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Abstract. We consider the tritangent circles of a triangle as the great circles of spheres in three dimensional space, and identify the spheres tangent externally to these four spheres.

In the plane of a triangle ABC we consider the tritangent circles, the incircle and the three excircles. It is well known that the nine-point circle is tangent to the excircles externally and to the incircle internally. Together with the sidelines of ABC, considered as degenerate circles, this is the only circle tangent to all four tritangent circles. Considering the tritangent circles as the sections of spheres by the plane containing their centers, we wonder if there are spheres quadritangent to these "tritangent spheres", apart from the one containing the nine-point circle. In this paper we identify the spheres tangent externally to the four tritangent spheres. We use methods similar to [4]. By symmetry it is enough to consider spheres on one side of the plane.

Let us start with the excircles $\mathcal{C}_a = I_a(r_a)$, $\mathcal{C}_b = I_b(r_b)$ and $\mathcal{C}_c = I_c(r_c)$, and the excircle-spheres $\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$ in 3-dimensional space with the same centers and radii. Consider a sphere with radius ρ, and center D at a distance d above the plane of triangle ABC, and tangent to the three excircle-spheres. Clearly, $\rho \geq \frac{R}{2}$, where R is the circumradius of triangle ABC. The orthogonal projection of the center onto the plane is the radical center of the circles $I_a(r_a+\rho)$, $I_b(r_b+\rho)$ and $I_c(r_c+\rho)$. For $\rho = \frac{R}{2}$, this is the nine-point center N. In general, this projection lies on the line joining N to the radical center of the excircles, namely, the Spieker center S_{p}. The power of S_{p} with respect to each excircle is $\frac{r^2+s^2}{4}$, where r and s are the inradius and semiperimeter of the triangle (see, for example, [2, Theorem 4]).

Let P be the reflection of S_{p} in N. A simple application of Menelaus' theorem (to triangle PIS_{p} with transversal GNH) shows that it is also the midpoint between the incenter I and the orthocenter H (see Figure 1).

Theorem 1. *The sphere $\mathcal{Q}$ with radius R, and center at $\frac{\sqrt{bc+ac+ab}}{2}$ above the point P, is tangent externally to the four tritangent spheres.*

Proof. Consider triangle I_aPS_{p} with median I_aN. Note that $I_aN = \frac{R}{2} + r_a$ and $NS_{\text{p}} = \frac{1}{2}OI$, where O is the circumcenter. It follows that $NS_{\text{p}}^2 = \frac{1}{4}R(R-2r)$ by Euler's formula. Since the power of S_{p} with respect to each excircle is $\frac{1}{4}(r^2+s^2)$,

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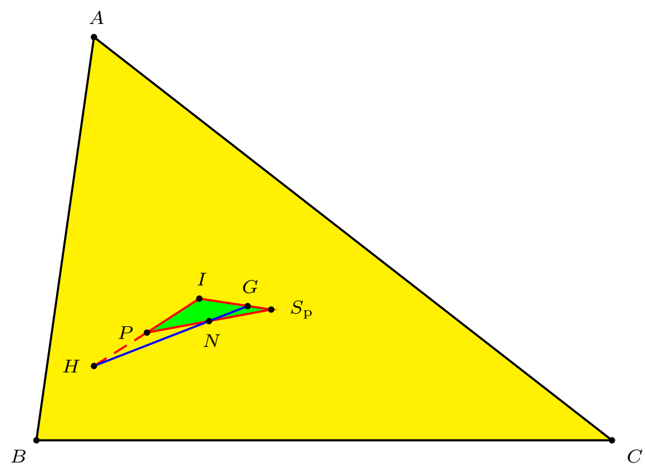

Figure 1.

$I_aS_{\rm p}^2 = \frac{r^2+s^2}{4} + r_a^2$. Applying Apollonius' theorem to triangle $I_aPS_{\rm p}$, we have

$$\begin{aligned} I_aP^2 &= 2I_aN^2 + 2NS_{\rm p}^2 - I_aS_{\rm p}^2 \\ &= 2\left(\frac{R}{2}+r_a\right)^2 + \frac{1}{2}R(R-2r) - \frac{r^2+s^2}{4} - r_a^2 \\ &= (R+r_a)^2 - \frac{r^2+s^2+4Rr}{4} \\ &= (R+r_a)^2 - \frac{ab+bc+ca}{4}. \end{aligned}$$

The last equality follows from $R = \frac{abc}{4\Delta}$, $r = \frac{\Delta}{s}$ and Heron's formula for the area Δ. Similarly,

$$I_bP^2 = (R+r_b)^2 - \frac{ab+bc+ca}{4} \quad \text{and} \quad I_cP^2 = (R+r_c)^2 - \frac{ab+bc+ca}{4}.$$

By letting D be the point at a distance $d := \frac{\sqrt{ab+bc+ca}}{2} = \frac{\sqrt{r^2+s^2+4Rr}}{2}$ above P, we have

$$I_aD = R + r_a, \quad I_bD = R + r_b, \quad I_cD = R + r_c.$$

Therefore the sphere $\mathcal{Q}$ with center D, radius R, is tangent to each of $\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$. Since the point P is also the midpoint of IH, and $IH^2 = 4R^2 + 4Rr + 3r^2 - s^2$ (see [1, p.50]), we have

$$DI^2 = \frac{r^2+s^2+4Rr}{4} + \frac{4R^2+4Rr+3r^2-s^2}{4} = (R+r)^2.$$

This shows that $\mathcal{Q}$ is also tangent to the incircle-sphere $\mathcal{S}$. □

The point P, which is the reflection of $S_{\rm p}$ in N (also the midpoint of IH), is the triangle center

$$X_{946} = (a^3(b+c) + (b-c)^2(a^2 - a(b+c) - (b+c)^2) : \cdots : \cdots)$$

in [3].

The orthogonal projections to the plane of ABC of the points of contact of $\mathcal{Q}$ with the excircle-spheres form a triangle $A'B'C'$. The point A', for instance, is the point that divides the segment PI_a in ratio $R : r_a$. Let AA' intersect the line IP at Q (see Figure 2). Applying Menelaus' theorem to triangle PII_a with transversal AXA', we have

$$\frac{PQ}{QI} \cdot \frac{IA}{AI_a} \cdot \frac{I_aA'}{A'P} = -1 \implies \frac{PQ}{QI} \cdot \frac{-r}{r_a} \cdot \frac{r_a}{R} = -1 \implies \frac{PQ}{QI} = \frac{R}{r}.$$

Similarly, the lines BB' and CC' intersect IP at the same point Q, which divides PI in the ratio $R : r$. This is the orthogonal projection of the point of tangency of $\mathcal{Q}$ with the incircle-sphere $\mathcal{S}$. It has barycentric coordinates

$$\left(\frac{b+c}{b+c-a} : \frac{c+a}{c+a-b} : \frac{a+b}{a+b-c}\right),$$

and is the triangle center X_{226} in [3].

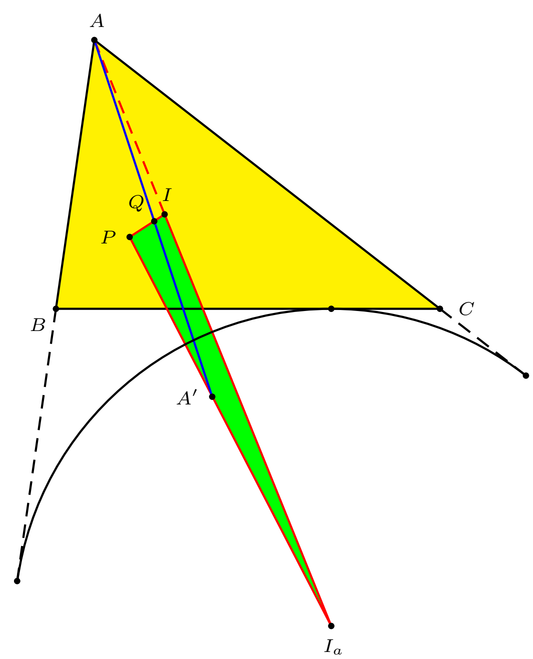

Figure 2.

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